Honam Mathematical J. **21**(1999), No. 1,pp. 75–95

SUBREGULAR POINTS FOR SOME CASES OF LIE ALGEBRAS

Y.K. Kim, K.H. So, J.W. Jeong, D.Y. Park, and S.H. Choi

Dept. of Mathematics,
Chonbuk National University, Chonju, Chonbuk 561-756, Korea.
E-mail : yanggon@moak.chonbuk.ac.kr.
Dept. of Mathematics,
Jeonju University, Chonju, Chonbuk 560-759, Korea.

Abstract Dimensions of irreducible $so_5(F)-$modules over an algebraically closed field F of characteristic $p > 2$ shall be obtained. It turns out that they should be coincident with p^m, where $2m$ is the dimension of coadjoint orbits of $\chi \in so_5(F)^* \setminus 0$ as Premet asserted. But there is no subregular point for $\mathfrak{g} = sp_4(F) = so_5(F)$ over F.

1. Introduction

In this paper, we let $\mathfrak{g} := so_5(F)$ over an algebraically closed field F of characteristic $p > 2$, i.e., $\mathfrak{g} = \mathcal{L}(SO_5(F))$ which is the Lie algebra of an algebraic group $G = SO_5(F)$; we are then mainly concerned with dimensions of all irreducible $\mathfrak{g}$-modules.

We use most notations and nomenclature appearing in [5], [6].

In 1954, Zassenhaus proved that any specialization of $\mathfrak{Z} = \mathfrak{Z}(\mathcal{U}(\mathfrak{g}))$ onto an F-algebra A determines a specialization of $\mathcal{U}(\mathfrak{g})$ onto a finitely generated A-ring B, which is unique up to isomorphisms over A. Moreover, according to him, except for a subvariety of $\mathfrak{Z}$ characterized by the vanishing of the specialized discriminant ideal of $\mathcal{U}(\mathfrak{g})$ over $\mathfrak{Z}$, the classes of equivalent absolutely

Received March 31, 1999.
1991 AMS Subject Classification : 17B50, 17B20.
Key words and phrases : modular Lie algebra, modular representation theory.
* Supported by the Basic Science Research Institute Program, Ministry of Education, 1998, Project No. BSRI-1998-015-D00009.

irreducible representations correspond to the specializations of the center $\mathfrak{Z}$ into F and the degree of those representations equals p^m with $[Q(\mathcal{U}(\mathfrak{g})) : Q(\mathfrak{g})] = p^{2m}$. $\mathfrak{Z}$ becomes just a normal algebraic variety of the same dimension as that of $\mathfrak{g}$ over F and $\mathcal{U}(\mathfrak{g})$ becomes a maximal order of a division algebra $Q(\mathcal{U}(\mathfrak{g}))$ of dimension p^{2m} over $Q(\mathfrak{Z})$ [10]. Such a variety is called a Zassenhaus variety.

Shafarevich and Rudakov showed in 1967 that there exists a correspondence between irreducible p-dimensional S-representations of $sl_2(F)$ and maximal points in $Spec(\mathfrak{Z})$ provided the point P of the manifold $Spec_m(\mathfrak{Z})$ is not equal to $(0,0,0,k^2)$, $k(\neq 0) \in F$; the points $P = (0,0,0,k^2)$, $k \neq 0$ correspond to two irreducible p-representations of degree k and $p-k$; $(0,0,0,0)$ is just the irreducible representation $V(p-1)$ [8]. Steinberg and Curtis classified p-representations for simple modular Lie algebras, but their dimension formulas are still under research by many Lie algebraists.

In this paper, the usual basis of $sl_2(F)$ is denoted by $\{e,f,h\}$ with $[eh] = -2e$, $[fh] = 2f$, $[fe] = -h$; $\mathfrak{Z}(sl_2(F))$ is then generated by $x = f^p$, $y = e^p$, $z = h^p - h$, $t = (h+1)^2 + 4fe$ and $Spec_m(\mathfrak{Z})$ is defined by the algebraic equation $z^2 - \prod_{k=0}^{p-1}(t-k^2) + 4xy = 0$ defined in $F[x,y,z,t]$ [8].

In §5, we shall define three kinds of points in the algebraic variety
$Spec_m(\mathfrak{Z})$ from the standpoint of dimensions and characters of their associated irreducible modules [5], [6].

Final results appear in §6, 7 stating that $\mathfrak{Z}(\mathcal{U}(\mathfrak{g}))$ has no subregular point. It is well known that $sl_2(F)$ has no subregular point as mentioned above.

2. Least upper bounds of dimensions

Let E_{ij} denote an elementary matrix whose (i,j)-th entry is 1 with all others zero. The base of the root system Φ of $B_2 = C_2$ consists of a long root α_1, a short root α_2 and $\Phi^+ = \{\alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1\}$, where $2\alpha_1 + \alpha_2$ is a maximal root as a short root. A standard basis of $\mathfrak{g}$ consists of : $h_{\alpha_2} := diag(1,0,-1,0)$, $h_{2\alpha_1+\alpha_2} := diag(0,1,0,-1)$, $x_{\alpha_1} = {}^tE_{12} - {}^tE_{43}$, $x_{\alpha_2} = E_{13}$, $x_{\alpha_1+\alpha_2} = E_{14} + E_{23}$, $x_{2\alpha_1+\alpha_2} = E_{24}$, $x_{-\alpha_1} = E_{12} - E_{43}$, $x_{-\alpha_2} =$

${}^tE_{13}$, $x_{-\alpha_1-\alpha_2} = {}^tE_{14} + {}^tE_{23}$, $x_{-2\alpha_1-\alpha_2} = {}^tE_{24}$. Let $\mathcal{O}(\mathfrak{g})$ be the p-center of $\mathcal{U}(\mathfrak{g})$. Denote the basis of $\mathfrak{g}$ by $\{u_i | 1 \leq i \leq 10\}$ in any fixed order. An obvious filtration $\mathcal{U}^{(k)} := F \cdot 1 \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^k$ exists for $\mathcal{U}(\mathfrak{g})$. Noting that $(ad\ u_i)^p = ad\ u_i^{[p]}$ for some $u_i^{[p]} \in \mathfrak{g}$, $z_i := u_i^p - u_i^{[p]}$ commutes with $\mathfrak{g}$ elementwise, and hence $z_i \in \mathfrak{Z}(\mathcal{U}(\mathfrak{g}))$. So $u_i^{[p]} = u_i^p - z_i \in \mathcal{U}^{(p-1)}$. Following §7 Chapter V [3], we see that all elements of the form $z_{i_1}^{\sigma_1} z_{i_2}^{\sigma_2} \cdots z_{i_{10}}^{\sigma_{10}} u_{i_1}^{\lambda_1} \cdots u_{i_{10}}^{\lambda_{10}}$ with $i_1 < i_2 < \cdots < i_{10}$, $\sigma_j \geq 0$ and $0 \leq \lambda_j < p$ also constitute a basis for $\mathcal{U}(\mathfrak{g})$ and that $\mathfrak{g}$ becomes a restricted Lie algebra with respect to the p-mapping. $\mathcal{O}(\mathfrak{g})$ becomes a polynomial ring in 10-variables and $\mathcal{U}(\mathfrak{g})$ is just a free $\mathcal{O}(\mathfrak{g})$-module of rank p^{10}. Furthermore, $Q(\mathcal{U}(\mathfrak{g})) := \{\mathfrak{Z}(\mathcal{U}(\mathfrak{g})) \setminus 0\}^{-1}\mathcal{U}(\mathfrak{g})$ equals $\{\mathcal{O}(\mathfrak{g}) \setminus 0\}^{-1}\mathcal{U}(\mathfrak{g})$ since $\mathcal{U}(\mathfrak{g})$ is a finitely generated $\mathcal{O}(\mathfrak{g})$-module (see §6.5 [9]). Hence $p^{2m} := dim_{Q(\mathfrak{Z}(\mathcal{U}(\mathfrak{g})))} Q(\mathcal{U}(\mathfrak{g})) \leq dim_{Q(\mathcal{O}(\mathfrak{g}))} Q(\mathcal{U}(\mathfrak{g})) = p^{10}$. So $m \leq 5$ is obtained implying that p^5 is an upper bound for the dimensions of all irreducible $\mathfrak{g}$-modules. The next proposition shows that p^4 is in fact the upper bound of these and so there exists an irreducible $\mathfrak{g}$-module of dimension p^4 since F is algebraically closed.

PROPOSITION (2.1). $[Q(\mathcal{U}(\mathfrak{g})) : Q(\mathfrak{Z})] = p^8$.

Proof. Consider the $Q(\mathfrak{Z})$-vector space generated by $\{x_\alpha^{i_1} \mid 0 \leq i_1 \leq p-1\}$. Then $Q(\mathfrak{Z}) \cdot 1 + Q(\mathfrak{Z})x_\alpha + \cdots + Q(\mathfrak{Z})x_\alpha^{p-1}$ for any $\alpha \in \Phi$ becomes a free $Q(\mathfrak{g})$-module in $Q(\mathcal{U}(\mathfrak{g}))$. For, if there is a linearly dependent relation with the least number of terms, then by multiplying h_α on both sides of this equation and by using $h_\alpha x_\alpha = x_\alpha(2 + h_\alpha)$, we can make a shorter relation than the given one. Next by the elementary theory for tensor products of free modules and by the fact that $\mathcal{O}(\mathfrak{g})$ is the Noether normalization of $\mathfrak{Z}$, we see easily that $\{\oplus_{i=0}^{p-1} Q(\mathfrak{Z})x_\alpha^i\} \otimes_{Q(\mathfrak{Z})} \{\oplus_{j=0}^{p-1} Q(\mathfrak{Z})x_{-\alpha}^j\}$ becomes a free $Q(\mathfrak{Z})$-module in $Q(\mathcal{U}(\mathfrak{g}))$ with basis $\{x_\alpha^{i_1} \otimes x_{-\alpha}^{i_2} \mid 0 \leq i_1, i_2 \leq p-1\}$. By induction, we see easily that $B = \{x_{\alpha_1}^{i_1} \otimes x_{-\alpha_1}^{i'_1} \otimes x_{\alpha_2}^{i_2} \otimes x_{-\alpha_2}^{i'_2} \otimes x_{\alpha_1+\alpha_2}^{i_3} \otimes x_{-\alpha_1-\alpha_2}^{i'_3} \otimes x_{2\alpha_1+\alpha_2}^{i_4} \otimes x_{-2\alpha_1-\alpha_2}^{i'_4} \mid 0 \leq i_j, i'_j \leq p-1\}$ spans a $Q(\mathfrak{Z})$-vector space with dimension p^8, which is just $Q(\mathcal{U}(\mathfrak{g}))$ since $[Q(\mathcal{U}(\mathfrak{g})) : Q(\mathfrak{Z})] = p^{2m} \leq p^8$ holds clearly.

