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ON GENERALIZED SYMMETRIC BI-DERIVATIONS IN PRIME RINGS

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ABSTRACT After the derivation was defined in [19] by Posner a lot of researchers studied the derivations in ring theory in different manners such as in [2], [4], [5], , etc Furthermore, many researches followed the definition of the generalized derivation ([3], [6], [7], .., etc). Finally, Maksa defined a symmetric bi-derivation and many researches have been done in ring theory by using this definition In this work, defining a symmetric bi-α-derivation, we study the mentioned researches above in the light of this new concept

1. Introduction

Throughout this work, R will represent an associative ring and Z will denote the center of R. We set $C_\alpha = \{c \in R \mid c\alpha(x) = xc, \forall x \in \text{R}\}$, and $[x,y]_\alpha = x\alpha(y) - yx$ and $(x,y)_\alpha = x\alpha(y) + yx$, where α is a non-zero mapping of R. In particular, $C_1 = Z$, and $[x,y]_1 = xy - yx$ and $(x,y)_1 = xy + yx = (x,y)$, in the usual sense. Furthermore, we use the relation:

$$[x, yz]_\alpha = y[x,z]_\alpha + [x,y]_\alpha \alpha(z)$$
$$[xy, z]_\alpha = x[y,z]_\alpha + [x,z]y = x[y,\alpha(z)] + [x,z]_\alpha y$$
$$(x, yz)_\alpha = y(x,z)_\alpha + [x,y]_\alpha \alpha(z)$$

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and

$$(xy,z)_\alpha = x(y,z)_\alpha - [x,z]y = x[y,\alpha(z)] + (x,z)_\alpha y.$$

A mapping $D(.,.) : R \times R \to R$ is called symmetric if $D(x,y) = D(y,x)$ holds for all $x, y \in R$. A mapping $d : R \to R$ defined by $d(x) = D(x,x)$ is the called trace of $D(.,.)$, where $D(.,.) : R \times R \to R$ is a symmetric mapping. It is obvious that, if $D(.,.) : R \times R \to R$ is a symmetric mapping which is also bi-additive (i.e. additive in both arguments) then d the trace of $D(.,.)$ satisfies the relation $d(x+y) = d(x) + d(y) + 2D(x,y)$ for all $x, y \in R$.

A symmetric bi-additive mapping $D(.,.) : R \times R \to R$ is called a symmetric bi-derivation if $D(xy,z) = D(x,z)y + xD(y,z)$ is fulfilled for all $x, y, z \in R$. Then the relation $D(x,yz) = D(x,y)z + yD(x,z)$ is also fulfilled for all $x, y, z \in R$.

2. The results

We shall need the following well-known and frequently used lemmas.

LEMMA 1. *([14, Lemma 2. (ii)]) Let R be a prime ring , $a \in R$ and $d : R \to R$ an α-derivation. If U is a non-zero ideal of R and $ad(U) = 0$ then $a = 0$ or $d = 0$.*

LEMMA 2. *([11, Lemma 1]) Let R be a prime ring and U a non-zero right ideal of R. If U is commutative, then R is commutative*

LEMMA 3. *([8, Lemma 1]) Let R be a semi-prime, 2-torsion free ring and U a Lie ideal of R. If $[U,U] \subset Z$, then $U \subset Z$.*

LEMMA 4 *([13, Lemma 3]) Let R be a prime ring, $a, b \in R$ and σ, τ an automorphism of R. If $b, ab \in C_{\sigma,\tau}$, then $a \in Z$ or $b = 0$.*

We shall start with the following definition.

DEFINITION 5. Let R be a ring. A symmetric bi-additive mapping $D(.,.) : R \times R \to R$ is called a symmetric bi-α-derivation if $D(xy,z) = D(x,z)\alpha(y) + xD(y,z)$ is fulfilled for all $x, y, z \in R$, where $\alpha : R \times R$ is a non-zero mapping. Then the relation $D(x,yz) = D(x,y)\alpha(z) + yD(x,z)$ is also fulfilled for all $x, y, z \in R$.

