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ON PRIME SUBMODULES

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Abstract The height of a prime submodule and a module version of the Krull dimension are studied.

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. A proper submodule N of a module M over a ring R is said to be prime (P-prime) if $ra \in N$ for $r \in R$ and $a \in M$ implies that either $a \in N$ or $r \in (N : M) = P$ (see, for example, [4], [6].)

Let K be a prime submodule of an R-module M. We say that K is minimal prime over a submodule N of M if $N \subseteq K$ and there does not exist a prime submodule L of M such that $N \subseteq L \subset K$. It is said that $ht\ K = n$, if there exists a chain $K_n \subset \cdots \subset K_2 \subset K_1 \subset K_o = K$ of prime submodules $K_i (o \leq i \leq n)$ of M, but there is no longer such chain.

It is said that the generalized principal ideal theorem (the GPIT) holds for M, if for every positive integer n and prime submodule N of M minimal over a submodule generated by n elements, $ht\ N \leq n$.

2. The generalized principal ideal theorem for modules

From now on, S is a multiplicatively closed subset of R.

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LEMMA 1. *Let P be a prime ideal of R such that $P \cap S = \emptyset$ and M be an R -module. Then there exists a one-to-one correspondence between the P-prime submodules of M and the $S^{-1}P$-prime submodules of $S^{-1}M$.*

Proof. See [5] §1 Proposition 1.

In the following lemma R is an integral domain and K is the field of quotients of R.

LEMMA 2. *If M is an R-module and B a submodule of M such that $KB \neq KM$, then*

(i) *$N = KB \cap M$ is a 0-prime submodule and $KN = KB$.*
(ii) *B is prime if and only if $B = KB \cap M$.*

Proof. See [5] §1 corollaries after Propositions 2 and 3.

LEMMA 3. *Let M be an R-module and B a submodule of M. If $S^{-1}B \neq S^{-1}M$, then $(B : M) \cap S = \emptyset$. Conversely, if $(B : M) \cap S = \emptyset$ and B is a prime submodule of M, then $S^{-1}B \neq S^{-1}M$, and $ht\ B = ht\ S^{-1}B$. Also the following conditions are equivalent.*

(i) *B is a prime submodule of M and $S^{-1}B \neq S^{-1}M$.*
(ii) *$S^{-1}B$ is a prime submodule of $S^{-1}M$ and $B = S^{-1}B \cap M$.*

Proof. The proof is easy by use of the above lemmas and [5], §1 Proposition 2.

We know that if N is a prime submodule of an R -module M, then $(N : M)$ is a prime ideal of R (see, for example, [4]).

THEOREM 2.1. *Let M be an R-module and B be a submodule of M which is generated by n elements. If N is a minimal prime submodule over B such that $(N : M)$ is a minimal prime ideal of R, then $ht\ N \leq n$.*

Proof. First let $(N : M) = 0$, that is R is an integral domain. Let K be the field of quotients of R. It is easy to see that the *rank* of the subspace KB of the vector space KA over the field K is not greater than n. That is, $rank KB \leq n$. Now by Lemmas 2 and 3, $KN \cap M = N$. Hence $B \subseteq KB \cap M \subseteq KN \cap M = N$ and since $KB \subseteq KN \subset KM$, by Lemma 2, $KB \cap M$ is a prime submodule of M. So $KB \cap M = N$ and so $KB = KN$. One can see that in a vector space every proper subspace W is prime and $ht\ W = rank\ W$. By Lemma 3 we have that $ht\ N = ht\ KN$. So $ht\ N = ht\ KN = ht\ KB = rank\ KB \leq n$.

Now we return to the general case. Let the following chain be a chain of prime submodules of M, $N_{n+1} \subset N_n \subset \cdots \subset N_1 \subset N_o = N$. As $(N : M)$ is a minimal prime ideal, $(N_i : M) = P\ \forall\ i = 1, 2, \cdots, n+1$. It is straightforward to prove the following,

(i) $\frac{N}{N_{n+1}}$ is a minimal prime submodule over the submodule $\frac{B+N_{n+1}}{N_{n+1}}$ of the $\frac{R}{P}$ -module $\frac{M}{N_{n+1}}$.