COROLLARY (2.2). $[Q(\mathfrak{Z}):Q(\mathcal{O}(\mathfrak{g}))]=p^2$.

Proof. Straightforward since $p^{10} = [Q(\mathcal{U}(\mathfrak{g})) : Q(\mathcal{O}(\mathfrak{g}))] = [Q(\mathcal{U}(\mathfrak{g})) : Q(\mathfrak{Z})][Q(\mathfrak{Z}) : Q(\mathcal{O}(\mathfrak{g}))] = p^8[Q(\mathfrak{Z}) : Q(\mathcal{O}(\mathfrak{g}))]$ by the proposition.

REMARK (2.3). *Evidently the above base B spans a free $\mathfrak{Z}$-submodule M of $\mathcal{U}(\mathfrak{g})$ with cardinality p^8, while $(\mathfrak{Z}\setminus 0)^{-1}M = Q(\mathfrak{Z})\otimes_{\mathfrak{Z}} M = Q(\mathfrak{Z})\otimes_{\mathfrak{Z}}\mathcal{U}(\mathfrak{g}) = Q(\mathcal{U}(\mathfrak{g}))$ does not always mean $M=\mathcal{U}(\mathfrak{g})$.*

3. Center $\mathfrak{Z}$ of $\mathcal{U}(\mathfrak{g})$

The natural representation $\varphi:\mathfrak{g}\to gl_4(F)$ has a Casimir element $s := (h_{\alpha_2}+1)^2 + (h_{2\alpha_1+\alpha_2}+1)^2 + 2h_{\alpha_2} + 4(x_{-\alpha_2}x_{\alpha_2} + x_{-2\alpha_1-\alpha_2}x_{2\alpha_1+\alpha_2}) + 2(x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2} + x_{\alpha_1}x_{-\alpha_1})$ belonging to $\mathfrak{Z} := \mathfrak{Z}(\mathcal{U}(\mathfrak{g}))$. With respect to the nondegenerate symmetric bilinear form $\beta(x,y) := tr(\varphi(x)\varphi(y))$, we have dual basis as follows :

Basis element(dual element) :	x_{α_2}	h_{α_2}	$x_{2\alpha_1+\alpha_2}$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
Dual element(basis element) :	$x_{-\alpha_2}$	$2^{-1}h_{\alpha_2}$	$x_{-2\alpha_1-\alpha_2}$

$h_{2\alpha_1+\alpha_2}$	$x_{\alpha_1+\alpha_2}$	$x_{-\alpha_1}$
$\updownarrow$	$\updownarrow$	$\updownarrow$
$2^{-1}h_{2\alpha_1+\alpha_2}$	$2^{-1}x_{-\alpha_1-\alpha_2}$	$2^{-1}x_{\alpha_1}$

Now we shall show that s becomes an integral element over $\mathcal{O}(\mathfrak{g})$ of degree p^2.

PROPOSITION (3.1). *The following hold :*
(i) $dim_{Q(\mathfrak{Z})}Q(\mathfrak{Z})(h_{\alpha_2})(h_{2\alpha_1+\alpha_2}) = p^2$.
(ii) $Q(\mathfrak{Z})(h_{\alpha_2})(h_{2\alpha_1+\alpha_2})$ *becomes a Galois field over* $Q(\mathfrak{Z})$.

Proof. (i) Since $Q(\mathfrak{Z})$ is the center of the simple Artinian algebra $Q(\mathcal{U}(\mathfrak{g}))$, it becomes a central simple $Q(\mathfrak{Z})$-algebra. Since $Q(\mathfrak{Z})(h_{\alpha_2})(h_{2\alpha_1+\alpha_2})$ is a finite dimensional simple $Q(\mathfrak{Z})$-subalgebra of $Q(\mathcal{U}(\mathfrak{g}))$ containing $Q(\mathfrak{Z})$, then by Skolem-Noether theorem, every automorphism of $Q(\mathfrak{Z})(h_{\alpha_2})(h_{2\alpha_1+\alpha_2})$ extends to an inner automorphism of $Q(\mathcal{U}(\mathfrak{g}))$. By direct computation, we have

$$\begin{aligned} &h_{\alpha_2}(x_{\alpha_2}x^i_{-\alpha_2})(x_{2\alpha_1+\alpha_2}x^j_{-2\alpha_1-\alpha_2}) = \\ &x_{\alpha_2}x^i_{-\alpha_2}(x_{2\alpha_1+\alpha_2}x^j_{-2\alpha_1-\alpha_2})(h_{\alpha_2} - 2(i-1)), \\ &h_{2\alpha_1+\alpha_2}(x_{\alpha_2}x^i_{-\alpha_2})(x_{2\alpha_1+\alpha_2}x^j_{-2\alpha_1-\alpha_2}) = \\ &x_{\alpha_2}x^i_{-\alpha_2}(x_{2\alpha_1+\alpha_2}x^j_{-2\alpha_1-\alpha_2})(h_{2\alpha_1+\alpha_2} - 2(j-1)) \end{aligned}$$

for $1 \leq i, j \leq p$. So conjugation by $x_{\alpha_2}x^i_{-\alpha_2}x_{2\alpha_1+\alpha_2}x^j_{-2\alpha_1-\alpha_2}$ in $Q(\mathcal{U}(\mathfrak{g}))$ gives p^2-different values to $Q(\mathfrak{Z})$ $(h_{\alpha_2}, h_{2\alpha_1+\alpha_2}) = Q(\mathfrak{Z})$ (h_{α_2}) $(h_{2\alpha_1+\alpha_2})$. Hence we have $[Q(\mathfrak{Z})(h_{\alpha_2}, h_{2\alpha_1+\alpha_2}) : Q(\mathfrak{Z})] = p^2$.
(ii) is an immediate consequence of the proof of (i).

PROPOSITION (3.2). *$Q(\mathcal{O}(\mathfrak{g}))(s) = Q(\mathfrak{Z})$ in $Q(\mathcal{U}(\mathfrak{g}))$.*

Proof. Since $\mathcal{O}(\mathfrak{g})$ is a Noether normalization of $\mathfrak{Z}(\mathcal{U}(\mathfrak{g}))$ and since $s \in \mathfrak{Z}(\mathcal{U}(\mathfrak{g})) \setminus \mathcal{O}(\mathfrak{g})$, our assertion is straightforward. Specifically, recall that $(h^p_{\alpha_2} - h_{\alpha_2})^2 - \prod_{k=0}^{p-1}\{(h_{\alpha_2}+1)^2 + 4x_{-\alpha_2}x_{\alpha_2} - k^2\} = -4x^p_{-\alpha_2}x^p_{\alpha_2}$ holds by virtue of [8]. Since $(h_{\alpha_2}+1)^2 + 4x_{-\alpha_2}x_{\alpha_2} = s - \{(h_{2\alpha_1+\alpha_2}+1)^2 + 4x_{-2\alpha_1-\alpha_2}x_{2\alpha_1+\alpha_2} + 2h_{\alpha_2} + 2x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2} + 2x_{\alpha_1}x_{-\alpha_1}\}$, we have

$$\begin{aligned} &(h^p_{\alpha_2} - h_{\alpha_2})^2 - \prod_{k=0}^{p-1}[s - \{2h_{\alpha_2} + (h_{2\alpha_1+\alpha_2}+1)^2 \\ &+ 4x_{-2\alpha_1-\alpha_2}x_{2\alpha_1+\alpha_2} + 2x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2} + 2x_{\alpha_1}x_{-\alpha_1}\} - k^2] \\ &+ 4x^p_{-\alpha_2}x^p_{\alpha_2} = 0, \end{aligned}$$

which is clearly an algebraic equation of s over the field $Q(\mathcal{O}(\mathfrak{g}))$ $(2h_{\alpha_2} + (h_{2\alpha_1+\alpha_2}+1)^2 + 4x_{-2\alpha_1-\alpha_2}x_{2\alpha_1+\alpha_2} + 2x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2} + 2x_{\alpha_1}x_{-\alpha_1})$. So if there exists an algebraic equation of s over $Q(\mathcal{O}(\mathfrak{g}))$, it should be of degree $> p$, i.e., of degree p^2 by Corollary (2.2).

Noting that $\mathcal{O}(\mathfrak{g})$ is a unique factorization domain and that $\mathfrak{Z}$ becomes integral over $\mathcal{O}(\mathfrak{g})$, we have $Irr(s, Q(\mathcal{O}(\mathfrak{g}))) = Irr(s, \mathcal{O}(\mathfrak{g}))$.

PROPOSITION (3.3). *$\mathcal{O}(\mathfrak{g})[s] = \mathfrak{Z}(\mathcal{U}(\mathfrak{g})) = \mathfrak{Z}$ holds.*

Proof. We see easily that $Q(\mathfrak{Z}) = Q(\mathcal{O}(\mathfrak{g})[s])$ by proposition (3.2). Since $\mathfrak{Z}$ becomes a finitely generated $\mathcal{O}(\mathfrak{g})$-module and $\mathcal{O}(\mathfrak{g})[s]$ is completely closed in $\mathfrak{Z}$, i.e., any nontrivial quotients of $\mathcal{O}(\mathfrak{g})[s]$ is not contained in $\mathfrak{Z} \setminus \mathcal{O}(\mathfrak{g})[s]$, our assertion holds. Explicitly, suppose that some $\mu \in \mathfrak{Z} \setminus \mathcal{O}(\mathfrak{g})[s]$ satisfies an equation $\mu \cdot \alpha\,(x^p_{\alpha_2}, x^p_{-\alpha_2}, h^p_{\alpha_2} - h_{\alpha_2}, \cdots, x^p_{-\alpha_1}, x^p_{\alpha_1}, s) = \beta(x^p_{\alpha_2}, x^p_{-\alpha_2}, h^p_{\alpha_2} - h_{\alpha_2}, \cdots, x^p_{-\alpha_1}, x^p_{\alpha_1}, s)$ with β/α reduced, where α, β are distinct polynomials in $F[x^p_{\alpha_2}, x^p_{-\alpha_2}, h^p_{\alpha_2} - h_{\alpha_2}, \cdots, x^p_{-\alpha_1}, x^p_{\alpha_1}, s]$ and μ, s must satisfy a nontrivial integral equation over $\mathcal{O}(\mathfrak{g})$. Note that $x^p_{\alpha_2}$, $x^p_{-\alpha_2}$, $h^p_{\alpha_2} - h_{\alpha_2}$, $\cdots$, $x^p_{-\alpha_1}$, $x^p_{\alpha_1}$ are all algebraically independent and so the above relation must be an identical equation with respect to these variables. Now comparing degrees of both sides yields a contradiction by P-B-W theorem.

