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THE UNITS AND IDEMPOTENTS IN THE GROUP RING OF ABELIAN GROUPS $Z_2 \times Z_2 \times Z_2$ AND $Z_2 \times Z_4$

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Abstract Let K be a algebraically closed field of characteristic 0 and G be abelian group $Z_2 \times Z_2 \times Z_2$ or $Z_2 \times Z_4$.

We find the conditions which the elements of the group ring KG are unit and idempotent respecting using the basic table matrix of G. We can see that if $\alpha = \sum r(g)g$ is an idempotent element of KG, then $r(1) = 0, \frac{1}{|G|}, \frac{2}{|G|}, \cdots, \frac{|G|-1}{|G|}, 1$.

1. Introduction

Kaplansky and Zalesskii say that if $\alpha = \sum r(g)g \in KG$ is a nontrivial idempotent element, then $r(1)$ is a rational number lying strictly between 0 and 1 when K is a field of characteristic 0 and G is any group.

Cliff and Sehgal say that if $\alpha = \sum r(g)g \in KG$ is a nontrivial idempotent element, then $r(1)$ is a rational number such that $r(1) = \frac{r}{s}$ and $(r, s) = 1$ when K is a field of characteristic 0 and G is a polycyclic - by - finite group.

Sehgal and Zassenhaus say that the group ring RG has no nontrivial idempotent elements if the commutative ring R has nontrivial idempotent elements and every prime divisors of $|G|$ is a non unit of R [6].

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In [3], [4] and [5], we found all idempotent elements in KG and thus we can see that if $\alpha = \sum^{|G|} r(g)g \in KG$ is an idempotent element, then $r(1) = 0, \frac{1}{|G|}, \frac{2}{|G|}, \cdots, \frac{|G|-1}{|G|}, 1$ when K is a algebraically closed field and G is a Klein's four group, finite cyclic group or $Z_n \times Z_n$.

In this paper, let K be a algebraically closed field of characteristic 0. We shall find the units and idempotent elements in the group ring KG and shall say that if $\alpha = \sum r(g)g \in KG$ is an idempotent element, then $r(1) = 0, \frac{1}{|G|}, \frac{2}{|G|}, \cdots, \frac{|G|-1}{|G|}, 1$ when G is an abelian group $Z_2 \times Z_2 \times Z_2$ or $Z_2 \times Z_4$.

Let R be a ring with unity and $G = \{g_0 = 1, g_1, g_2, \cdots, g_{n-1}\}$ be a finite group. From the element $\alpha = \sum_{i=0}^{n-1} r(g_i)g_i$ of the group ring RG, we obtain a following matrix M_α by putting $r(g_i)$ in the place of g_i in the basic group table matrix of G.

$$M_\alpha = \begin{pmatrix} r(1) & r(g_1) & \cdots & r(g_{n-1}) \\ r(g_1^{-1}) & r(1) & \cdots & \cdot \\ \vdots & & \ddots & \vdots \\ r(g_{n-1}^{-1}) & \cdot & \cdots & r(1) \end{pmatrix}$$

2. $Z_2 \times Z_2 \times Z_2$

In abelian group $Z_2 \times Z_2 \times Z_2$, let $G_0 = (0,0,0), g_1 = (0,0,1)$, $g_2 = (0,1,0), g_3 = (0,1,1), g_4 = (1,0,0), g_5 = (1,0,1), g_6 = (1,1,0), g_7 = (1,1,1)$. Then the represented matrix M_α of the element $\alpha = \sum_{i=0}^{7} r_i g_i$ of the group ring $K(Z_2 \times Z_2 \times Z_2)$ is as following

$$M_\alpha = \begin{pmatrix} A & \vdots & B \\ \cdots & \cdots & \cdots \\ B & \vdots & A \end{pmatrix}$$

where

$$A = \begin{pmatrix} r_0 & r_1 & r_2 & r_3 \\ r_1 & r_0 & r_3 & r_2 \\ r_2 & r_3 & r_0 & r_1 \\ r_3 & r_2 & r_1 & r_0 \end{pmatrix}, \quad B = \begin{pmatrix} r_4 & r_5 & r_6 & r_7 \\ r_5 & r_4 & r_7 & r_6 \\ r_6 & r_7 & r_4 & r_5 \\ r_7 & r_6 & r_5 & r_4 \end{pmatrix}.$$

And thus

$$M_\alpha = \begin{pmatrix} I & \vdots & I \\ \cdots & \cdots & \cdots \\ I & \vdots & -I \end{pmatrix}^{-1} \begin{pmatrix} A+B & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & A-B \end{pmatrix} \begin{pmatrix} I & \vdots & I \\ \cdots & \cdots & \cdots \\ I & \vdots & -I \end{pmatrix}.$$