(ii) $\frac{B+N_{n+1}}{N_{n+1}}$ is generated by n elements.

(iii) $(\frac{N}{N_{n+1}} :_{\frac{R}{P}} \frac{M}{N_{n+1}}) = \frac{(N:_R M)}{P} = 0$.

Therefore, by the first case, $ht\ \frac{N}{N_{n+1}} \leq n$ which is a contradiction with the following chain of prime submodules

$$0 = \frac{N_{n+1}}{N_{n+1}} \subset \frac{N_n}{N_{n+1}} \subset \cdots \subset \frac{N_2}{N_{n+1}} \subset \frac{N_1}{N_{n+1}} \subset \frac{N}{N_{n+1}}.$$

Corollary 1. *The GPIT holds for every divisible module over an integral domain.*

Proof. Let M be a divisible R-module and N be a proper submodule of M. Then easily one can show that $(N : M) = 0$.

In [2] it is proved that if R is an integral domain, then the PIT holds for every R-module if and only if R is a field. $(*)$

Now we prove the following result.

Corollary 2. *Let R be a ring. Then the GPIT (or the PIT) holds for every R- module if and only if dim $R = 0$.*

Proof. We only need to prove that if the PIT holds for every R module, then dim $R = 0$. If P is a prime ideal of R, we show that the PIT holds for every $\frac{R}{P}$-module, and so by $(*)$ in above $\frac{R}{P}$ is a field.

Let B be a cyclic submodule of the $\frac{R}{P}$-module M and N be minimal prime over B. It is obvious that N is minimal prime over the cyclic submodule B of M as an R-module. So $ht\ _RN \leq 1$ and hence $ht\ _{\frac{R}{P}}N \leq 1$.

The next lemma is due to McCasland and Moore [6], however, we shall provide a simpler proof for it.

LEMMA 4. *Let M be a finitely generated R-module and B a submodule of M. If $(B : M) \subseteq P$, where P is a prime ideal of R, then there exists a prime submodule N of M containing B such that $(N : M) = P$.*

Proof. Let $S = \{C \leq M : B \subseteq C, (C : M) \subseteq P\}$. By Zorn's Lemma S has a maximal element N. We show that N is prime and $(N : M) = P$. Let $ra \in N$ such that $r \notin (N : M)$ and $a \notin N$. So $N \subset N + rM$ and $N \subset N + Ra$. Let $r_1 \in (N + Ra : M) - P$, and $r_2 \in (N + rM : M)$. Then $r_1M \subseteq N + Ra$. Hence $r_1rM \subseteq N$. Since $r_2M \subseteq N + rM$, $r_1r_2M \subseteq N + rr_1M \subseteq N$. So $r_1r_2 \in (N : M) \subseteq P$ and hence $r_1 \in P$ or $r_2 \in P$, which is a contradiction. Now if $\bar{M} = \frac{M}{N}$, we have $(N + PM : M) = (P\bar{M} : \bar{M}) = Ann(\frac{\bar{M}}{P\bar{M}}) \subseteq \sqrt{Ann(\frac{\bar{M}}{P\bar{M}})} = \sqrt{Ann(\bar{M}) + P} = \sqrt{(N : M) + P} = P$. So $(N + PM : M) \subseteq P$. Thus $N = N + PM$. Hence $PM \subseteq N + PM = N$. That is, $(N : M) = P$.

Recall that an R-module M is called a weak multiplication module provided that for every prime submodule N of M there exists an ideal I of R such that $N = IM$ [1]. In this case we have $N = (N : M)M$ and it is easy to see that $ht\ N \leq ht\ (N : M)$.

