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RECURRENCE RELATIONS FOR POLYNOMIALS OF HYPERGEOMETRIC CHARACTER

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ABSTRACT We derive various pure and differential recurrence relations for polynomials of terminating hypergeometric character by making use of Sister Celine's method

1. Introduction

The pure recurrence relation for hypergeometric polynomials received probably its first systematic attack at the hand of Sister Mary Celine Fasenmyer in a Michigan thesis in 1945. She introduced the tool in her study of a certain class of hypergeometric polynomials [2] Our object in the present paper is to obtain various recurrence relations of well known polynomials by using Sister Celine's method.

The generalized hypergeometric function with p numerator and q denominator parameters is defined by

$$(1.1)\qquad {}_pF_q(\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q,z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ \cdots(\alpha_p)_n}{(\beta_1)_n\ \cdots(\beta_q)_n}\frac{z^n}{n!},$$

where $(\alpha)_n$ denotes the Pochhammer symbol defined by

$$(1.2)\qquad (\alpha)_n=\begin{cases}\alpha(\alpha+1)\dots(\alpha+n-1) & \text{if n=1,2,3,\dots,}\\ 1 & \text{if n=0,}\end{cases}$$

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for any complex number α.

Equation (1.2) yields

$$(\alpha)_{m+n} = (\alpha)_m(\alpha+m)_n, \tag{1.3}$$

and

$$(\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}, \qquad 0 \le k \le n. \tag{1.4}$$

For $\alpha = 1$ in (1.4), we have

$$(n-k)! = \frac{(-1)^k n!}{(-n)_k}, \qquad 0 \le k \le n. \tag{1.5}$$

2. Pure recurrence relations

At first, consider Rice's polynomials [5]

$$H_n = H_n(\zeta, p, \nu) = {}_3F_2(-n, n+1, \zeta; 1, p; \nu), \tag{2.1}$$

which, for $p = \frac{1}{2}$ and $\zeta = a$, becomes Sister Celine's polynomial

$$f_n(\nu) = {}_3F_2(-n, n+1, a, 1, \frac{1}{2}; \nu).$$

Now, with the aid of (1.3) and (1.5), we have

$$H_n = \sum_{k=0}^{\infty} \frac{(-n)_k(n+1)_k(\zeta)_k}{(1)_k(p)_k k!}\nu^k = \sum_{k=0}^{\infty} \frac{(-1)^k(n+k)!(\zeta)_k\nu^k}{(n-k)!(p)_k(k!)^2}, \tag{2.2}$$

which, for convenience, is rewritten as in the following form

$$H_n = \sum_{k=0}^{\infty} \epsilon(k,n), \tag{2.3}$$

where $\epsilon(k,n)$ denotes the general term of the right-hand most summation part of (2.2).

Sister Celine's technique is to express H_{n-1}, H_{n-2}, νH_{n-1}, etc., as series involving $\epsilon(k,n)$, and then to find a combination of coefficients which vanishes identically.

We observe that, in view of (2.3),

$$H_{n-1} = \sum_{k=0}^{\infty} \frac{n-k}{n+k}\epsilon(k,n), \tag{2.4}$$

$$H_{n-2} = \sum_{k=0}^{\infty} \frac{(n-k)(n-k-1)}{(n+k)(n+k-1)}\epsilon(k,n), \tag{2.5}$$

$$H_{n-3} = \sum_{k=0}^{\infty} \frac{(n-k)(n-k-1)(n-k-2)}{(n+k)(n+k-1)(n+k-2)}\epsilon(k,n), \tag{2.6}$$

$$\nu H_{n-1} = \sum_{k=0}^{\infty} \frac{-k^2(p+k-1)}{(\zeta+k-1)(n+k)(n+k-1)}\epsilon(k,n), \tag{2.7}$$

$$\nu H_{n-2} = \sum_{k=0}^{\infty} \frac{-k^2(p+k-1)(n-k)}{(\zeta+k-1)(n+k)(n+k-1)(n+k-2)}\epsilon(k,n) \tag{2.8}$$