Let ξ be a primitive 2th root of unity in K and $\bar{\xi}$ be conjugate to ξ in K.
Let $V = \frac{1}{\sqrt{2}}V(1\xi)$, $\bar{V} = \frac{1}{\sqrt{2}}V(1\bar{\xi})$, $P_1 = diag(p_1(1)p_1(\xi))$, $P_2 = diag(p_2(1)p_2(\xi))$, $H_1 = diag(h_1(1)h_1(\xi))$
and $H_2 = diag(h_2(1)h_2(\xi))$ where $V(1\xi)$ is a Vandermonde matrix, $p_1(x) = (r_0 + r_4) + (r_1 + r_5)x, p_2(x) = (r_0 - r_4) + (r_1 - r_5)x$, $h_1(x) = (r_2+r_6)+(r_3+r_7)x$ and $h_2(x) = (r_2-r_6)+(r_3-r_7)x$.

Then

$$A+B = \begin{pmatrix} \bar{V} & \vdots & \bar{V} \\ \cdots & \cdots & \cdots \\ \bar{V} & \vdots & -\bar{V} \end{pmatrix}^{-1} \begin{pmatrix} P_1+H_1 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & P_1-H_1 \end{pmatrix} \begin{pmatrix} \bar{V} & \vdots & \bar{V} \\ \cdots & \cdots & \cdots \\ \bar{V} & \vdots & -\bar{V} \end{pmatrix}$$

$$A-B = \begin{pmatrix} \bar{V} & \vdots & \bar{V} \\ \cdots & \cdots & \cdots \\ \bar{V} & \vdots & -\bar{V} \end{pmatrix}^{-1} \begin{pmatrix} P_2+H_2 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & P_2-H_2 \end{pmatrix} \begin{pmatrix} \bar{V} & \vdots & \bar{V} \\ \cdots & \cdots & \cdots \\ \bar{V} & \vdots & -\bar{V} \end{pmatrix}.$$

Thus $M_\alpha =$

$$\begin{pmatrix} \bar{V} \vdots & \bar{V} \vdots & \bar{V} \vdots & \bar{V} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{V} \vdots & -\bar{V} \vdots & \bar{V} \vdots & -\bar{V} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{V} \vdots & \bar{V} \vdots & -\bar{V} \vdots & -\bar{V} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{V} \vdots & -\bar{V} \vdots & -\bar{V} \vdots & \bar{V} \end{pmatrix}^{-1} \begin{pmatrix} P_1+H_1 \vdots & 0 & \vdots & 0 & \vdots & 0 \\ \cdots & \cdots & & \cdots & & \cdots \\ 0 & \vdots P_1-H_1 \vdots & & 0 & \vdots & 0 \\ \cdots & \cdots & & \cdots & & \cdots \\ 0 & \vdots & 0 & \vdots P_2+H_2 \vdots & & 0 \\ \cdots & \cdots & & \cdots & & \cdots \\ 0 & \vdots & 0 & \vdots & 0 & \vdots P_2-H_2 \end{pmatrix} \begin{pmatrix} \bar{V} \vdots & \bar{V} \vdots & \bar{V} \vdots & \bar{V} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{V} \vdots & -\bar{V} \vdots & \bar{V} \vdots & -\bar{V} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{V} \vdots & \bar{V} \vdots & -\bar{V} \vdots & -\bar{V} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{V} \vdots & -\bar{V} \vdots & -\bar{V} \vdots & \bar{V} \end{pmatrix}$$

Since $det M_\alpha =$

$[p_1(1)^2 - h_1(1)^2][p_1(\xi)^2 - h_1(\xi)^2][p_2(1)^2 - h_2(1)^2][p_2(\xi)^2 - h_2(\xi)^2]$, we have that M_α is a unit if and only if

$$\begin{aligned} &p_1(1) \neq \pm h_1(1), \\ &p_1(\xi) \neq \pm h_1(\xi), \\ &p_2(1) \neq \pm h_2(1) \quad \text{and} \\ &p_2(\xi) \neq \pm h_2(\xi). \end{aligned}$$

Therefore we have the following Theorem.

THEOREM 2.1. *Let $\alpha = \sum_{i=0}^{7} r_i g_i \in K(Z_2 \times Z_2 \times Z_2)$. α is a unit if and only if*

$$\begin{aligned} &(r_0 + r_1) + (r_2 + r_3) \neq \pm[(r_4 + r_5) + (r_6 + r_7)], \\ &(r_0 + r_1) - (r_2 + r_3) \neq \pm[(r_4 + r_5) - (r_6 + r_7)], \\ &(r_0 - r_1) + (r_2 - r_3) \neq \pm[(r_4 - r_5) + (r_6 - r_7)] \quad \text{and} \\ &(r_0 - r_1) - (r_2 - r_3) \neq \pm[(r_4 - r_5) - (r_6 - r_7)]. \end{aligned}$$