For any fixed $y \in R$, a mapping $x \mapsto D(x,y)$ is an α-derivation, where $D(.,.)$ is a symmetric bi-α-derivation of R.

EXAMPLE 6. For a commutative ring R, let
$M := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in R \right\}$, it is obvious that M is a ring under matrix addition and multiplication. $D(,) : M \times M \rightarrow M$, defined by $\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix} \right) \rightarrow \begin{pmatrix} 0 & a_1 a_2 \\ 0 & 0 \end{pmatrix}$ is a symmetric bi-α-derivation, where $\alpha : M \rightarrow M$ defined by $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is a mapping.

REMARK 7. *Let R be a 2-torsion free (i.e. $2x = 0$ implies $x = 0$) prime ring and α a homomorphism of R. If α is an even function, then $\alpha = 0$. Therefore, when R is 2-torsion free ring, the homomorphism $\alpha : R \rightarrow R$ assumed to be an odd function.*

THEOREM 8. *Let R be a prime ring which is 2-torsion free. Let $D(.,.)$ be a symmetric bi-α-derivation of R and d the trace of $D(,.)$, where α is an automorphism of R. If $[d(x), x]_\alpha = 0$ for all x in$R$, then R is commutative or $D = 0$.*

PROOF. We assume that R is non-commutative. From the hypothesis, for any $x, y \in R$

$$[d(x+y), x+y]_\alpha + [d(-x+y), -x+y]_\alpha = 0$$

and since R is 2-torsion free we have, for all $x, y \in R$

$$[d(x), y]_\alpha + 2[D(x,y), x]_\alpha = 0. \tag{1}$$

Writing xy for y in (1), from hypothesis and (1), since R is 2-torsion free we obtain, for all $x, y \in R$

$$d(x)\alpha([x, y]) = 0. \tag{2}$$

From (2) and Lemma 1 one can conclude that, for $x \notin Z$ and since α is automorphism of R we have $d(x) = 0$ (note that, for any fixed $x \in R$ a mapping $y \mapsto [x, y]$ is a derivation). Let $x \in Z, y \notin Z$. Then $-y, x+y, x+(-y) \notin Z$. Thus, $0 = d(x+y) = d(x) + 2D(x,y)$ and $0 = d(x+(-y)) = d(x) - 2D(x,y)$ which implies that $d(x) = 0$. Therefore we have proved that $d(x) = 0$ for all $x \in R$, which means that $D(x, y) = 0$ for all $x, y \in R$. Namely, D is zero

THEOREM 9. *Let R be a prime ring which is 2-torsion free. Let $D(.,.)$ be a symmetric bi-α-derivation of R and d the trace of $D(.,.)$, where α is an automorphism of R. If $[d(x),x]_\alpha \in C_\alpha$ for all $x \in R$, then R is commutative or $D = 0$.*

PROOF. We assume that R is non-commutative. In this case, from the hypothesis, for any $x, y \in R$

$$[d(x+y), x+y]_\alpha + [d(-x+y), -x+y]_\alpha \in C_\alpha$$

and since R is 2-torsion free we have, for all $x, y \in R$

$$[d(x), y]_\alpha + 2[D(x,y), x]_\alpha \in C_\alpha. \tag{3}$$

Replacing y by x^2 in (3), from hypothesis and (3), since R is 2-torsion free we get, for all $x, y \in R$

$$x[d(x), x]_\alpha \in C_\alpha. \tag{4}$$

Thus, from hypothesis and (4), we have $yx[d(x),x]_\alpha = x[d(x),x]_\alpha \alpha(y)$ and so we get, for all $x, y \in R$

$$[x,y][d(x),x]_\alpha = 0. \tag{5}$$

Now, the relation above makes it possible to conclude, using the same arguments as in the proof of Theorem 8, that for any $x \notin Z$ we have $[d(x),x]_\alpha = 0$. Thus, from Theorem 8 we obtain $D = 0$.