In equations (2.3)-(2.8) the coefficients of $\epsilon(k,n)$ have a lowest common denominator $(\zeta+k-1)(n+k)(n+k-1)(n+k-2)$. When that denominator is used in each coefficient, the maximum degree with respect to k of the numerators is four. Then there exist constants (functions of n but not of k or ν) A, B, C, D, and E such that

$$H_n + (A+B\nu)H_{n-1} + (C+D\nu)H_{n-2} + EH_{n-3} = 0 \tag{2.9}$$

is an identity. We find that (2.9) is written equivalently to the following identity in k.

$$\begin{aligned}(\zeta+k-1)&(n+k)(n+k-1)(n+k-2)\\ &+ A(n-k)(\zeta+k-1)(n+k-1)(n+k-2)\\ -Bk^2(p+k-1)&(n+k-2) + C(n-k)(n-k-1)(\zeta+k-1)(n+k-2)\\ -Dk^2(p+k-1)&(n-k) + E(n-k)(n-k-1)(n-k-2)(\zeta+k-1) = 0\end{aligned} \tag{2.10}$$

The choice $k = n,\ \ k = 1-\zeta,\ \ k = 2-n,\ \ k = 1-n,$ the coefficients of ξ^4 in (2.10) yield

$$B = \frac{2(2n-1)(\zeta+n-1)}{n(p+n-1)}, D = \frac{2(2n-1)(\zeta-n+1)}{n(p+n-1)} \tag{2.11}$$

$$E = \frac{(n-2)(2n-1)(p-n+1)}{n(2n-3)(p+n-1)}, C = \frac{(2n-3)[2(n-1)^2 - n(p-n+1)]}{n(2n-3)(p+n-1)},$$

$$A = -\frac{(2n-1)[2(n-1)(2n-3) + (n-2)(p-n+1)]}{n(2n-3)(p+n-1)},$$

which, with replaced in (2.9), yields a pure recurrence relation of the Rice's polynomials H_n:

$$\begin{aligned} &n(2n-3)(p+n-1)H_n - (2n-1)[(n-2)(p-n+1) \\ &+2(n-1)(2n-3) - 2(2n-3)(\zeta+n-1)\nu]H_{n-1} \\ &+(2n-3)[2(n-1)^2 - n(p-n+1) + 2(2n-1)(\zeta-n+1)\nu]H_{n-2} \\ &+(n-2)(2n-1)(p-n+1)H_{n-3} = 0. \end{aligned} \tag{2.12}$$

Next, consider polynomials

$$f_n(x) = {}_1F_2(-n; 1+\alpha, 1+\beta; x), \tag{2.13}$$

which is intimately related to Bateman's polynomials $J_n^{u,v}$ (cf., e.g., Rainville [4]). Now

$$f_n(x) = \sum_{k=0}^{\infty} \frac{(-n)_k x^k}{(1+\alpha)_k (1+\beta)_k k!} = \sum_{k=0}^{\infty} \frac{(-1)^k n! x^k}{(n-k)! k! (1+\alpha)_k (1+\beta)_k}. \tag{2.14}$$

Put

$$\gamma_n(x) = \frac{f_n(x)}{n!}, \tag{2.15}$$

where

$$\gamma_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(n-k)! k! (1+\alpha)_k (1+\beta)_k} := \sum_{k=0}^{\infty} \epsilon(k, n), \tag{2.16}$$

from which it follows that, by using the same procedure as before,

$$\gamma_{n-1}(x) = \sum_{k=0}^{\infty}(n-k)\epsilon(k,n), \tag{2.17}$$

$$\gamma_{n-2}(x) = \sum_{k=0}^{\infty}(n-k)(n-k-1)\epsilon(k,n), \tag{2.18}$$

$$\gamma_{n-3}(x) = \sum_{k=0}^{\infty}(n-k)(n-k-1)(n-k-2)\epsilon(k,n), \tag{2.19}$$

$$x\gamma_{n-1}(x) = -\sum_{k=0}^{\infty}k(\alpha+k)(\beta+k)\epsilon(k,n), \tag{2.20}$$

$$x\gamma_{n-2}(x) = -\sum_{k=0}^{\infty}k(n-k)(\alpha+k)(\beta+k)\epsilon(k,n), \tag{2.21}$$

and so there exists a relation

$$\gamma_n(x) + (A+Bx)\gamma_{n-1}(x) + (C+Dx)\gamma_{n-2}(x) + E\gamma_{n-3}(x) = 0 \tag{2.22}$$

where A, B, C, D, and E are determined by the identity in k

$$\begin{aligned} 1 + A(n-k) - Bk(\alpha+k)(\beta+k) + C(n-k)(n-k-1) \\ -Dk(n-k)(\alpha+k)(\beta+k) + E(n-k)(n-k-1)(n-k-2) = 0. \end{aligned} \tag{2.23}$$