THEOREM 2.2. *Let R be a Noetherian domain, M be a finitely generated weak multiplication R -module and N be a minimal prime submodule over Ra for $a \in M$. Then $ht\ N = 1$, if $(Ra : M)M = Ra$ and $Ann(a) = 0$.*

Proof. i) Let $(N : M) = P$ and $S = R - P$. Then $S^{-1}M$ is a finitely generated weak multiplication module over the Noetherian domain $S^{-1}R$ and $Ann_{S^{-1}R}(\frac{a}{1}) = 0$. Moreover, $ht\ N = ht\ S^{-1}N$ by Lemma 3, and $S^{-1}N$ is a minimal prime over $S^{-1}R(\frac{a}{1})$. So we can assume that R is a local domain with the maximal ideal m. We show that $(Ra : M)$ is a principal ideal and $(N : M)$ is a minimal prime ideal over $(Ra : M)$. Since $(Ra : M)M = Ra$, let

$a = \sum_{i=1}^{n} r_i m_i$, where $r_i \in (Ra : M)$ and $m_i \in M$. We consider two cases.

Case 1: $n = 1$, we claim that $(Ra : M) = Rr_1$. Let $r \in (Ra : M)$. So $rM \subseteq Ra$. Hence $rm_1 = ta = tr_1 m_1$ for some $t \in R$ and so $(r - tr_1)m_1 = 0$. Thus $(r - tr_1)a = r_1(r - tr_1)m_1 = 0$. Therefore, $(r - tr_1) \in Ann(a) = 0$, and so $r = tr_1$. That is, $(Ra : M) \subseteq Rr_1$, and $r_1 \in (Ra : M)$. Therefore, $(Ra : M) = Rr_1$.

Case 2: $n > 1$, let $r_i m_i = t_i a$, for some $t_i \in R$ for all i. One can assume that $t_j \notin m$ for some j, since if $t_i \in m$ for all i, then $a = \sum_{i=1}^{n} t_i a$ and hence $1 = \sum_{i=1}^{n} t_i \in m$ which is a contradiction. Now if $t_j \notin m$, for some j, we have $a = t_j^{-1} r_j m_j$ and the result follows by case 1.

Now we show that $(N : M)$ is a minimal prime ideal over the principal ideal $(Ra : M)$. If $(Ra : M) \subseteq Q \subseteq (N : M)$, where Q is a prime ideal, by Lemma 4 there is a prime submodule N_1 of M containing Ra such that $(N_1 : M) = Q$. Since M is weak multiplicative, $N_1 = (N_1 : M)M = QM \subseteq N$, and hence $N_1 = N$. So $Q = (N_1 : M) = (N : M)$. Now $(N : M)$ is a minimal prime ideal over the principal ideal $(Ra : M)$, so by the Krull Principal Ideal Theorem $ht\ (N : M) \leq 1$ and obviously we have $ht\ N \leq ht\ (N : M) \leq 1$. Since $AnnM \subseteq Ann(a) = 0$, by Lemma 4 there exists a prime submodule T such that $(T : M) = 0 \subset (N : M)$ and hence $T = (T : M)M \subset (N : M)M = N$, so $ht\ N = 1$.

PROPOSITION 1. *Let R be a PID, F a free R-module of finite rank and B a submodule of F which has a minimal generator with n elements. Let N be minimal prime over B. If $(N : F) = 0$, then $ht\ N = n$.*

Proof. We know that there exists a basis $\{x_1, x_2, \cdots, x_m\}$ of F, an integer $d (1 \leq d \leq m)$ and nonzero elements $r_1, r_2, \cdots, r_d$ of R such that $r_1 | r_2 | \cdots | r_d$ and $\{r_1 x_1, r_2 x_2, \cdots, r_d x_d\}$ is a basis of N.

For all $1 \leq i \leq d$, $r_i x_i \in N$ and $0 \neq r_i \notin (N : F) = 0$. Then $x_i \in N$ and so $\{x_1, x_2, \cdots, x_d\}$ is a basis of N. Let $N_k = Rx_1 + Rx_2 + \cdots + Rx_k$, $1 \leq k \leq d$. We show that N_k is a prime submodule of F. If $ry \in N_k$, $y = \sum_{i=1}^{m} t_i x_i$. Then $ry = \sum_{i=1}^{m} rt_i x_i \in N_k$, and hence $rt_i = 0$ for all i, $i = k + 1, \cdots, m$. If

$r = 0$, then $r \in (N_k : F)$, otherwise $t_i = 0$ for all $i = k+1, \cdots, n$. Therefore, $y \in N_k$. By the following chain,

$$0 \subset N_1 \subset \cdots \subset N_{d-2} \subset N_{d-1} \subset N_d = N$$

we have $d \leq ht\ N$. So $n \leq d \leq ht\ N \leq n$ by Theorem 2.1.