4. Irreducible polynomial of s over $\mathcal{O}(\mathfrak{g})$

Here we want to find out the irreducible polynomial of s over $Q(\mathcal{O}(\mathfrak{g}))$, which is just the irreducible integral equation of s over $\mathcal{O}(\mathfrak{g})$ by the unique factorization domain property.

PROPOSITION (4.1). *We have the following :*
(i) $Irr(s, \mathcal{O}(\mathfrak{g}))$ is obtained by expanding out

$$N_{Q(\mathfrak{Z})}^{Q(\mathfrak{Z})(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})}\{s - (h_{\alpha_2}+1)^2 - (h_{2\alpha_1+\alpha_2}+1)^2 - 2h_{\alpha_2}\}$$
$$= N_{Q(\mathfrak{Z})}^{Q(\mathfrak{Z})(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})}\{4(x_{-\alpha_2}x_{\alpha_2} + (x_{-2\alpha_1-\alpha_2}x_{2\alpha_1+\alpha_2})$$
$$+ 2(x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2} + x_{\alpha_1}x_{-\alpha_1})\}$$

and its degree is p^2.
(ii) s is separable over $\mathcal{O}(\mathfrak{g})$ and so over $Q(\mathcal{O}(\mathfrak{g}))$.

Proof. (i) Left hand side $= s^{p^2} + a_1 s^{p^2-1} + \cdots + a_{p^2-1}s + a_{p^2}$ for some $a_i \in Q(\mathfrak{Z})(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})$, so we show that $a_i \in Q(\mathcal{O}(\mathfrak{Z}))$. For, choose any distinct p^2-elements $s_i \in \mathcal{O}(\mathfrak{g})$, and take norms as :

$$N_{Q(\mathfrak{Z})}^{Q(\mathfrak{Z})(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})}\{s_i - (h_{\alpha_2}+1)^2 - (h_{2\alpha_1+\alpha_2}+1)^2 - 2h_{\alpha_2}\}$$
$$= N_{Q(\mathfrak{Z})}^{Q(\mathfrak{Z})(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})}\{s_i - s + 4(x_{-\alpha_2}x_{\alpha_2} + (x_{-2\alpha_1-\alpha_2}x_{2\alpha_1+\alpha_2})$$
$$+ 2(x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2} + x_{\alpha_1}x_{-\alpha_1})\} \in Q(\mathfrak{Z}).$$

Since $Q(\mathfrak{Z})$ is the center of the simple Artinian algebra $Q(\mathcal{U}(\mathfrak{g}))$, $Q(\mathcal{U}(\mathfrak{g}))$ becomes a $Q(\mathfrak{Z})$-algebra. Since $Q(\mathfrak{Z})(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})$ is a finite dimensional simple $Q(\mathfrak{Z})$-subalgebra of $Q(\mathcal{U}(\mathfrak{g}))$ containing $Q(\mathfrak{Z})$, then by Skolem-Noether theorem, every automorphism of $Q(\mathfrak{Z})(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})$ extends to an inner automorphism of $Q(\mathcal{U}(\mathfrak{g}))$. By direct calculation, we have

$$\begin{aligned}
&h_{\alpha_2}(x_{\alpha_2}x^i_{-\alpha_2})(x_{2\alpha_1+\alpha_2}x^j_{-2\alpha_1-\alpha_2})\\
&= (x_{\alpha_2}x^i_{-\alpha_2})(x_{2\alpha_1+\alpha_2}x^j_{-2\alpha_1-\alpha_2})(h_{\alpha_2} - 2(i-1)),\\
&h_{2\alpha_1+\alpha_2}(x_{\alpha_2}x^i_{-\alpha_2})(x_{2\alpha_1+\alpha_2}x^j_{-2\alpha_1-\alpha_2})\\
&= (x_{\alpha_2}x^i_{-\alpha_2})(x_{2\alpha_1+\alpha_2}x^j_{-2\alpha_1-\alpha_2})(h_{2\alpha_1+\alpha_2} - 2(j-1)).
\end{aligned}$$

Hence conjugation by $x_{\alpha_2}x^i_{-\alpha_2}x_{2\alpha_1+\alpha_2}x^j_{-2\alpha_1-\alpha_2}$ in $Q(\mathcal{U}(\mathfrak{g}))$ gives p^2-distinct values to $Q(\mathfrak{Z})$ $(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})$. Next, since $[Q(\mathfrak{Z})$ $(h_{\alpha_2}, h_{2\alpha_1+\alpha_2}) : Q(\mathfrak{Z})] = [Q(\mathcal{O}(\mathfrak{g}))$ $(h_{\alpha_2}, h_{2\alpha_1+\alpha_2}) : Q(\mathcal{O}(\mathfrak{g}))] = p^2$ and since isomorphisms of $Q(\mathfrak{Z})$ $(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})$ over $Q(\mathfrak{Z})$ are the same as those of $Q(\mathcal{O}(\mathfrak{g}))(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})$ over $Q(\mathcal{O}(\mathfrak{g}))$ which are precisely conjugations by $x_{\alpha_2}x^i_{-\alpha_2}x_{2\alpha_1+\alpha_2}x^j_{-2\alpha_1-\alpha_2}$, we obtain

$$\begin{aligned}
&N^{Q(\mathfrak{Z})(h_{\alpha_2},h_{2\alpha_1+\alpha_2})}_{Q(\mathfrak{Z})}\{s_i - (h_{\alpha_2}+1)^2 - (h_{2\alpha_1+\alpha_2}+1)^2 - 2h_{\alpha_2}\} =\\
&N^{Q(\mathcal{O}(\mathfrak{g}))(h_{\alpha_2},h_{2\alpha_1+\alpha_2})}_{Q(\mathcal{O}(\mathfrak{g}))}\{s_i - (h_{\alpha_2}+1)^2 - (h_{2\alpha_1+\alpha_2}+1)^2 - 2h_{\alpha_2}\}\\
&\in Q(\mathcal{O}(\mathfrak{g})),
\end{aligned}$$

so we see that $N^{Q(\mathfrak{Z})(h_{\alpha_2},h_{2\alpha_1+\alpha_2})}_{Q(\mathfrak{Z})}$ $\{s_i - s + 4(x_{-\alpha_2}x_{\alpha_2} + x_{-2\alpha_1-\alpha_2}x_{2\alpha_1+\alpha_2}) + 2(x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2} + x_{\alpha_1}x_{-\alpha_1})\}$ actually belongs to $Q(\mathcal{O}(\mathfrak{g}))$.

On the other hand,

$$\begin{aligned}
&N^{Q(\mathcal{O}(\mathfrak{g}))(h_{\alpha_2},h_{2\alpha_1+\alpha_2})}_{Q(\mathcal{O}(\mathfrak{g}))}\{s_i - (h_{\alpha_2}+1)^2 - (h_{2\alpha_1+\alpha_2}+1)^2 - 2h_{\alpha_2}\}\\
&= \prod_{j,k}(x_{\alpha_2}x^j_{-\alpha_2}x_{2\alpha_1+\alpha_2}x^k_{-2\alpha_1-\alpha_2})^{-1}\{s_i - (h_{\alpha_2}+1)^2\\
&\quad - (h_{2\alpha_1+\alpha_2}+1)^2 - 2h_{\alpha_2}\}(x_{\alpha_2}x^j_{-\alpha_2}x_{2\alpha_1+\alpha_2}x^k_{-2\alpha_1-\alpha_2})\\
&= \prod_{j,k}\{s_i - (h_{\alpha_2} - 2(j-1)+1)^2 - (h_{2\alpha_1+\alpha_2} - 2(k-1)+1)^2\\
&\quad - 2(h_{\alpha_2} - 2(j-1))\}\\
&=: s_i^{p^2} + b_1 s_i^{p^2-1} + \cdots + b_{p^2-1}s_i + b_{p^2} = k_i
\end{aligned}$$

for some $b_l (l = 1, 2, \cdots, p^2) \in Q(\mathcal{O}(\mathfrak{g}))(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})$ and $k_i \in Q(\mathcal{O}(\mathfrak{g}))$. Hence there arises a linear system in indeterminates b_l of p^2- equations with the determinant of coefficients

$$\begin{vmatrix} 1 & s_1 & \cdots & s_1^{p^2-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & s_{p^2} & \cdots & s_{p^2}^{p^2-1} \end{vmatrix} = \prod_{i<j} (s_j - s_i) \neq 0,$$

which is nothing but the well known Vandermonde determinant. Hence by the Cramer's method, we may obtain solutions $b_l \in Q(\mathcal{O}(\mathfrak{g}))$. But since $b_i = a_i$ for $i = 1, 2, \cdots, p^2$, we have $a_i \in Q(\mathcal{O}(\mathfrak{g}))$.

On the other hand, since s is integral over $\mathcal{O}(\mathfrak{g})$, right hand side of (i) $= N_{Q(\mathfrak{Z})}^{Q(\mathfrak{Z})(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})}\{4(x_{-\alpha_2}x_{\alpha_2} + (x_{-2\alpha_1-\alpha_2}x_{2\alpha_1+\alpha_2}) + 2(x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2} + x_{\alpha_1}x_{-\alpha_1})\}$ also belongs to $Q(\mathcal{O}(\mathfrak{g}))$. This is so because of the following fact.