The choice $k=n$, the coefficients of k^4, $k=n-1$, $k=n-2$, $k=0$ in (2.23) yield

$$\begin{aligned} B &= \frac{1}{n(\alpha+n)(\beta+n)}, \qquad D = 0, \\ A &= -\frac{[3n^2-3n+1+(2n-1)(\alpha+\beta)+\alpha\beta]}{n(\alpha+n)(\beta+n)}, \\ C &= \frac{3n-3+\alpha+\beta}{n(\alpha+n)(\beta+n)}, \qquad E = -\frac{1}{n(\alpha+n)(\beta+n)}. \end{aligned} \tag{2.24}$$

Thus the polynomials $\gamma_n(x)$ satisfy a pure recurrence relation

$$\begin{aligned}(2.25)\quad & n(\alpha+n)(\beta+n)\gamma_n(x)-[3n^2-3n+1+(2n-1)(\alpha+\beta)\\ &+\alpha\beta-x]\gamma_{n-1}(x)+(3n-3+\alpha+\beta)\gamma_{n-2}(x)-\gamma_{n-3}(x)=0,\end{aligned}$$

which, in terms of (2.15), yields a pure recurrence relation for $f_n(x)$:
(2.26)

$$\begin{aligned}&(\alpha+n)(\beta+n)f_n(x)-[3n^2-3n+1+(2n-1)(\alpha+\beta)+\alpha\beta-x]f_{n-1}(x)\\ &+(n-1)(3n-3+\alpha+\beta)f_{n-2}(x)-(n-1)(n-2)f_{n-3}(x)=0.\end{aligned}$$

Consider Shively's polynomials [6]

$$\sigma_n(x)=\frac{(2n)!}{(n!)^2}{}_1F_2(-n,\,1+n,1;\,x)$$

which are related to the $f_n(x)$. If we set $\alpha=n$ and $\beta=0$ in (2.13), $f_n(x)$ becomes $\frac{(n!)^2}{(2n)!}\sigma_n(x)$ and (2.26) yields a pure recurrence relation for Shively's polynomials:

$$\begin{aligned}(2.27)\quad & n^3\sigma_n(x)-(2n-1)(5n^2-4n+1-x)\sigma_{n-1}(x)\\ &+2(4n-3)(2n-1)(2n-3)\sigma_{n-2}(x)-4(2n-1)(2n-3)(2n-5)\sigma_{n-3}=0.\end{aligned}$$

The pseudo-Laguerre polynomials (see [1]) $g_n(x)$ are defined, for nonintegral λ, by

$$(2.28)\qquad g_n(x)=\frac{(-\lambda)_n}{n!}{}_1F_1(-n;\,1+\lambda-n;\,x)=\sum_{k=0}^{n}\frac{(-\lambda)_{n-k}x^k}{k!(n-k)!}.$$

When $\lambda=a+2n-1$ the polynomials $g_n(x)$ become $(-1)^nR_n(a,x)$, where Shively's polynomials $R_n(a,x)$ are defined by

$$(2.29)\qquad R_n(a,x)=\frac{(a)_{2n}}{n!(a)_n}{}_1F_1(-n,\,a+n;\,x).$$

In the same manner, using Sister Celine's method, $g_n(x)$ satisfies the pure recurrence relation

$$(2.30)\qquad ng_n(x)=(x+n-1-\lambda)g_{n-1}(x)-xg_{n-2}(x).$$

From (2.30) Shively's polynomials $R_n(a,x)$ satisfy the relation

$$(2.31)\qquad nR(a,x)=-(x-a-n)R_{n-1}(a,x)-xR_{n-2}(a,x).$$

3. Mixed recurrence relations

Define the polynomial $v_n(x)$ by

$$v_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n! P_k(x)}{(k!)^2 (n-k)!} \tag{3.1}$$

in terms of the Legendre polynomial $P_k(x)$, where $P_k(x)$ is defined by the generating relation