Now we show that the condition $(N : F) = 0$ in Proposition 9 is necessary.

EXAMPLE. Let $\{x_1, x_2, \cdots, x_m\}, m > 2$ be a basis of F and N be generated by $x_1, px_2, px_3, \cdots, px_m$. One can easily show that $ht\ N = 2$, although N is minimal over N which is generated by m elements.

3. A module version of Krull dimension

DEFINITION 3.1. The dimension of a module M $(dimM)$ is defined by

$$\sup\{ht\ N : N \text{ is a prime submodule of } M\}.$$

if spec$(M) \neq \emptyset$, otherwise it is defined to be -1.

Let S be a multiplicatively closed set. Then by Lemmas 1 to 3 one can easily show that $dimS^{-1}M \leq dimM$. Also if M is a finitely generated faithful module, then by Lemma 4, if $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_m$ is a chain of prime ideals in the ring R, then there exists a chain of prime submodules $N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_m$ in M and hence $dimR \leq dimM$.

We recall that if R is an integral domain with the quotient field K, the rank of an R-module M is defined to be the maximal number of elements of M linearly independent over R. We have rank $M=$ the dimension of the vector space KM over K [8].

THEOREM 3.1. *If R is a Dedekind domain which is not a field and M is a finitely generated torsion-free R-module, then if N is a prime submodule,*

([)(i)] $dimM = rankM$. ([)(ii)] $ht\ N + dim\frac{M}{N} = dimM$.

Proof. i) First let R be a PID and M be a free module of finite rank over R. If N is a prime submodule of M, we have two cases. If $(N : M) = 0$, by the proof of Proposition 1, $ht\ N \leq rankM$. If $(N : M) \neq 0$, let $(N : M) =< p >$. One can prove that there is a basis $\{x_1, x_2, \cdots, x_m\}$ for M such that $N =< x_1, x_2, \cdots, x_k, px_{k+1}, px_{k+2}, \cdots, px_m >$ and $k \leq m - 1$, besides $ht\ N = k + 1$. So $dimM \leq rankM$.

If $\{x_1, x_2, \cdots, x_m\}$ is a basis of M, one can easily prove that $N =< x_1, x_2, \cdots, x_{m-1}, px_m >$ is a prime submodule and $ht\ N = m$ and hence $m = rankM \leq dimM$.

Now we prove (i) for every finitely generated torsion-free module M over the Dedekind domain R. Let $0 \neq P$ be a prime ideal of R, and $S = R - P$, then by Lemmas 2 and 3, we have $dimS^{-1}M \leq dimM$. Since $S^{-1}M$ is finitely generated torsion-free over the PID $S^{-1}R$, it is a free module, so by the first step $rankS^{-1}M = dimS^{-1}M$. Also obviously one can check that $rankM = rankS^{-1}M$. Hence $rankM \leq dimM$.

Let N be a prime submodule of M. If $(N : M) = P \neq 0$, then let $S = R - P$, so $ht\ N = ht\ S^{-1}N \leq dimS^{-1}M$, and since $S^{-1}M$ is free, $dimS^{-1}M = rankS^{-1}M = rankM$. If $(N : M) = 0$, then there exists a prime ideal P such that $(N : M) \subset P$. By Lemma 4 there exists a prime submodule N_1 such that $N \subset N_1$ and $(N_1 : M) = P$. From above we have $ht\ N < ht\ N_1 \leq rankM$.

ii) Evidently $ht\ N + dim\frac{M}{N} \leq dimM$. If $(N : M) = 0$, then $\frac{M}{N}$ is a finitely generated torsion-free R-module, so $dim\frac{M}{N} = rank\frac{M}{N} = rank_K\frac{KM}{KN} = rank_K KM - rank_K KN = rankM - rank_K KN = dimM - rank_K KN$. One can show that in a vector space V, for every proper subspace W, $rankW = ht\ W$, and hence $rank_K KN = ht\ KN$. Also by Lemma 3 $ht\ KN = ht\ N$. So $dim\frac{M}{N} = dimM - ht\ N$.