THEOREM 10. *Let R be a 2-torsion free prime ring, α be an automorphism of R, $D(.,.)$ a symmetric bi-α-derivation of R and d the trace of $D(.,.)$. If $(d(x),x)_\alpha = 0$ for all $x \in R$, then $[d(x),x]_\alpha = 0$ for all $x \in R$. Furthermore, if R is non-commutative, then $D = 0$.*

PROOF. From the hypothesis, for any $x, y \in R$

$$(d(x+y), x+y)_\alpha + (d(-x+y), -x+y)_\alpha = 0$$

and since R is 2-torsion free we have, for all $x, y \in R$

$$(d(x), y)_\alpha + 2(D(x,y), x)_\alpha = 0. \tag{6}$$

Replacing y by yx in (6), from (6) we get, for all $x, y \in R$

$$[d(x), y]_\alpha + 2(D(x,y), x)_\alpha \alpha(x) + 2[x,y]d(x) = 0. \tag{7}$$

Now, right multiplication of the relation (6) by $\alpha(x)$ gives for all $x, y \in R$

$$(d(x), y)_\alpha \alpha(x) + 2(D(x,y), x)_\alpha \alpha(x) = 0. \tag{8}$$

Combining (7) and (8), from the hypothesis we have, for all $x, y \in R$

$$xyd(x) = 0. \tag{9}$$

Replacing y by $d(x)yx$ in (9) and since R is prime ring we have, for all $x, y \in R$

$$xd(x) = 0. \tag{10}$$

Consequently, from the hypothesis and (10) we get $[d(x), x]_\alpha = 0$ for all $x \in R$. In this case, if R is non-commutative, then $D = 0$ by Theorem 8.

THEOREM 11. *Let R be a prime ring which is 2, 3-torsion free and α an automorphism of R. Let $D(.,.)$ be a symmetric bi-α-derivation of R and d the trace of $D(.,.)$. If $(d(x), x)_\alpha \in C_\alpha$ for all $x \in R$, then R is commutative or $D = 0$.*

PROOF. We assume that R is non-commutative. In this case, from the hypothesis, for any $x, y \in R$

$$(d(x+y), x+y)_\alpha + (d(-x+y), -x+y)_\alpha \in C_\alpha$$

and since R is 2-torsion free we have, for all $x, y \in R$

$$(d(x), y)_\alpha + 2(D(x,y), x)_\alpha \in C_\alpha. \tag{11}$$

Replacing y by x^2 in (11) and from (11) we get, for all $x \in R$

$$[d(x), x]_\alpha \alpha(x) \in C_\alpha. \tag{12}$$

Thus, from (12)

$$\begin{aligned} 0 &= [[d(x), x]_\alpha \alpha(x), y]_\alpha \\ &= [d(x), x]_\alpha [\alpha(x), \alpha(y)] + [[d(x), x]_\alpha, y]_\alpha \alpha(x) \end{aligned}$$

and so from the hypothesis we get, for all $x, y \in R$

$$[d(x), x]_\alpha \alpha([x, y]) = 0. \tag{13}$$

From (13) and Lemma 1 one can conclude that, for $x \notin Z$ and since α is automorphism of R we have $[d(x), x]_\alpha = 0$. Thus, we have $d(x) = 0$ for all $x \notin Z$ by Theorem 8. Now, let $x \in Z, y \notin Z$. Then $-y, x+y, x+(-y) \notin Z$. Thus, $0 = d(x+y) = d(x)+2D(x,y)$ and $0 = d(x+(-y)) = d(x)-2D(x,y)$ which implies that $d(x) = 0$. Therefore we have proved that $d(x) = 0$ for all $x \in R$, which means that $D(x,y) = 0$ for all $x, y \in R$. Namely, D is zero.