THEOREM 2.2. *Let $\alpha = \sum_{i=0}^{7} r_i g_i \in K(Z_2 \times Z_2 \times Z_2)$. Then α is an idempotent element if and only if*

$$\begin{aligned} &(r_0 + r_1) + (r_2 + r_3) + (r_4 + r_5) + (r_6 + r_7) = 0 \quad or \quad 1, \\ &(r_0 + r_1) + (r_2 + r_3) - (r_4 + r_5) - (r_6 + r_7) = 0 \quad or \quad 1, \\ &(r_0 + r_1) - (r_2 + r_3) + (r_4 + r_5) - (r_6 + r_7) = 0 \quad or \quad 1, \\ &(r_0 + r_1) - (r_2 + r_3) - (r_4 + r_5) + (r_6 + r_7) = 0 \quad or \quad 1, \\ &(r_0 - r_1) + (r_2 - r_3) + (r_4 - r_5) + (r_6 - r_7) = 0 \quad or \quad 1, \\ &(r_0 - r_1) + (r_2 - r_3) - (r_4 - r_5) - (r_6 - r_7) = 0 \quad or \quad 1, \\ &(r_0 - r_1) - (r_2 - r_3) - (r_4 - r_5) + (r_6 - r_7) = 0 \quad or \quad 1 \quad and \\ &(r_0 - r_1) - (r_2 - r_3) - (r_4 - r_5) + (r_6 - r_7) = 0 \quad or \quad 1. \end{aligned}$$

Proof. Since $\alpha = \sum_{i=0}^{n} r_i g_i \in K(Z_2 \times Z_2 \times Z_2)$ is an idempotent element if and only if $M_\alpha^2 = M_\alpha$, we have that α is an idempotent element if and only if

$$\begin{aligned}
&p_1(1) + h_1(1) = 0 \quad \text{or} \quad 1,\\
&p_1(1) - h_1(1) = 0 \quad \text{or} \quad 1,\\
&p_1(\xi) + h_1(\xi) = 0 \quad \text{or} \quad 1,\\
&p_1(\xi) - h_1(\xi) = 0 \quad \text{or} \quad 1,\\
&p_2(\xi^2) + h_2(\xi^2) = 0 \quad \text{or} \quad 1,\\
&p_2(\xi^2) - h_2(\xi^2) = 0 \quad \text{or} \quad 1,\\
&p_2(\xi^3) + h_2(\xi^3) = 0 \quad \text{or} \quad 1 \quad \text{and}\\
&p_2(\xi^3) - h_2(\xi^3) = 0 \quad \text{or} \quad 1.
\end{aligned}$$

where $p_1(x) = (r_0+r_4)+(r_1+r_5)x$, $p_2(x) = (r_0-r_4)+(r_1-r_5)x$, $h_1(x) = (r_2+r_6)+(r_3+r_7)x$, $h_2(x) = (r_2-r_6)+(r_3-r_7)x$ and ξ is a primitive 2th root of unity in K.

From theorem2.2 we have the following theorem.

THEOREM 2.3. *Let $\alpha = \sum_{i=0}^{7} r_i g_i \in K(Z_2 \times Z_2 \times Z_2)$. Then if α is an idempotent element, then*

$$r_0 = 0, \frac{1}{8}, \frac{2}{8}, \cdots, \frac{7}{8}, 1$$

3. $Z_2 \times Z_4$

In $Z_2 \times Z_4$, let $g_0 = (0,0), g_1 = (0,1), g_2 = (0,2), g_3 = (0,3), g_4 = (1,0), g_5 = (1,1), g_6 = (1,2), g_7 = (1,3)$, then the represented matrix M_α of the element $\alpha = \sum_{i=0}^{7} r_i g_i$ of the group ring $K(Z_2 \times Z_4)$ is an following

$$M_\alpha = \begin{pmatrix} A & \vdots & B \\ \cdots & \cdots & \cdots \\ B & \vdots & A \end{pmatrix}$$

where

$$A = \begin{pmatrix} r_0 & r_1 & r_2 & r_3 \\ r_3 & r_0 & r_1 & r_2 \\ r_2 & r_3 & r_0 & r_1 \\ r_1 & r_2 & r_3 & r_0 \end{pmatrix}, \quad B = \begin{pmatrix} r_4 & r_5 & r_6 & r_7 \\ r_7 & r_4 & r_5 & r_6 \\ r_6 & r_7 & r_4 & r_5 \\ r_5 & r_6 & r_7 & r_4 \end{pmatrix}.$$

$$A = \frac{1}{2}V(1\xi\xi^2\xi^3) diag(p(1)p(\xi)p(\xi^2)p(\xi^3)) \frac{1}{2}V(1\bar{\xi}\bar{\xi}^2\bar{\xi}^3)$$
$$B = \frac{1}{2}V(1\xi\xi^2\xi^3) diag(h(1)h(\xi)h(\xi^2)h(\xi^3)) \frac{1}{2}V(1\bar{\xi}\bar{\xi}^2\bar{\xi}^3)$$

where ξ is a primitive 4th root of unity in K, $\bar{\xi}$ is conjugate to ξ in K, $V(1\xi\xi^2\xi^3)$ is a Vandermonde matrix, $p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$ and $h(x) = r_4 + r_5 x + r_6 x^2 + r_7 x^3$.