LEMMA (4.2). *Let $s \in \mathfrak{Z}(\mathcal{U}(\mathfrak{g})) =: \mathfrak{Z}$ be as before and let s satisfy an algebraic equation $f(X) := X^{p^2} + a_1 X^{p^2-1} + \cdots + a_{p^2-1}X + a_{p^2} \in Q(\mathfrak{Z})[X]$ with $a_j \in Q(\mathcal{O}(\mathfrak{g}))$ for $1 \leq j \leq p^2 - 1$ the same as in the above argument for the left hand side of (i); we have then $a_{p^2} \in Q(\mathcal{O}(\mathfrak{g}))$.*

Proof. Since $\mathcal{O}(\mathfrak{g})$ becomes the Noether normalization of $\mathfrak{Z}$, the integral equation of s is itself $Irr(s, Q(\mathcal{O}(\mathfrak{g})))$. Let $Irr(s, Q(\mathcal{O}(\mathfrak{g}))) =: X^{p^k} + b_1 X^{p^k-1} + \cdots + b_{p^k-1}X + b_{p^k}$. Put $\sigma_{m,n} :=$ conjugation by $x_{\alpha_2} x^m_{-\alpha_2} x_{2\alpha_1+\alpha_2} x^n_{-2\alpha_1-\alpha_2}$ for $1 \leq m, n \leq p$. Then $s \mapsto \sigma_{m,n}(2h_{\alpha_2} + (h_{\alpha_2+1})^2 + (h_{2\alpha_1+\alpha_2} + 1)^2) + \sigma_{m',n'}(4(x_{-\alpha_2}x_{\alpha_2} + x_{-2\alpha_1-\alpha_2}x_{2\alpha_1+\alpha_2}) + 2(x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2} + x_{\alpha_1}x_{-\alpha_1}))$ for $1 \leq m$, n, m', $n' \leq p$ yields an isomorphism of $Q(\mathcal{O}(\mathfrak{g}))(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})(s)$ over $Q(\mathcal{O}(\mathfrak{g}))$. Hence $Irr(s, Q(\mathcal{O}(\mathfrak{g}))) \mid f(X)$, and so $f(X) = Irr(s, Q(\mathcal{O}(\mathfrak{g})))\ g(X)$ for some unique $g(X) := X^{p^2-p^k} + c_1 X^{p^2-p^k-1} + \cdots + c_{p^2-p^k-1}X + c_{p^2-p^k}$ with $c_j \in Q(\mathfrak{Z})$. Since c_j's are uniquely determined only by the coefficients of $Irr(s, Q(\mathcal{O}(\mathfrak{g})))$ and those of terms X^{p^2}, X^{p^2-1}, $\cdots$, X^{p^k+1}, X^{p^k} in $f(X)$ and since these coefficients belong to $Q(\mathcal{O}(\mathfrak{g}))$, we have our assertion.

Returning to our main proof, we have to show that the algebraic equation of s obtained above is just the $Irr(s, Q(\mathcal{O}(\mathfrak{g})))$, which is also the integral equation of s over $\mathcal{O}(\mathfrak{g})$. Such a fact is due to the following lemma.

LEMMA (4.3). *Let E be a finite extension of F with characteristic p and let α, β, γ be in $E \setminus F$ with $[F(\alpha)(\gamma) : F] = p^n$. Let $f(X) = Irr(\alpha, F) = (X - \sigma_1(\alpha)) \cdots (X - \sigma_{p^m}(\alpha))$ for some distinct $\sigma_i(\alpha) \in F(\alpha)$, $i = 1, 2, \cdots, p^m \leq p^n$ with $\sigma_i \in$ {isomorphisms of E over F} and let $\beta = \alpha + \gamma$ be an element such that β satisfies $\sigma_i(\beta) = \beta$ for all σ_i. Suppose that $\prod_{i=1}^{p^m} \sigma_i(\gamma) \in F$ and $g(X) := \prod_{i=1}^{p^m}(X - \sigma_i(\alpha)) - \prod_{i=1}^{p^m} \sigma_i(\gamma) \in F[X]$. We have then $g(X) = Irr(\beta, F)$ and is separable over F.*

Proof. We see easily that $\gamma \notin F(\alpha)$ and so at least p^{m+1} distinct isomorphisms of $F(\alpha)(\gamma)(\ni \beta)$ over F exist.

Now consider a field lattice diagram :

$$\begin{array}{c} F(\alpha)(\gamma) \ni \beta \\ \Big\uparrow \text{at least } p\text{-dimensional} \\ F(\alpha) \not\ni \gamma \\ \Big\uparrow p^m\text{-dimensional} \\ F \end{array}$$

Then for any nontrivial isomorphism τ of $F(\alpha)(\gamma)$ over $F(\alpha)$, $\tau(\beta) = \tau(\sigma_i(\alpha) + \sigma_i(\gamma)) = \tau(\sigma_i(\alpha)) + \tau(\sigma_i(\gamma)) = \sigma_i(\alpha) + \bar{\tau}(\gamma)$ holds for some isomorphism $\bar{\tau}$ of $F(\alpha)(\gamma)$ over $F(\alpha)$ and for all i with $1 \leq i \leq p^m$. Hence there are at least p^m-distinct conjugates of β over F since $F(\alpha)$ is a Galois extension of F. But since $deg\ g(X) = p^m$ and since $g(\beta) = 0$ obviously, $Irr(\beta, F) = g(X)$ should hold.

Finally to complete our proof of the proposition (4.1), put

$$\begin{aligned} F &= Q(\mathcal{O}(\mathfrak{g})), \\ \beta &= s, \\ \alpha &= (h_{\alpha_2} + 1)^2 + (h_{2\alpha_1+\alpha_2} + 1)^2 + 2h_{\alpha_2}, \\ \gamma &= 4(x_{-\alpha_2}x_{\alpha_2} + x_{-2\alpha_1-\alpha_2}x_{2\alpha_1+\alpha_2}) \\ &\quad + 2(x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2} + x_{\alpha_1}x_{-\alpha_1}) \end{aligned}$$

and use the proof of proposition (3.1), where isomorphisms of $F(\alpha)$ over F are well specified. Evidently, (ii) is obtained from Lemma (4.3).

COROLLARY (4.4).

$$Q(\mathcal{O}(\mathfrak{g}))(h_{\alpha_2}, h_{2\alpha_1+\alpha_2}) =$$
$$Q(\mathcal{O}(\mathfrak{g}))((h_{\alpha_2}+1)^2 + (h_{2\alpha_1+\alpha_2}+1)^2 + 2h_{\alpha_2}).$$

Proof. Straightforward from isomorphisms of $Q(\mathcal{O}(\mathfrak{g}))$ $(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})$ over $Q(\mathcal{O}(\mathfrak{g}))$.

REMARK. *From the proof of proposition (4.1), we must see that* $Q(\mathfrak{Z})(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})$ *is a Galois extension of* $Q(\mathfrak{Z})$ *and that the division algebra* $Q(\mathcal{U}(\mathfrak{g}))$ *is a crossed product of* $Q(\mathfrak{Z})$ $(h_{\alpha_2}, h_{2\alpha_1+\alpha_2})$ $(x_{\alpha_2}x_{-\alpha_2} + x_{2\alpha_1+\alpha_2}\ x_{-2\alpha_1-\alpha_2} + x_{\alpha_1+\alpha_2}\ x_{-\alpha_1-\alpha_2} + x_{-\alpha_1}x_{\alpha_1})$.

5. Points in the Zassenhaus Variety

Let V be any finite dimensional irreducible $\mathfrak{g}$-module, i.e. $\forall v (\neq 0) \in V$, $\mathcal{U}(\mathfrak{g})v = V$. By virtue of [1], its irreducible representation $\varphi : \mathcal{U}(\mathfrak{g}) \to End_F(\mathcal{U}(\mathfrak{g})v)$ is uniquely determined up to isomorphisms by $\mathfrak{Z}/(Ker\,\varphi \cap \mathfrak{Z})$ and by 1-dimensional representation of a Cartan subalgebra. Note that for $\Delta = C_{End_F(V)}(\varphi(\mathcal{U}(\mathfrak{g})))$, $\varphi(\mathcal{U}(\mathfrak{g}))$ is dense in $End_\Delta(V)$. Since $\Delta = F$ by Schur's lemma and since $[Q(\mathcal{U}(\mathfrak{g})) : Q(\mathfrak{Z})] = p^8$ by proposition (2.1), we have $End_\Delta(V) \cong End_F(V) \cong F_n$ for $n \leq p^4$ as F-algebras, where $F_n = M_n(F)$, which boils down to $\varphi(\mathcal{U}(\mathfrak{g})) \cong F_n$ after all. Furthermore, the irreducible representations of the same dimension are equivalent so long as their kernels meet $\mathfrak{Z}$ in the same part and $dim_F(\mathcal{U}(\mathfrak{g})v) = dimV = p^4$ [10].

Let ρ be a (regular) left maximal ideal of $\mathcal{U}(\mathfrak{g})$ and put $(\rho : \mathcal{U}(\mathfrak{g})) := \{x \in \mathcal{U}(\mathfrak{g}) | x \cdot \mathcal{U}(\mathfrak{g}) \subset \rho\}$. Then the annihilator $A(\mathcal{U}(\mathfrak{g})/\rho)$ $= (\rho : \mathcal{U}(\mathfrak{g}))$ becomes the largest two sided ideal contained in ρ, and so a maximal ideal $\mathfrak{m}$ of $\mathcal{U}(\mathfrak{g})$. Hence using the above notation, we have $\varphi(\mathcal{U}(\mathfrak{g})) \cong \mathcal{U}(\mathfrak{g})/\mathfrak{m}$.

Furthermore,

$$\begin{aligned}&\rho \supset A(\mathcal{U}(\mathfrak{g})/\rho) \supset \mathcal{U}(\mathfrak{g})(x^p_{\alpha_2} - \xi_1)\\ &+\mathcal{U}(\mathfrak{g})(x^p_{-\alpha_2} - \xi_2) + \mathcal{U}(\mathfrak{g})(h^p_{\alpha_2} - h_{\alpha_2} - \xi_3) + \mathcal{U}(\mathfrak{g})(x^p_{2\alpha_1+\alpha_2} - \xi_4)\\ &+\mathcal{U}(\mathfrak{g})(x^p_{-2\alpha_1-\alpha_2} - \xi_5) + \mathcal{U}(\mathfrak{g})(h^p_{2\alpha_1+\alpha_2} - h_{2\alpha_1+\alpha_2} - \xi_6)\\ &+\mathcal{U}(\mathfrak{g})(x^p_{\alpha_1+\alpha_2} - \xi_7) + \mathcal{U}(\mathfrak{g})(x^p_{-\alpha_1-\alpha_2} - \xi_8)\\ &+\mathcal{U}(\mathfrak{g})(x^p_{-\alpha_1} - \xi_9) + \mathcal{U}(\mathfrak{g})(x^p_{\alpha_1} - \xi_{10}) + \mathcal{U}(\mathfrak{g})(s - \xi_{11})\end{aligned}$$

holds, where $\xi_i (i = 1, 2, \cdots, 10)$ is an independent value in F for the corresponding indeterminate and $\xi_1, \cdots, \xi_{10}, \xi_{11}$ must satisfy $Irr(s, \mathcal{O}(\mathfrak{g}))$; $A(\mathcal{U}(\mathfrak{g})/\rho) \cap \mathfrak{Z} = \mathfrak{Z}(x^p_{\alpha_2} - \xi_1) + \mathfrak{Z}(x^p_{-\alpha_2} - \xi_2) + \mathfrak{Z}(h^p_{\alpha_2} - h_{\alpha_2} - \xi_3) + \mathfrak{Z}(x^p_{2\alpha_1+\alpha_2} - \xi_4) + \mathfrak{Z}(x^p_{-2\alpha_1-\alpha_2} - \xi_5) + \mathfrak{Z}(h^p_{2\alpha_1+\alpha_2} - h_{2\alpha_1+\alpha_2} - \xi_6) + \mathfrak{Z}(x^p_{\alpha_1+\alpha_2} - \xi_7) + \mathfrak{Z}(x^p_{-\alpha_1-\alpha_2} - \xi_8) + \mathfrak{Z}(x^p_{-\alpha_1} - \xi_9) + \mathfrak{Z}(x^p_{\alpha_1} - \xi_{10}) + \mathfrak{Z}(s - \xi_{11})$ becomes a maximal ideal of $\mathfrak{Z}$ by going-up theorem in [9].