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} P_k(x) t^n, \tag{3.2}$$

which $(1 - 2xt + t^2)^{-\frac{1}{2}}$ denotes the particular branch which $\to 1$ as $t \to 0$. Put

$$S_n(x) := \frac{v_n(x)}{n!} \quad \text{and} \quad \frac{(-1)^k P_k(x)}{(k!)^2} := u_k(x). \tag{3.3}$$

Then

$$S_n(x) = \sum_{k=0}^{\infty} \frac{u_k(x)}{(n-k)!}. \tag{3.4}$$

From the known relation

$$(1 - x^2) P_k''(x) - 2x P_k'(x) + k(k+1) P_k(x) = 0, \tag{3.5}$$

we determine a recurrence relation for $u_k(x)$ which may be used to find a relation for $S_n(x)$, and finally (3.3) is employed to transform our result into a relation for $v_n(x)$.

In (3.5) put $P_k(x) = (-1)^k (k!)^2 u_k(x)$. The resulting relation for $u_k(x)$ is

$$(1 - x)^2 u_k''(x) - 2x u_k'(x) + k(k+1) u_k(x) = 0. \tag{3.6}$$

First we set out to use (3.4) to obtain series

$$\sum_{k=0}^{\infty} \frac{k(k+1)u_k(x)}{(n-k)!}. \tag{3.7}$$

From (3.4) we get

$$S_n''(x) = \sum_{k=0}^{\infty} \frac{u_k''(x)}{(n-k)!}, \quad S_n'(x) = \sum_{k=0}^{\infty} \frac{u_k'(x)}{(n-k)!}, \tag{3.8}$$

$$S_{n-1}(x) = \sum_{k=0}^{\infty} \frac{(n-k)u_k(x)}{(n-k)!}, \quad S_{n-2}(x) = \sum_{k=0}^{\infty} \frac{(n-k)(n-k-1)u_k(x)}{(n-k)!}.$$

As before we observe that there exist constants A, B, and C such that

$$AS_n(x) + BS_{n-1}(x) + CS_{n-2}(x) = \sum_{k=0}^{\infty} \frac{k(k+1)u_k(x)}{(n-k)!} \tag{3.9}$$

From (3.8) and (3.9) we obtain

$$A = n(n+1), \qquad B = -2n, \qquad C = 1, \tag{3.10}$$

which, in conjunction with (3.9), yields

$$n(n+1)S_n(x) - 2nS_{n-1}(x) + S_{n-2}(x) = \sum_{k=0}^{\infty} \frac{k(k+1)u_k(x)}{(n-k)!}. \tag{3.11}$$

From (3.8)-(3.11) it follows that

(3.12)

$$(1-x^2)S_n''(x) - 2xS_n'(x) + n(n+1)S_n(x) - 2nS_{n-1}(x) + S_{n-2}(x) = 0.$$

With (3.3) and (3.12) we obtain the following mixed recurrence relation for $v_n(x)$:

(3.13)

$$(1-x^2)v_n''(x) - 2xv_n'(x) + n(n+1)v_n(x) = 2n^2 v_{n-1}(x) - n(n-1)v_{n-2}(x).$$

We conclude this note by remarking that the Sister Celine method can be applied to get a pure or mixed recurrence relation for given polynomials systematically, before her it seemed customary upon entering the study of a new set of polynomials to seek recurrence relations, pure or mixed (with or without derivatives involved) by essentially a hit-and-miss process (see [3]).

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REFERENCES

[1] R.P.,Jr Boas and R. C. Buck. *Polynomial Expansion of Analytic Functions*, Springer-Verlag, Berlin, 1958

[2] S. M. Celine, *Some generalized hypergeometric polynomials*, Bull. Amer. Math. Soc. **53** (1947), 806-812

[3] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960

[4] E. D. Rainville, *The contiguous function relations for ${}_pF_q$ with applications to Bateman's $J_n^{u,v}$ and Rice's $H_n(\zeta, p, v)$*, Bull. Amer. Math. Soc. **51** (1945), 714-723

[5] S. O. Rice, *Some properties of ${}_3F_2(-n, n+1, \zeta; 1, p, v)$*, Duke Math. J. **6** (1940), 108-119

[6] R. L. Shively, *On pseudo Laguerre polynomials*, Michigan thesis, 1953

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