If $(N : M) \neq 0$, then let $(N : M) = P$ and $S = R - P$. By Lemma 3, $ht\ N = ht\ S^{-1}N$. Also we show that $dimS^{-1}\frac{M}{N} = dim\frac{M}{N}$. As we said $dimS^{-1}\frac{M}{N} \leq dim\frac{M}{N}$. Let $dim\frac{M}{N} = t$. So there is a chain of prime submodules $N \subset N_1 \subset \cdots \subset N_t$. So by Lemma 3, $S^{-1}N \subset S^{-1}N_1 \subset \cdots \subset S^{-1}N_t$ is a chain of prime submodules of $S^{-1}M$. So $dimS^{-1}\frac{M}{N} = dim\frac{M}{N}$. Similarly it is proved that $dimM = dimS^{-1}M$. Thus we can assume that M ia a free module of finite rank over the local PID R. Let $dimM = m$. So $0 \neq (N : M) =< p >$ where p is a prime element in R and by part (i) $dimM = rankM = m$. It is easy to show that there is a basis $\{x_1, x_2, \cdots, x_m\}$ such that $N =< x_1, x_2, \cdots, x_k, px_{k+1}, px_{k+2}, \cdots, px_m >$ and $ht\ N = k+1$. Then the following is a chain of prime submodules of $\frac{M}{N}$, $\frac{N}{N} \subset \frac{<x_1, \cdots, x_{k+1}, px_{k+2}, \cdots, px_m>}{N} \subset \frac{<x_1, \cdots, x_{k+2}, px_{k+3}, \cdots, px_m>}{N} \subset \cdots \subset \frac{<x_1, \cdots, x_{m-1}, px_m>}{N}$. Therefore $dim\ \frac{M}{N} \geq m - k - 1$, then $ht\ N + dim\frac{M}{N} \geq m = dimM$ as required.

For a submodule B of an R-module M, the envelope of $B, E(B)$ is defined to be the set of all $x \in M$ for which there exist $r \in R$, $a \in M$ such that $x = ra$ and $r^n a \in B$ for some non-negative integer n. The intersection of all prime submodules of M containing B is denoted by rad B. We say that M satisfies the radical formula (s.t.r.f.) if for every submodule B of M, $rad\ B =< E(B) >$.

Proposition 2. *Let R be an integral domain and M a divisible R-module. Let B be a proper submodule of M. Then*

(i) *$E(B)$ is a submodule of M.*

(ii) *If $E(B) \neq M$, then $E(B)$ is the only minimal prime submodule of M over B.*

(iii) *Let $E(Rx) \neq M$. If $Ann(x) = 0$, then $ht\ E(Rx) = 1$ and if $Ann(x) \neq 0$, then $ht\ E(Rx) = 0$.*

(iv) *M s.t.r.f.*

(v) *If M is finitely generated, then rank M=$dimM + 1$.*

Proof. i) Let $ra, sb \in E(B)$, for nonzero elements $r, s \in R$ and $a, b \in M$ and $n \in \mathbf{N}$ such that $r^n a$, $s^n b \in B$. Since M is divisible, there exists $c \in M$ such that $rsc = ra - sb$. Hence $(rs)^{n+1}c = s^n r^{n+1} a - r^n s^{n+1} b \in B$. It means that $ra - sb \in E(B)$. Hence $E(B)$ is a submodule of M.

ii) First we show that if $ra \in E(B)$, where $0 \neq r \in R$, $a \in M$ and $r^n a \in B$ for some $n \in \mathbf{N}$, then $a \in E(B)$. (**)

There exists an element c of M such that $rc = a$. So $r^{n+1}c = r^n a \in B$, that is, $a = rc \in E(B)$. Now let $ra \in E(B)$ and $0 \neq r \in R$ and $a \in M$. Then there exist $s \in R$ and $b \in M$ such that $ra = sb$ and $s^n b \in B$ for some $n \in \mathbf{N}$.