So, if $\mathfrak{m} \cap \mathfrak{Z} = \bar{\mathfrak{m}}$ corresponds to $(\xi_1, \cdots, \xi_{10}, \xi_{11})$ in the Zassenhaus variety, $dim_F \mathcal{U}(\mathfrak{g})/\rho$ may be easily computed through independence of some elements of $\mathcal{U}(\mathfrak{g})/\rho$. It is noteworthy that a character $\chi : \mathfrak{Z} \to F$ is given if and only if $(\xi_1, \cdots, \xi_{10}, \xi_{11})$ is given.

By the way, [5] and [6] say that any maximal ideal $\mathfrak{m}$ of $\mathcal{U}(\mathfrak{g})$ must contain some $\{\sum_{i=1}^{10} \mathcal{U}(\mathfrak{g})(x_i^p - x_i^{[p]} - \xi_i)\} + \mathcal{U}(\mathfrak{g})(s - \xi_{11})$, where x_i ranges over $\{x_{\alpha_2}, x_{-\alpha_2}, h_{\alpha_2}, x_{2\alpha_1+\alpha_2}, x_{-2\alpha_1-\alpha_2}, h_{2\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2}, x_{-\alpha_1-\alpha_2}, x_{-\alpha_1}, x_{\alpha_1}\}$ and $dim_F \mathcal{U}(\mathfrak{g})/\mathfrak{m} \leq p^{2\times 4}$ considering proposition (2.1).

PROPOSITION (5.1). *Suppose that all* $\xi_i = 0$ *for* $1 \leq i \leq 10$; *then for all left maximal ideals* $\mathfrak{m}_j$ *containing* $\sum_{i=1}^{10} \mathcal{U}(\mathfrak{g})(x_i^p - x_i^{[p]}) + \mathcal{U}(\mathfrak{g})(s - \xi_{11})$, $\mathcal{U}(\mathfrak{g})/\mathfrak{m}_j$ *become irreducible p-representation modules.*

Proof. It is well known that the associated representations φ_j with
$\mathcal{U}(\mathfrak{g})/\mathfrak{m}_j$ satisfy $\varphi_j(x^{[p]}) - \varphi_j(x)^p = S_j(x)^p \cdot Id$ for some $S_j \in \mathfrak{g}^*$ and for all $x \in \mathfrak{g}$ [9]. But then $x_i^p \equiv 0 \Rightarrow \varphi_j(0) - \varphi_j(x_i)^p \equiv \varphi_j(x_i)^p \equiv 0 \,(\text{mod } \mathfrak{m})$, and $x_i^p - x_i \equiv 0 \Rightarrow \varphi_j(x_i) - \varphi_j(x_i)^p \equiv 0 \,(\text{mod } \mathfrak{m})$ for $1 \leq i \leq 10$, where $\mathfrak{m}$ is the maximal ideal contained in $\mathfrak{m}_j$ for a fixed j. So we have $S_j(x) = 0 \; \forall x \in \mathfrak{g}$.

In our situation, we encounter 3 possible cases of $(\xi_1, \cdots, \xi_{10}, \xi_{11}) \in F^{11}$ as follows :

(I) all $\xi_i = 0$ for $1 \leq i \leq 10$: There may exist finitely many left maximal ideals $\mathfrak{m}_j$ containing $\{\sum_{i=1}^{10} \mathcal{U}(\mathfrak{g})(x_i^p - x_i^{[p]} - \xi_i)\} + \mathcal{U}(\mathfrak{g})(s - \xi_{11})$ so that $\mathcal{U}(\mathfrak{g})/\mathfrak{m}_j$ become p-representation modules for $\mathfrak{g}$ with dimension $\leq p^4$ in view of proposition (5.1) and [10]. Here we suggest that we call such a point $(\xi_1, \cdots, \xi_{10}, \xi_{11}) \in F^{11}$ p-point ; in particular, we mean, by a regular p-point, that it is a p-point and its associated irreducible module has dimension p^4.

(II) not all $\xi_i = 0$ for $1 \leq i \leq 10$:

(i)For any left maximal ideals $\mathfrak{m}_j$ containing $\{\sum_{i=1}^{10} \mathcal{U}(\mathfrak{g})(x_i^p - x_i^{[p]} - \xi_i)\} + \mathcal{U}(\mathfrak{g})(s - \xi_{11})$, $\mathcal{U}(\mathfrak{g})/\mathfrak{m}_j$ may all become p^4-dimensional $S-$representation modules for $\mathfrak{g}$ and are isomorphic. Since F is algebraically closed of characteristic $p > 2$, such a module must necessarily exist [9]. We shall call such a point $(\xi_1, \cdots, \xi_{10}, \xi_{11})$ in F^{11} a regular point.

(ii)For all left maximal ideals $\mathfrak{m}_j$ containing $\{\sum_{i=1}^{10} \mathcal{U}(\mathfrak{g})(x_i^p - x_i^{[p]} - \xi_i)\} + \mathcal{U}(\mathfrak{g})(s - \xi_{11})$, $\mathcal{U}(\mathfrak{g})/\mathfrak{m}_j$ may have F-dimension $< p^4$ and are possibly nonisomorphic and are possibly of different dimensions. So, we call in this case such a point $(\xi_1, \cdots, \xi_{10}, \xi_{11}) \in F^{11}$ a subregular point.

Note that we have an obvious surjective mapping $\psi : Spec_{\mathfrak{m}}(\mathfrak{Z}) \twoheadrightarrow$ {Irreducible L − module classes}, where $Spec_{\mathfrak{m}}(\mathfrak{Z})$ denotes the maximal spectrum of $\mathfrak{Z}$.

PROPOSITION (5.2). *There exists no subregular point for* $sl_2(F)$.

Proof. See §1. Introduction of this paper.

6. Main result

We want in this section to show that there is no subregular point for $\mathfrak{g} = sp_4(F)$ like $sl_2(F)$. For this, we should like to find out dimensions for irreducible $\mathfrak{g}$-modules corresponding to points $(\xi_1, \cdots, \xi_{10}, \xi_{11}) \in F^{11} \leftrightarrow \{\sum_{i=1}^{10} \mathfrak{Z}(x_i^p - x_i^{[p]} - \xi_i)\} + \mathfrak{Z}(s - \xi_{11})$ which are maximal ideals of $\mathfrak{Z}$.

Let φ be a finite dimensional irreducible representation of $\mathfrak{g}$ as before. Let $\{x_\alpha, y_\alpha, h_\alpha\}$ be a standard basis corresponding to a root α which spans a copy of $sl_2(F)$, i.e., $[x_\alpha y_\alpha] = h_\alpha$, $[h_\alpha, x_\alpha] = 2x_\alpha$, $[h_\alpha y_\alpha] = -2y_\alpha$. For $\mathfrak{g} = sp_4(F)$, we have 4 kinds of such subalgebras corresponding to 4 positive roots α_2, $2\alpha_1 + \alpha_2$, $\alpha_1 + \alpha_2$, x_{α_1}. Specifically, they are respectively

$$\begin{aligned}
S_{\alpha_2} &= Fx_{\alpha_2} + Fx_{-\alpha_2} + Fh_{\alpha_2},\\
S_{2\alpha_1+\alpha_2} &= Fx_{2\alpha_1+\alpha_2} + Fx_{-2\alpha_1-\alpha_2} + Fh_{2\alpha_1+\alpha_2},\\
S_{\alpha_1+\alpha_2} &= Fx_{\alpha_1+\alpha_2} + Fx_{-\alpha_1-\alpha_2} + F(h_{\alpha_2} + h_{2\alpha_1+\alpha_2}),\\
S_{\alpha_1} &= Fx_{-\alpha_1} + Fx_{\alpha_1} + F(h_{\alpha_2} - h_{2\alpha_1+\alpha_2}).
\end{aligned}$$

Now put

$$\begin{aligned}
\omega_{\alpha_2} &= (h_{\alpha_2} + 1)^2 + 4x_{-\alpha_2}x_{\alpha_2},\\
\omega_{2\alpha_1+\alpha_2} &= (h_{2\alpha_1+\alpha_2} + 1)^2 + 4x_{-2\alpha_1-\alpha_2}x_{2\alpha_1+\alpha_2},\\
\omega_{\alpha_1+\alpha_2} &= (h_{\alpha_2} + h_{2\alpha_1+\alpha_2} + 1)^2 + 4x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2},\\
\omega_{\alpha_1} &= (h_{\alpha_2} - h_{2\alpha_1+\alpha_2} + 1)^2 + 4x_{\alpha_1}x_{-\alpha_1}
\end{aligned}$$

and put

$$\begin{aligned}
g_{\alpha_2} &= x_{\alpha_2}^{p-1} - x_{-\alpha_2},\\
g_{2\alpha_1+\alpha_2} &= x_{2\alpha_1+\alpha_2}^{p-1} - x_{-2\alpha_1-\alpha_2},\\
g_{\alpha_1+\alpha_2} &= x_{\alpha_1+\alpha_2}^{p-1} - x_{-\alpha_1-\alpha_2},\\
g_{\alpha_1} &= x_{\alpha_1}^{p-1} - x_{-\alpha_1}.
\end{aligned}$$

Let $\mathcal{U}(\mathfrak{g})/\rho$ for a left maximal ideal ρ of $\mathcal{U}(\mathfrak{g})$ be an irreducible $\mathfrak{g}$-module; then for each positive root $\beta_i \in \{\alpha_2, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_1\}$, we have an irreducible S_{β_i}-module V_{β_i} in $\mathcal{U}(\mathfrak{g})/\rho$. By choosing an appropriate basis of $\mathcal{U}(\mathfrak{g})/\rho$, we see by virtue of [8] that $\omega_{\beta_i} \in \mathfrak{Z}(\mathcal{U}(S_{\beta_i}))$ acts on V_{β_i} as a constant matrix respectively, where β_i represents any of positive roots.