Let $V = \frac{1}{2}V(1\xi\xi^2\xi^3)$, $\bar{V} = \frac{1}{2}V(1\bar{\xi}\bar{\xi}^2\bar{\xi}^3)$, $P = diag(p(1)p(\xi)p(\xi^2)$ $p(\xi^3))$ and $H = diag(h(1)h(\xi)h(\xi^2)h(\xi^3))$. Then

$$M_\alpha = \begin{pmatrix} VP\bar{V} & \vdots & VH\bar{V} \\ \cdots & \cdots & \cdots \\ VH\bar{V} & \vdots & VP\bar{V} \end{pmatrix}$$
$$= \begin{pmatrix} \bar{V} & \vdots & \bar{V} \\ \cdots & \cdots & \cdots \\ \bar{V} & \vdots & -\bar{V} \end{pmatrix}^{-1} \begin{pmatrix} P+H & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & P-H \end{pmatrix} \begin{pmatrix} \bar{V} & \vdots & \bar{V} \\ \cdots & \cdots & \cdots \\ \bar{V} & \vdots & -\bar{V} \end{pmatrix}$$

Since

$$P \pm H = \begin{pmatrix} p(1) \pm h(1) & \vdots & 0 & \vdots & 0 & \vdots & 0 \\ \cdots & & \cdots & & \cdots & & \cdots \\ 0 & \vdots & p(\xi) \pm h(\xi) & \vdots & 0 & & 0 \\ \cdots & & \cdots & & \cdots & & \cdots \\ 0 & \vdots & 0 & \vdots & p(\xi^2) \pm h(\xi^2) & \vdots & 0 \\ \cdots & & \cdots & & \cdots & & \cdots \\ 0 & \vdots & 0 & \vdots & 0 & \vdots & p(\xi^3) \pm h(\xi^3) \end{pmatrix}$$

$$\begin{aligned} det M_\alpha &= |P+H||P-H| \\ &= [p(1)^2 - h(1)^2][p(\xi)^2 - h(\xi)^2][p(\xi^2)^2 - h(\xi^2)^2][p(\xi^3)^2 - h(\xi^3)^2] \end{aligned}$$

Therefore we have the following theorems.

THEOREM 3.1. *Let* $\alpha = \sum_{i=0}^{7} r_i g_i \in K(Z_2 \times Z_4)$. *Then* α *is a unit if and only if*

$$\begin{aligned} &p(1) \neq \pm h(1), \\ &p(\xi) \neq \pm h(\xi), \\ &p(\xi^2) \neq \pm h(\xi^2) \quad \text{and} \\ &p(\xi^3) \neq \pm h(\xi^3), \end{aligned}$$

where ξ *is a primitive 4th root of unity in* K, $p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$ *and* $h(x) = r_4 + r_5 x + r_6 x^2 + r_7 x_3$.

THEOREM 3.2. *Let* $\alpha = \sum_{i=0}^{7} r_i g_i \in K(Z_2 \times Z_4)$. *Then* α *is an idempotent element if and only if*

$$\begin{aligned} &p(1) + h(1) = 0 \quad \text{or} \quad 1, \\ &p(1) - h(1) = 0 \quad \text{or} \quad 1, \\ &p(\xi) + h(\xi) = 0 \quad \text{or} \quad 1, \\ &p(\xi) - h(\xi) = 0 \quad \text{or} \quad 1, \\ &p(\xi^2) + h(\xi^2) = 0 \quad \text{or} \quad 1, \\ &p(\xi^2) - h(\xi^2) = 0 \quad \text{or} \quad 1, \\ &p(\xi^3) + h(\xi^3) = 0 \quad \text{or} \quad 1 \quad \text{and} \\ &p(\xi^3) - h(\xi^3) = 0 \quad \text{or} \quad 1, \end{aligned}$$

where ξ *is a primitive 4th root of unity in* K, $p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$ *and* $h(x) = r_4 + r_5 x + r_6 x^2 + r_7 x_3$.

From Theorem3.2, we have the following theorem.

THEOREM 3.3. *If* $\alpha = \sum_{i=0}^{7} r_i g_i \in K(Z_2 \times Z_4)$ *is and idempotent element, then* $r_0 = 0, \frac{1}{8}, \frac{2}{8}, \cdots \frac{7}{8}, 1$.

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