LEMMA 12. *Let R be a 2-torsion free prime ring, U a non-zero right (or left) ideal of R and α a mapping (or one-to-one homomorphism) of R. Let $D(.,.)$ be a symmetric bi-α-derivation of R and d the trace of $D(.,.)$. If $d(U) = 0$, then $D = 0$.*

PROOF For any $u, v \in U$

$$d(u+v) = d(u) + d(v) + 2D(u,v)$$

and since R is 2-torsion free we have $D(u,v) = 0$ for all $u, v \in U$. Writing $vr, r \in R$ for v in this relation. From this relation and since R is prime ring we have $D(u,r) = 0$ for all $u \in U$ and $r \in R$. In this relation, writing us, $s \in R$ for u and since R is prime ring we have $D(s,r) = 0$ for all $r, s \in R$.

THEOREM 13. *Let R be a 2-torsion free prime ring, U a non-zero ideal of R and α an automorphism of R. Let $D(.,.)$ be a symmetric bi- α-derivation of R and d the trace of $D(.,.)$. If $d(u) \in C_\alpha$ for all $u \in U$, then R is commutative or $D = 0$.*

PROOF. We assume that R is non-commutative. Then, replacing u by $u+v, v \in U$ in the hypothesis, from the hypothesis and since R is 2-torsion free we have $D(u,v) \in C_\alpha$ for all $u,v \in U$. In this relation, writing u^2 for u, from this relation and since R is 2-torsion free, we have $uD(u,v) \in C_\alpha$ for all $u,v \in U$. Thus we have $u \in Z$ for all $u \in$ or we have $D(u,v)=0$ for all $u,v \in U$ by Lemma 4.

In other words, U is the union of its subsets $A = \{u \in U \mid D(u,v) = 0 \text{ for all } u \in U\}$ and $B = \{u \in U \mid u \in Z \text{ for all } u \in U\}$. Note that A and B are the additive subgroups of U. If $U = B$ then $U \subset Z$ and so R is commutative. This contradicts the our assumption. So $U \neq B$. Therefore, by Brauer trick, we have $U = A$ which implies that $D(u,v) = 0$ for all $u,v \in U$. Finally, we get $D = 0$ by Lemma 12.

THEOREM 14. *Let R be a prime ring, U a non-zero ideal of R and α an automorphism of R. Let $D(\,,.)$ be a symmetric bi-α- derivation of R such that $d(U) \subset U$ and d the trace of $D(.,\,)$.*

i) *If R is 2-torsion free and $[d(u),u]_\alpha = 0$ for all $u \in U$, then R is commutative or $D = 0$.*

ii) *If R is 2, 3-torsion free and $[d(u),u]_\alpha \in C_\alpha$ for all $u \in U$, then R is commutative or $D = 0$*

PROOF i) We assume that R is non-commutative. In this case, U isn't a commutative ideal of R by Lemma 2. Since U is a non-zero ideal of a prime ring R which is 2-torsion free, U itself is a non-commutative prime ring which is 2-torsion free Therefore, $d(u) = 0$ for all $u \in U$ by Theorem 8. Thus, we have $D = 0$ by Lemma 12

ii) We assume that R is non-commutative Then, since U is 2, 3-torsion free, we have $[d(u),u]_\alpha = 0$ for all $u \in U$ by the proof of Theorem 9. Hence $D = 0$ by (i).

THEOREM 15. *Let R be a prime ring, U a non-zero ideal of R and α an automorphism of R. Let $D(.,.)$ be a symmetric bi-α -derivation of R such that $d(U) \subset U$ and d the trace of $D(\,,.)$.*

i) *If R is 2-torsion free and $(d(u),u)_\alpha = 0$ for all $u \in U$, then R is commutative or $D = 0$.*

ii) *If R is 2, 3-torsion free and $(d(u), u)_\alpha \in C_\alpha$ for all $u \in U$, then R is commutative or $D = 0$.*

PROOF. Similar to the Theorem 14.