We note here that the equations

$$\begin{aligned}
h_{\alpha_2}g_{\alpha_2} &= g_{\alpha_2}h_{\alpha_2} - 2g_{\alpha_2} = g_{\alpha_2}(h_{\alpha_2} - 2),\\
h_{2\alpha_1+\alpha_2}g_{2\alpha_1+\alpha_2} &= g_{2\alpha_1+\alpha_2}h_{2\alpha_1+\alpha_2} - 2g_{2\alpha_1+\alpha_2}\\
&= g_{2\alpha_1+\alpha_2}(h_{2\alpha_1+\alpha_2} - 2),\\
(h_{\alpha_2} + h_{2\alpha_1+\alpha_2})g_{\alpha_1+\alpha_2} &= g_{\alpha_1+\alpha_2}(h_{\alpha_2} + h_{2\alpha_1+\alpha_2}) - 2g_{\alpha_1+\alpha_2},\\
(h_{\alpha_2} - h_{2\alpha_1+\alpha_2})g_{\alpha_1} &= g_{\alpha_1}(h_{\alpha_2} - h_{2\alpha_1+\alpha_2}) - 2g_{\alpha_1}
\end{aligned}$$

are obtained without difficulty and that g_{α_2} is invertible in $\mathcal{U}(\mathfrak{g})/\mathfrak{m}$ if $\xi_1 = \xi_2 = 0$ but $\xi_3 \neq 0$; similarly $g_{2\alpha_1+\alpha_2}$ is also invertible in $\mathcal{U}(\mathfrak{g})/\mathfrak{m}$ if $\xi_4 = \xi_5 = 0$ but $\xi_6 \neq 0$.

PROPOSITION (6.1). *Suppose that $\xi_1 \neq 0$ or $\xi_2 \neq 0$ or $\xi_3 \neq 0$; $\sum_{i=0}^{p-1} F(h^i_{\alpha_2}+\mathfrak{m})$ becomes then a free F-submodule of $\mathcal{U}(\mathfrak{g})/\mathfrak{m}$. Suppose similarly that $\xi_4 \neq 0$ or $\xi_5 \neq 0$ or $\xi_6 \neq 0$; $\sum_{i=0}^{p-1} F(h^i_{2\alpha_1+\alpha_2}$ $+\mathfrak{m})$ becomes then a free F-submodule of $\mathcal{U}(\mathfrak{g})/\mathfrak{m}$.*

Proof. We recall that $\bar{\mathfrak{m}} := \mathfrak{Z}\cap\mathfrak{m}$ in §5 and $(\mathfrak{Z}+\mathfrak{m})/\mathfrak{m} \cong \mathfrak{Z}/\bar{\mathfrak{m}} \cong F \subset \mathcal{U}(\mathfrak{g})/\mathfrak{m}$. If there exists a dependence relation of least degree with one term $a_i h^i_j \in \mathfrak{m}$ with $a_i(\neq 0) \in \mathfrak{Z}/\bar{\mathfrak{m}}$, $j = \alpha_2$, $2\alpha_1 + \alpha_2$ and $0 \leq i \leq p-1$, we observe that $\varphi(a_i)(\neq 0)$ becomes a contant in F and so we may assume that $h^i_j \in \mathfrak{m}$; in other words, $\varphi(a_i h^i_j) = \varphi(a_i)\varphi(h^i_j) \equiv 0 \,(\mathrm{mod}\ \mathfrak{m})$, where φ is the corresponding representation of the irreducible module $\mathcal{U}(\mathfrak{g})/\rho$, so that $\varphi(h^i_j) \equiv 0 \,(\mathrm{mod}\ \mathfrak{m})$ since $\varphi(a_i)$ is a nonzero constant by Schur's lemma. Evidently $i \geq 1$. But then $x_j h^i_j = (h_j - 2)^i x_j \in \mathfrak{m}$, $g_j h^i_j = (h_j + 2)^i g_j \in \mathfrak{m}$, $x_{-j} h^i_j = (h_j + 2)^i x_{-j} \in \mathfrak{m}$ for $j = \alpha_2$, $2\alpha_1 + \alpha_2$. By our hypothesis and [8], $\varphi(x^p_{\alpha_2}) \neq 0$ or $\varphi(x^p_{-\alpha_2}) \neq 0$ or $g^p_{\alpha_2} \neq 0$, i.e., there exists some invertible element among these in $\mathcal{U}(\mathfrak{g})/\mathfrak{m}$ if $\xi_1 \neq 0$ or $\xi_2 \neq 0$ or $\xi_3 \neq 0$. Similar assertion is obtained for the other case. Note that we have chosen a composition series of S_j-module $\mathcal{U}(\mathfrak{g})/\rho$ if $\xi_1 = \xi_2 = 0$ for $j = \alpha_2$ and if $\xi_4 = \xi_5 = 0$ for $j = 2\alpha_1 + \alpha_2$ respectively. So applying some invertible element on the relations modulo $\mathfrak{m}$, we meet an equation with a lower degree with respect to h_j than the first one, a contradiction.

If there exists a dependence relation of least degree with more than one term, apply some invertible element as before on both sides and get a dependence relation of lower degree than the given one. So arises another contradiction.

PROPOSITION (6.2). *Suppose that $\xi_1 \neq 0$; we have then a free F-module with rank p^8 in $\mathcal{U}(\mathfrak{g})/\mathfrak{m}$, i.e., $dim\,\mathcal{U}(\mathfrak{g})/\mathfrak{m} = p^8$.*

Proof. We claim that we have a basis :

$$
\begin{aligned}
&\{(a_1h_{\alpha_2}+b_1(c_1+h_{2\alpha_1+\alpha_2}))+x_{\alpha_2}\}^{i_1}\otimes\\
&\{(a_2h_{\alpha_2}+b_2(c_2+h_{2\alpha_1+\alpha_2}))+h_{2\alpha_1+\alpha_2}x_{\alpha_2}\}^{i_2}\otimes\\
&\{(a_3h_{\alpha_2}+b_3(c_3+h_{2\alpha_1+\alpha_2}))+x_{2\alpha_1+\alpha_2}\}^{i_3}\otimes\\
&\{(a_4h_{\alpha_2}+b_4(c_4+h_{2\alpha_1+\alpha_2}))+h_{2\alpha_1+\alpha_2}x_{2\alpha_1+\alpha_2}\}^{i_4}\otimes\\
&\{(a_5h_{\alpha_2}+b_5(c_5+h_{2\alpha_1+\alpha_2}))+x_{-2\alpha_1-\alpha_2}\}^{i_5}\otimes\\
&\{(a_6h_{\alpha_2}+b_6(c_6+h_{2\alpha_1+\alpha_2}))+x_{\alpha_1+\alpha_2}\}^{i_6}\otimes\\
&\{(a_7h_{\alpha_2}+b_7(c_7+h_{2\alpha_1+\alpha_2}))+x_{-\alpha_1}\}^{i_7}\otimes\\
&\{(a_8h_{\alpha_2}+b_8(c_8+h_{2\alpha_1+\alpha_2}))\\
&\qquad\quad+(h_{\alpha_2}+x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2}+x_{\alpha_1}x_{-\alpha_1})\}^{i_8}
\end{aligned}
$$

with $0\le i_j\le p-1$, where (a_i,b_i) are chosen so that $(a_ih_{\alpha_2}+b_i(c_i+h_{2\alpha_1+\alpha_2}))x_{\alpha_2}\not\equiv x_{\alpha_2}(a_ih_{\alpha_2}+b_i(c_i+h_{2\alpha_1+\alpha_2}))\ (\mathrm{mod}\ \mathfrak{m})$, $a_ih_{\alpha_2}+b_i(c_i+h_{2\alpha_1+\alpha_2})\not\equiv c(a_jh_{\alpha_2}+b_j(c_j+h_{2\alpha_1+\alpha_2}))$ for any $c\in F$ (which is possible considering $\mathbb{P}^1(F)$) and c_j is chosen in F so that $c_j+h_{2\alpha_1+\alpha_2}$ is invertible modulo $\mathfrak{m}$. Furthermore we choose (a_i,b_i,c_i) so that any three distinct $(a_i,b_i,c_i)'$s are linearly independent. We first show that $h_{\alpha_2}+x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2}+x_{\alpha_1}x_{-\alpha_1}\notin\mathfrak{m}$. Suppose not; we have then $x_{-\alpha_1}(h_{\alpha_2}+x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2}+x_{\alpha_1}x_{-\alpha_1})-(h_{\alpha_2}+x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2}+x_{\alpha_1}x_{-\alpha_1})x_{-\alpha_1}\in\mathfrak{m}$, so $-x_{-\alpha_1}-2x_{-2\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2}+2x_{-\alpha_1-\alpha_2}x_{\alpha_2}+(h_{\alpha_2}-h_{2\alpha_1+\alpha_2})\ x_{-\alpha_1}\equiv 0\ (\mathrm{mod}\ \mathfrak{m})$. Hence $x_{\alpha_2}\ (-x_{-\alpha_1}-2x_{-2\alpha_1-\alpha_2}\ x_{\alpha_1+\alpha_2}+2x_{-\alpha_1-\alpha_2}\ x_{\alpha_2}+(h_{\alpha_2}-h_{2\alpha_1+\alpha_2})\ x_{-\alpha_1})\ x_{\alpha_2}^{-1}\equiv 2x_{\alpha_2}\ x_{-\alpha_1-\alpha_2}\ -2x_{-\alpha_1}\equiv 2(x_{-\alpha_1}\ -x_{-\alpha_1-\alpha_2}\ x_{\alpha_2})\ -2x_{-\alpha_1}\equiv x_{-\alpha_1-\alpha_2}\ x_{\alpha_2}\equiv x_{-\alpha_1-\alpha_2}\equiv 0\ (\mathrm{mod}\ \mathfrak{m})$, a contradiction since non-P-point always yields an irreducible $\mathfrak{g}$-module ofdimension $>p$. Note that $h_{\alpha_2}+x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2}+x_{\alpha_1}x_{-\alpha_1}$ commutes with x_{α_2}.