If $s = 0$, then $ra = 0 \in B$ and so by $(**)$, $a \in E(B)$. Otherwise since $rs \neq 0$ and $(rs)^{n+1}a = r^n s^{n+2} b \in B$, by $(**)$ we have $a \in E(B)$. Therefore, $E(B)$ is a prime submodule of M.

Let N be a minimal prime submodule of M over B. So $B \subseteq E(B) \subseteq N$. Since $E(B)$ is prime, $N = E(B)$.

iii) By Corollary 1 and ii) we have $ht\ E(Rx) \leq 1$. Also $E(0) \subseteq E(Rx)$.

Now if $Ann(x) = 0$ and $ht\ E(Rx) = 0$, then $E(Rx) = E(0)$. Since $x \in E(Rx) = E(0)$, there exist $r \in R$ and $n \in \mathbf{N}$ such that $x = ra$ and $r^n a = 0$. So $r^n \in Ann(x) = 0$ and hence $r = 0$. That is, $x = 0$ and hence $Ann(x) = R$ which is a contradiction.

Let $Ann(x) \neq 0$, and $0 \neq r \in Ann(x)$. Then $rx \in E(0)$ and by the proof of Corollary 1, $(E(0) : M) = 0$. So $x \in E(0)$, since $E(0)$ is prime. That is, $Rx \subseteq E(0) \subseteq E(Rx)$ and by (ii) $E(0) = E(Rx)$. Therefore, $ht\ E(Rx) = 0$.

iv)Let B be a submodule of M. If $E(B) = M$, then $M = E(B) \subseteq rad\ B \subseteq M$. So let $E(B) \neq M$. From (i) and (ii) we have

$$E(B) \subseteq radB = \bigcap_{\substack{N prime \\ B \subseteq N}} N \subseteq E(B).$$

v) Obviously $rankM = rankKM$. Also for the vector space KM over K we have $rankKM = dimKM + 1$ and as we said $dimKM \leq dimM$. So $rankM \leq dimM + 1$. If $N_0 \subset N_1 \subset \cdots \subset N_t$ is a chain of prime submodules of M, then since $(N_i : M) = 0$, by Lemma 3, $KN_0 \subset KN_1 \subset \cdots \subset KN_t$ is a chain of prime submodules of KM, so $dimM \leq dimKM = rankKM - 1 = rankM - 1$. That is, $rankM = dimM + 1$ as required.

We recall that if R is a Prüfer domain and S is a multiplicatively closed subset of R, then $S^{-1}R$ is a valuation ring [3].

THEOREM 3.2. *Let R be a Prüfer domain and M a torsion-free weak multiplication R-module.*

(i) *If N is a prime submodule of M, then $ht\ N = ht\ (N : M)$.*

(ii) *M s.t.r.f.*

(iii) *If M is finitely generated, then $dimM$ =dim R.*

Proof. i) Obviously $ht\ N \leq ht\ (N : M)$. Let $P = (N : M)$ and $S = R - P$. Then $S^{-1}M$ is a torsion-free weak multiplication module over the valuation ring $S^{-1}R$ and $ht\ N = ht\ S^{-1}N$ by Lemma 3. By [5], Corollary 1 of Proposition 1, $(S^{-1}N : S^{-1}M) = S^{-1}(N : M) = S^{-1}P$. Moreover, $ht\ S^{-1}P = ht\ P$. So by localization we can assume that M is a torsion-free weak multiplication module over the valuation ring R. First, we show that:

If P is a prime ideal of R and $PM \neq M$, then PM is a prime submodule of M and $(PM : M) = P$, indeed, we show that if $ra \in PM$, then $r \in P$ or $a \in PM$. $(***)$

Let $ra \in PM$, so $ra = \sum_{i=1}^{n} p_i m_i$, $p_i \in P$, $m_i \in M$. If $r \notin P$, then $P \subseteq < r >$. So let $p_i = rr_i$, for all $1 \leq i \leq n$, $p_i \in P$. So $r_i \in P$. Now $ra = \sum_{i=1}^{n} rr_i m_i$ and hence $a = \sum_{i=1}^{n} r_i m_i \in PM$. Let $m \in M - PM$. If $r \in (PM : M)$, then $rm \in PM$, so $r \in P$. That is, $(PM : M) = P$.