LEMMA 16. *Let R be a prime ring, $a \in R$, U a non-zero Lie ideal of R and α a homomorphism of R. Let $D(.,.)$ be a symmetric bi-α-derivation of R such that $d(U) \subset U$ and d the trace of $D(.,.)$. If $[a, u]_\alpha = 0$ for all $u \in U$, then $a \in C_\alpha$ or $U \subset Z$.*

PROOF. Writing $[r, u], r \in R$ for u in the hypothesis, from the hypothesis we have, for all $u \in U$ and $r \in R$

$$[[a, r]_\alpha, u]_\alpha = 0. \tag{14}$$

Writing $vr, v \in U$ for r in (14) and so, from the hypothesis and (14) we get, for all $u, v \in U$ and $r \in R$

$$[u, v][a, r]_\alpha = 0. \tag{15}$$

Writing $sr, s \in R$ for r in (15) and from (15) we get, for all $u, v \in U$ and $s, r \in R$

$$[u, v]s[a, r]_\alpha = 0. \tag{16}$$

From (16) and since R is prime ring we get $U \subset Z$ by Lemma 3 or $a \in C_\alpha$.

THEOREM 17. *Let R be a 2-torsion free prime ring, U a non-zero Lie ideal of R and α a homomorphism of R Let $D(.,.)$ be a symmetric bi-α-derivation of R and d the trace of $D(.,)$*

i) *If $d(u) = 0$ for all $u \in U$, then $U \subset Z$ or $D = 0$ or R is commutative.*
ii) *If $d(u) \in C_\alpha$ and $u^2 \in U$ for all $u \in U$, then $U \subset Z$ or $D = 0$ or R is commutative.*

PROOF i) The linearization of the hypothesis and from the hypothesis we have, for all $u, v \in U$

$$D(u,v) = 0. \tag{17}$$

Writing $[u,r], r \in R$ for u in (17) and from (17) we have, for all $u, v \in U$ and $r \in R$

$$[D(r,v),u]_\alpha = 0. \tag{18}$$

Writing $rw, w \in U$ for r in (18),from (17) and (18) we have, for all $u, v, w \in U$ and $r \in R$

$$D(r,v)\alpha([w,u]) = 0 \tag{19}$$

Replacing r by $rs, s \in R$ in (19), from (19) we get, for all $u, v, w \in U$ and $r, s \in R$

$$D(r,v)\alpha(s)\alpha([w,u]) = 0 \tag{20}$$

Thus, since R is prime ring, from (20) we have, for all $u, v, w \in U$ and $r \in R$

$$D(r,v) = 0 \text{ or } [w,u] = 0 \tag{21}$$

In this case, from (21) and Lemma 3 we have $U \subset Z$ or we have, for all $v \in U$ and $r \in R$

$$D(r,v) = 0. \tag{22}$$

Now, replacing v by $[r,v], r \in R$ in (22), from (22) we have $[d(r),v]_\alpha = 0$ for all $v \in U$ and $r \in R$. Thus, we get $d(r) \in C_\alpha$ for all $r \in R$ or we get $U \subset Z$ by Lemma 16. If $d(r) \in C_\alpha$ for all $r \in R$, then $D = 0$ or R is commutative by Theorem 14.

ii) We assume that $U \subset Z$. In this case, from the hypothesis we have, for all $u \in U$

$$d(u) \in C_\alpha. \tag{23}$$

Replacing u by $u+v, v \in U$ in (23), from (23) and since R is 2-torsion free we have, for all $u, v \in U$

$$(24) \qquad D(u,v) \in C_\alpha.$$

Now, replacing u by u^2 in (24), from (24) and since R is 2-torsion free we have, for all $u, v \in U$

$$(25) \qquad uD(u,v) \in C_\alpha.$$

Thus, from Lemma 4 we have $U \subset Z$ or $D(u,v) = 0$ for all $u, v \in U$. In other words, U is the union of its subsets $A = \{u \in U \mid u \in Z$ for all $u \in U\}$ and $B = \{u \in U \mid (D(u,v) = 0$ for all $u \in U\}$. Note that A and B are the additive subgroups of U. If $U = A$, then $U \subset Z$ and so this contradicts the our assumption. Thus, $U \neq A$. Therefore, by Brauer trick, we have $U = B$ which implies that $D(u,v) = 0$ for all $u, v \in U$. Finally, $D = 0$ or R is commutative by (i).

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