Similarly we have $x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2}+x_{\alpha_1}x_{-\alpha_1}\notin\mathfrak{m}$ as a byproduct.

Next, it is not difficult to show that $x_{-\alpha_1}$, x_{α_2}, $h_{2\alpha_1+\alpha_2}\ x_{\alpha_2}$, $x_{2\alpha_1+\alpha_2}$, $h_{2\alpha_1+\alpha_2}x_{2\alpha_1+\alpha_2}$, $x_{-2\alpha_1-\alpha_2}$, $x_{\alpha_1+\alpha_2}\notin\mathfrak{m}$ and these commute with x_{α_2}. Here we observe that the above elements of the basis candidate are F-linearly independent by P-B-W theorem.

Now we have to show that they are linearly independent modulo m.

Suppose that we have a dependence equation which is of least degree with respect to h_{α_2} and the number of whose highest degree terms is also least. If there is an exponent ≥ 2 in any place of the dependence equation, then conjugation by x_{α_2} yields a nontrivial dependence equation of lower degree than the given one, a contradiction. So we assume that we have a dependence equation whose terms contain only one exponent. Suppose further that the highest degree terms are of degree ≥ 2; it should then contain terms of the form, say $\{(a_1h_{\alpha_2}+b_1(c_1+h_{2\alpha_1+\alpha_2}))+X_1\}\{(a_2h_{\alpha_2}+b_2(c_2+h_{2\alpha_1+\alpha_2}))+X_2\}\times$ some factors $+\{(a_2h_{\alpha_2}+b_2(c_2+h_{2\alpha_1+\alpha_2}))+X_2\}\{(a_3h_{\alpha_2}+b_3(c_3+h_{2\alpha_1+\alpha_2}))+X_3\}\times$ some factors $+\{(a_3h_{\alpha_2}+b_3(c_3+h_{2\alpha_1+\alpha_2}))+X_3\}\{(a_4h_{\alpha_2}+b_4(c_4+h_{2\alpha_1+\alpha_2}))+X_4\}\times$ some factors $+\{(a_1h_{\alpha_2}+b_1(c_1+h_{2\alpha_1+\alpha_2}))+X_1\}\{(a_4h_{\alpha_2}+b_4(c_4+h_{2\alpha_1+\alpha_2}))+X_4\}\times$ some factors, i.e., some factor, say $(a_3h_{\alpha_2}+b_3(c_3+h_{2\alpha_1+\alpha_2}))+X_3$ arises as aformer factor as well as a latter factor of some terms; otherwise conjugation by x_{α_2} leads to a contradiction, where $X_i \in \{x_{\alpha_2}, h_{2\alpha_1+\alpha_2}x_{\alpha_2}, x_{2\alpha_1+\alpha_2}, h_{2\alpha_1+\alpha_2}x_{2\alpha_1+\alpha_2}, x_{-2\alpha_1-\alpha_2}, x_{\alpha_1+\alpha_2}, x_{-\alpha_1}, h_{\alpha_2}+x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2}+x_{\alpha_1}x_{-\alpha_1}\}$. But since $a_ih_{\alpha_2}+b_i(c_i+h_{2\alpha_1+\alpha_2}) \not\equiv c(a_jh_{\alpha_2}+b_j(c_j+h_{2\alpha_1+\alpha_2}))$ for any $c \in F$ and $i \neq j$, the supposed linearly dependent equation reduces to a nontrivial linearly dependent one of lower degree than the first one as in proposition (6.1) if it is conjugated by x_{α_2}, a contradiction. So it remains to show that

$$\begin{aligned}W_1 &= d_1x_{\alpha_2}+d_2h_{2\alpha_1+\alpha_2}x_{\alpha_2}+d_3x_{2\alpha_1+\alpha_2}+d_4h_{2\alpha_1+\alpha_2}x_{2\alpha_1+\alpha_2}+\\ &d_5x_{-2\alpha_1-\alpha_2}+d_6x_{\alpha_1+\alpha_2}+d_7x_{-\alpha_1}+\\ &d_8(h_{\alpha_2}+x_{-\alpha_1-\alpha_2}x_{\alpha_1+\alpha_2}+x_{\alpha_1}x_{-\alpha_1})\\ &\equiv 0 \,(\text{mod m})\end{aligned}$$

with $d_i \in F \Rightarrow d_i = 0\ \forall i$. We proceed in several steps :

(i) $d_1 = d_2 = 0$:

Otherwise $W_2 = (h_{\alpha_2}W_1 - W_1h_{\alpha_2}) - \{h_{\alpha_2}(h_{\alpha_2}W_1 - W_1h_{\alpha_2}) - (h_{\alpha_2}W_1 - W_1h_{\alpha_2})h_{\alpha_2}\} \equiv d_1x_{\alpha_2}+d_2h_{2\alpha_1+\alpha_2}x_{\alpha_2} \equiv 0\,(\text{mod m})$; $d_1 \neq 0$, $d_2 \neq 0$ yields $x_{\alpha_2}x_{2\alpha_1+\alpha_2} \equiv 0$ from $x_{2\alpha_1+\alpha_2}W_2 - W_2x_{2\alpha_1+\alpha_2}$, a contradiction. So $d_2 = 0$, and hence $d_1 = 0$.

(ii) $d_3 = d_4 = d_5 = d_6 = d_7 = 0$: Otherwise $W_3 := h_{2\alpha_1+\alpha_2}W_1 - W_1h_{2\alpha_1+\alpha_2}$ yields $2d_3x_{2\alpha_1+\alpha_2}+2d_4h_{2\alpha_1+\alpha_2}x_{2\alpha_1+\alpha_2}-2d_5x_{-2\alpha_1-\alpha_2}+d_6x_{\alpha_1+\alpha_2}+d_7x_{-\alpha_1} \equiv 0$, so $h_{2\alpha_1+\alpha_2}W_3-W_3h_{2\alpha_1+\alpha_2} \equiv 4d_3x_{2\alpha_1+\alpha_2}+4d_4h_{2\alpha_1+\alpha_2}x_{2\alpha_1+\alpha_2}+4d_5h_{-2\alpha_1-\alpha_2}+d_6x_{\alpha_1+\alpha_2}+d_7x_{-\alpha_1} \equiv 0$. Subtracting the last two equations, we have $W_4 := d_3x_{2\alpha_1+\alpha_2} + d_4h_{2\alpha_1+\alpha_2}x_{2\alpha_1+\alpha_2} + 3d_5x_{-2\alpha_1-\alpha_2} \equiv 0$. Suppose that $d_4 \neq 0$; $x_{2\alpha_1+\alpha_2}W_4 - W_4x_{2\alpha_1+\alpha_2}$ then yields $-2d_4x^2_{2\alpha_1+\alpha_2} + 3d_5h_{2\alpha_1+\alpha_2} \equiv 0$. If $char\, F = p \neq 3$, then $x_{2\alpha_1+\alpha_2}\ (-2d_4x^2_{2\alpha_1+\alpha_2} + 3d_5h_{2\alpha_1+\alpha_2}) - (-2d_4x^2_{2\alpha_1+\alpha_2} + 3d_5h_{2\alpha_1+\alpha_2})\ x_{2\alpha_1+\alpha_2} \equiv d_5x_{2\alpha_1+\alpha_2} \equiv 0$, so $d_5 = 0$ and so $d_4 = 0$ and $d_3 = 0$. If $char F = p = 3$, we have $d_4x^2_{2\alpha_1+\alpha_2} \equiv 0$. If $d_4 \neq 0$, then $x^2_{2\alpha_1+\alpha_2} \equiv 0$ yields

$$
\begin{aligned}
x_{-\alpha_1}x^2_{2\alpha_1+\alpha_2} &= (x_{\alpha_1+\alpha_2} + x_{2\alpha_1+\alpha_2}x_{-\alpha_1})x_{2\alpha_1+\alpha_2} \\
&= x_{\alpha_1+\alpha_2}x_{2\alpha_1+\alpha_2} + x_{2\alpha_1+\alpha_2}(x_{\alpha_1+\alpha_2} + x_{2\alpha_1+\alpha_2}x_{-\alpha_1}) \\
&\equiv x_{\alpha_1+\alpha_2}x_{2\alpha_1+\alpha_2} \equiv 0 \pmod{\mathfrak{m}} \equiv x^3_{\alpha_1+\alpha_2} \equiv 0
\end{aligned}
$$

since $x_{-\alpha_1}x^2_{2\alpha_1+\alpha_2} \equiv x_{2\alpha_1+\alpha_2}x_{\alpha_1+\alpha_2} \equiv 0$.

Now $x_{-\alpha_1-\alpha_2}x^3_{\alpha_1+\alpha_2} - x^3_{\alpha_1+\alpha_2}x_{-\alpha_1-\alpha_2} \equiv (h_{\alpha_2} + h_{2\alpha_1+\alpha_2} - 2)x^2_{\alpha_1+\alpha_2} \equiv 0$ by easy computation, which yields $x^2_{\alpha_1+\alpha_2} \equiv 0$ by conjugation by x_{α_2}. Hence $(h_{\alpha_2} + h_{2\alpha_1+\alpha_2} - 1)x_{\alpha_1+\alpha_2} \equiv 0$ is obtained, so $x_{\alpha_1+\alpha_2} \equiv 0$, a contradiction. So $d_4 = 0 = d_3$; applying $h_{2\alpha_1+\alpha_2}$ to W_3, we have $d_5 = d_7 = d_6 = 0$.

(iii) From the foregoing remark preceding (i), we finally have $d_8 = 0$.

Along the way we used the formula : $\forall\alpha \in \Phi,\ x^k_\alpha \in \mathfrak{m}(k \geq 1) \Rightarrow \{h_\alpha - (k-1)\}x^{k-1}_\alpha \in \mathfrak{m}$.

PROPOSITION (6.3). *Suppose that* $\xi_1 = \xi_2 = 0$, *but* $\xi_3 \neq 0$; *we have then a free* F*-module with rank* p^8 *in* $\mathcal{U}(\mathfrak{g})/\mathfrak{m}$, *i.e.,* $\dim \mathcal{U}(\mathfrak{g})/\mathfrak{m} = p^8$.