Now let $ht\ P = n$ and the following be a chain of prime ideals in R

$$P_0 \subset P_1 \subset \cdots \subset P_{n-1} \subset P_n = P.$$

Hence $P_i M \subseteq PM = (N : M)M = N \subset M$. So by $(***)$ for all $0 \leq i \leq n$, $P_i M$ is a prime submodule of M and if $P_i M = P_j M$, then by $(***)$, $P_i = (P_i M : M) = (P_j M : M) = P_j$. Hence $i = j$ and we have the following chain of prime submodules in M

$$P_0 M \subset P_1 M \subset \cdots \subset P_{n-1} M \subset P_n M = (N : M)M = N.$$

So $ht\ (N : M) = n \leq ht\ N$.

ii) By [7] we know that if the $S^{-1}R$-module $S^{-1}M$ s.t.r.f., then the R-module M s.t.r.f. So by localization we can assume that M is a torsion-free weak multiplication R-module, where R is a valuation ring. Let B be a submodule of M. We consider two cases.

Case 1. There exists a minimal prime ideal P over $(B : M)$ such that $PM = M$.

In this case, as R is a valuation ring, $rad(B : M) = P$. So

$$M = PM = (rad(B : M))M \subseteq < E(B) > \subseteq rad\ B \subseteq M.$$

Case 2. For every minimal prime ideal P over $(B : M)$, $PM \neq M$.

Since M is a weak multiplication module, by $(***)$ we have $\{P : P \text{ is minimal prime over } (B : M)\} \subseteq \{(N : M) : B \subseteq N,\ N \text{ is prime in } M\} \subseteq \{P : P \text{ is prime containing } (B : M)\}$. Thus,

$$(rad\, B : M) = (\bigcap_{\substack{B\subseteq N\\ N prime}} N : M) = \bigcap_{\substack{B\subseteq N\\ N prime}} (N : M)$$
$$= \bigcap_{\substack{P minimal\ prime\\ over (B:M)}} P = rad(B : M).$$

Now we show that $rad\ B$ is a prime submodule. Let $ra \in rad\ B$. If $a \notin rad\ B$ and $r \notin (rad\ B : M)$, then there exist prime submodules N and T containing B such that $a \notin N$ and $r \notin (T : M)$. So $r \in (N : M)$ and $a \in T$. If $(T : M) \subseteq (N : M)$, then $a \in T = (T : M)M \subseteq (N : M)M = N$, but if $(N : M) \subseteq (T : M)$, then $r \in (N : M) \subseteq (T : M)$ which is impossible. Hence $rad\ B = (rad\ B : M)M = (rad(B : M))M \subseteq < E(B) > \subseteq rad\ B$.

iii) Since M is a weak multiplication module, $dimM \leq dimR$ and the proof follows easily by Lemma 4.

Note that R can be an arbitrary ring in (iii).

We know that if R is a Noetherian ring and $dim\ R = 0$, then R is Artinian. Now we prove a generalisation of this theorem for modules.

PROPOSITIION 3. *If M is a Noetherian module and $dimM = 0$, then M is an Artinian module.*

Proof. First let M be a Noetherian faithful R-module. Since $\frac{R}{AnnM} = R$, R is a Noetherian ring. We show that $dimR = 0$. If $P_1 \subset P_2$ is a chain of prime ideals of R, then by Lemma 4 there is a chain of prime submodules $N_1 \subset N_2$ such that $(N_i : M) = P_i$ and this is a contradiction. So R or indeed $(\frac{R}{AnnM})$ is an Artinian ring, thus M is an Artinian module.

Now M is a Noetherian faithful $\frac{R}{AnnM}$-module, and it is easy to show that $dim_{\frac{R}{AnnM}} M = 0$, so by the above, $dim \frac{R}{AnnM} = 0$ and since $\frac{R}{AnnM}$ is a Noetherian ring, then it is an Artinian ring.

Therefore, M is an Artinian module as an $\frac{R}{AnnM}$-module and obviously M is an Artinian R-module.

The converse of Proposition 3 is not true even if M is a finitely generated module, for example, if M is a vector space of rank n, where $n > 1$, then $dimM = n - 1$.

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