Proof. We claim that we have a basis :

$$\{a'_1 h_{\alpha_2} + b'_1(c'_1 + h_{2\alpha_1+\alpha_2}) + g_{\alpha_2}\}^{i_1} \otimes$$
$$\{a'_2 h_{\alpha_2} + b'_2(c'_2 + h_{2\alpha_1+\alpha_2}) + h_{2\alpha_1+\alpha_2} g_{\alpha_2}\}^{i_2} \otimes$$
$$\{a'_3 h_{\alpha_2} + b'_3(c'_3 + h_{2\alpha_1+\alpha_2}) + x_{2\alpha_1+\alpha_2}\}^{i_3} \otimes$$
$$\{a'_4 h_{\alpha_2} + b'_4(c'_4 + h_{2\alpha_1+\alpha_2}) + \omega_{\alpha_2}\}^{i_4} \otimes$$
$$\{a'_5 h_{\alpha_2} + b'_5(c'_5 + h_{2\alpha_1+\alpha_2}) + x_{\ alpha_1+\alpha_2} x_{-\alpha_1-\alpha_2} + x_{\alpha_1} x_{-\alpha_1}\}^{i_5} \otimes$$
$$\{a'_6 h_{\alpha_2} + b'_6(c'_6 + h_{2\alpha_1+\alpha_2}) + x_{-\ alpha_1-\alpha_2} x_{\alpha_1+\alpha_2} + x_{-\alpha_1} x_{\alpha_1}\}^{i_6} \otimes$$
$$\{a'_7 h_{\alpha_2} + b'_7(c'_7 + h_{2\alpha_1+\alpha_2}) + x_{-2\ alpha_1-\alpha_2}\}^{i_7} \otimes$$
$$\{a'_8 h_{\alpha_2} + b'_8(c'_8 + h_{2\alpha_1+\alpha_2}) + h_{2\alpha_1+\alpha_2} x_{-2\alpha_1-\alpha_2}\}^{i_8}$$

with $0 \leq i_j \leq p-1$, where c'_j is chosen in F so that $c'_j + h_{2\alpha_1+\alpha_2}$is invertible modulo $\mathfrak{m}$ and $(a'_i,\ b'_i)$ are chosen so that $(a'_i h_{\alpha_2} + b'_i(c'_i + h_{2\alpha_1+\alpha_2}))g_{\alpha_2} \not\equiv g_{\alpha_2}(a'_i h_{\alpha_2} + b'_i(c'_i + h_{2\alpha_1+\alpha_2}))\,(\text{mod } \mathfrak{m})$, and $a'_i h_{\alpha_2} + b'_i(c'_i + h_{2\alpha_1+\alpha_2}) \not\equiv c'(a'_j h_{\alpha_2} + b'_j(c'_j + h_{2\alpha_1+\alpha_2}))$ for any $c' \in F$ (which is possibleconsidering $\mathbb{P}^1(F)$). Furthermore we choose (a'_i, b'_i, c'_i) as in the proof of proposition (6.2). It is easy to show that g_{α_2} commutes with $h_{2\alpha_1+\alpha_2} g_{\alpha_2}$, $x_{2\alpha_1+\alpha_2}$, ω_{α_2}, $x_{\alpha_1+\alpha_2}\ x_{-\alpha_1-\alpha_2} + x_{\alpha_1}\ x_{-\alpha_1}$, $x_{-\alpha_1-\alpha_2}\ x_{\alpha_1+\alpha_2} + x_{-\alpha_1} x_{\alpha_1}$, $x_{-2\alpha_1-\alpha_2}$, and $h_{2\alpha_1+\alpha_2} x_{-2\alpha_1-\alpha_2}$ respectively. Here we should observe that the above elements of the basis candidate are F-linearly independent by P-B-W theorem.

Now we have to show that they are linearly independent modulo $\mathfrak{m}$. Suppose that we have a dependence equation which is of least degree with respect to h_{α_2} and the number of whose highest degree terms is also least. If there is an exponent ≥ 2 in any place of the dependence equation, then conjugation by g_{α_2} yields a nontrivial dependence equation of lower degree than the given one, a contradiction. So we assume that we have a dependence equation whose terms contain only one exponent. By virtue of [8], $\omega_{\alpha_2} \not\equiv 0\,(\text{mod } \mathfrak{m})$ and h_{α_2} commutes with it. So proceeding in the same spirit as in the proof of the preceding proposition (6.2), we should have a trivial dependence equation from scratch. Hence we have our assertion.

PROPOSITION (6.4). *Suppose that* $\xi_2 \neq 0$; *we have then a free* F*-module with rank* p^8 *in* $\mathcal{U}(\mathfrak{g})/\mathfrak{m}$, *i.e.,* $dim\,\mathcal{U}(\mathfrak{g})/\mathfrak{m} = p^8$.

Proof. Since there is a Lie algebra isomorphism induced from an isomorphism of ordered bases $\{\alpha_1 + \alpha_2, -\alpha_2\} \to \{\alpha_1, \alpha_2\}$, we have our assertion by virtue of proposition (6.2).

PROPOSITION (6.5). *Suppose that* $\xi_1 = \xi_2 = \xi_3 = 0$, *but one of* ξ_4, ξ_5, ξ_6 *is nonzero; we have then* $dim\,\mathcal{U}(\mathfrak{g})/\mathfrak{m} = p^8$.

Proof. Since there is a Lie algebra isomorphism induced from an isomorphism of ordered bases $\{-\alpha_1, 2\alpha_1 + \alpha_2\} \to \{\alpha_1, \alpha_2\} \to \{\alpha_1 + \alpha_2, -2\alpha_1 - \alpha_2\}$, we have our assertion by virtue of propositions (6.2), (6.3), (6.4).

PROPOSITION (6.6). *Suppose that* $\xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_5 = \xi_6 = 0$, *but* $\xi_7 \neq 0$; *we have then* $dim\,\mathcal{U}(\mathfrak{g})/\mathfrak{m} = p^8$.

Proof. We shall show that g_{α_2} is invertible in $\mathcal{U}(\mathfrak{g})/\mathfrak{m}$. Choosing new bases for S_{α_2}-irreducible modules in a composition series, we can make h_{α_2} diagonal in each irreducible block of $\varphi(\mathcal{U}(\mathfrak{g}))$ as in the form :

$(\mathcal{U}(\mathfrak{g})/\mathfrak{m} \cong \varphi(\mathcal{U}(\mathfrak{g})) \ni)$

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & & & & & & \\
0 & 1 & \cdots & 0 & & & * & & & * \\
\vdots & \vdots & \ddots & \vdots & & & & & & \\
0 & 0 & \cdots & p-1 & & & & & & \\
 & & & & 0 & 0 & \cdots & 0 & & \\
 & & & & 0 & 1 & \cdots & 0 & & * \\
 & 0 & & & \vdots & \vdots & \ddots & \vdots & & \\
 & & & & 0 & 0 & \cdots & p-1 & & \\
 & \cdots & & \cdots & & & \cdots & & & \ddots
\end{pmatrix}
\begin{array}{cc}
\uparrow & \uparrow \\
| & | \\
| & | \\
\downarrow & | \\
\uparrow & | \\
| & | \\
| & | \\
\downarrow & | \\
\vdots & \downarrow
\end{array},
$$

where short arrows denote S_{α_2}-irreducible parts and the long arrow denotes the irreducible $\mathfrak{g}$-module part. For, each irreducible block must have eigenvalues $0, 1, \cdots, p-1$ of h_{α_2} from the equation $x_{\alpha_1+\alpha_2}^{-1} h_{\alpha_2} x_{\alpha_1+\alpha_2} = h_{\alpha_2} + 1$. Hence g_{α_2} becomes invertible by virtue of [8]. But then proposition (6.3) ensures our assertion.

PROPOSITION (6.7). *Suppose that* $\xi_1 = \xi_2 = \cdots = \xi_7 = 0$, *but* $\xi_8 \neq 0$; *we have then* $dim\mathcal{U}(\mathfrak{g})/\mathfrak{m} = p^8$.

Proof. This easily comes from the isomorphism of ordered bases : $\{-\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2\} \to \{\alpha_1 + \alpha_2, -\alpha_2\}$.

PROPOSITION (6.8). *Suppose that* $\xi_1 = \xi_2 = \cdots = \xi_8 = 0$, *but either* $\xi_9 \neq 0$ *or* $\xi_{10} \neq 0$; *we have then* $dim\mathcal{U}(\mathfrak{g})/\mathfrak{m} = p^8$.

Proof. Also straightforward from isomorphisms of ordered bases :

$$\{-\alpha_1, -\alpha_2\} \to \{\alpha_1 + \alpha_2, -\alpha_2\}, \{\alpha_1, \alpha_2\} \to \{-\alpha_1, -\alpha_2\}.$$

7. Conclusion

Recalling now our definitions in §5 and combining main results in §6, we have our conclusion which boils down to the next.

THEOREM. *A point* $(\xi_1, \cdots, \xi_{10}, \xi_{11}) \in F^{11}$ *with* $\xi_i (1 \leq i \leq 10)$ *not all zero corresponds in one to one fashion up to isomorphism to a* p^4*-dimensional irreducible* S*-representation. In other words, there does not exist a subregular point for* $\mathfrak{g} = sp_4(F)$ *over an algebraically closed field* F *of characteristic* $p > 2$.

References

1. E.M. Friedlander and B.J. Parshall, *Modular representation theory of Lie algebras*, American J. of Math. 110 (1988), pp1055-1094.
2. J.E. Humphreys, *Introduction to Lie algebras and Representation theory*, Springer- Verlag, 1980.
3. N. Jacobson, *Lie algebras*, Interscience Publishers, 1979.
4. J.C. Jantzen, *Kohomologie von p-Lie algebren und nilpotente Elemente*, Abh. Math. Sem. Univ. Hamburg **Vol.56** (1989), pp191-219.
5. Y. Kim, K. So, G. Seo, D. Park, and S. Choi, *On subregular points for some cases of Lie algebra*, Honam Mathematical Journal **Vol.19, No.1** (1997), pp21-27.
6. ——, *Some remarks on four kinds of points associated to Lie algebras*, Honam Mathematical Journal **Vol.20, No.1** (1998), pp31-43.
7. A. Premet, *Support varieties of nonrestricted modules over Lie algebras of reductive groups*, Journal of the London Mathematical Society **Vol.55, Part 2** (1997), pp236-250.
8. I.R. Shafarevich amd A.N. Rudakov, *Irreducible representations of a simple three dimensional Lie algebra over a field of finite characteristic*, Math. Notes Acad. Sci. USSR 2 (1967), pp760-767.

9. H. Strade and R. Farnsteiner, *Modular Lie algebras*, Marcel Dekker, 1988.
10. H. Zassenhaus, *The representations of Lie algebras of prime characteristic*, Proceedings of Glasgow Math. Assoc. **No. 2** (1954), pp1-36.