Honam Mathematical J. 21(1999), No. 1,pp. 97–103

MULTIPLIERS OF $L^1(G,A) \cap L^p(G,A)$ TO $L^1(G,A)$

BIRSEN SAĞIR

ONDOKUZ MAYIS ÜNİVERSİTESİ,
Fen-Edebiyat Fakültesi, Matematik Bölümü, SAMSUN 55139, TURKEY.

1. Introduction

Let A be a commutative Banach algebra with identity of norm 1, X a Banach space and G be a locally compact Abelian group with Haar measure. In this paper we will characterize the following multipliers of module homomorphism forms under same appropriate conditions:

$$Hom_{L^1(G,A)}(L^1(G,A) \cap L^p(G,A),\ L^1(G,A)) = M(G,A)$$
$$Hom_{L^1(G,A)}(L^1(G,A) \cap C_0(G,A),\ L^1(G,A)) = M(G,A)$$

Throughout we let G be a locally compact Abelian group with Haar measure dt, A a commutative Banach algebra and X a Banach space. Based on Dinculeanu[1,2], Thomas[9] and Johnson[4], the spaces

$$L^1(G,X),\ M(G,X),\ L^p(G,X),\ 1 \leq p \leq \infty$$

are defined in the usual sense. Denote by $L^1(G,X)$ the space of all Bochner integrable X-valued functions defined on G. If A is a commutative semisimple Banach algebra with identity of norm 1 , then the space $L^1(G,A)$ is a commutative Banach algebra under convolution [3] and [4].

$$f * g(t) = \int_G f(ts^{-1})g(s)ds = \int_G f(s)g(ts^{-1})ds$$

for $f, g \in L^1(G,A)$ and norm

$$||f||_{1,A} = \int_G ||f(t)||_A dt$$

Received April 8, 1999.

for $f \in L^1(G, A)$.

Let $L^p(G, X)$ is the set of strong measurable functions $f : G \to X$ such that $\|f(t)\|_X^p$, $1 \leq p < \infty$, is integrable, that is, $\|f(t)\|_X^p \in L^1(G)$. The norm of $f \in L^p(G, X)$ is given by

$$\|f\|_{pX} = \left\{ \int_G \|f(t)\|_X^p dt \right\}^{1/p}, 1 \leq p < \infty$$

and $f = g$ if $f(t) = g(t)$ in X a.e. $t \in G$. It follows that $L^p(G, X)$ is a Banach space for $1 \leq p < \infty$. If $X = \mathbb{C}$, the space of complex numbers, then we write $L^p(G, X) = L^p(G)$.

Denote $C_0(G, X)$ the space of all X-valued continuous functions vanishing at infinity of G, and supply the norm as

$$\|f\|_{\infty X} = \sup_{t \in G} \|f(t)\|_X, \ f \in C_0(G, X).$$

Denote by $M(G, X)$ the space of X-valued regular Borel measures of bounded variation on G. $C_0(G, X)$ is a Banach space with the dual identified by $M(G, X^*)$ in usual form(see [2],[8]) where X^* is the Banach dual of X. If X is a Banach A-module, then $C_0(G, X)$ and $L^p(G, X)$,$(1 \leq p < \infty)$ are $L^1(G, X)$- module under the convolution product.

For Banach A-modules X and Y, we denote the multiplier space

$$Hom_A(X, Y) = \{T \in L(X, Y) \,|\, T(ax) = aT(x), a \in A, x \in X\}$$

where $L(X, Y)$ denotes the space of all continuous linear operators T on X to Y. If $X = Y = A$, then $Hom_A(A, A)$ is the usual multiplier space of A, and is denoted by $M(A)$.

In the case of scalar function space on G, the multipliers are identified with the translation invariant operators, but in the case of vector-valued function space on G, an invariant operator may not be a multiplier(see Lai and Chang [6, Thm. 6]). If A has an identity norm 1, Tewari, Dutta and Vaidya have been shown in [8] that

$$\text{(i) [8,Thm. 4]} \quad Hom_{L^1(G,A)}(L^1(G,A), L^1(G,A)) \cong M(G,A).$$

It is known that the space $C_0(G,A)$ is a $L^1(G,A)$-module and is characterized as the module multipliers of $C_0(G,A)$ by Lai[5].

$$\text{(ii) [5, Thm. 1]} \quad Hom_{L^1(G,A)}(C_0(G,A), C_0(G,A)) \cong M(G,A).$$

where A is a Banach algebra with identity of norm 1.

2. Multipliers of $L^1(G,A) \cap L^p(G,A)$ to $L^1(G,A), 1 < p < \infty$

If $1 < p < \infty$, then it is easily verified that the linear space $L^1(G,A) \cap L^p(G,A)$ is a Banach space with the norm

$$\|f\|_{1,pA} = \|f\|_{1,A} + \|f\|_{p,A} \text{ for } f \in L^1(G,A) \cap L^p(G,A).$$

We denote by $\tau_s f$ the s-translation of f on G, that is

$$\tau_s f(t) = f(ts^{-1}) \text{ for } t, s \in G.$$

First, we shall prove a lemma which will be again in theorem 1.

LEMMA 1. *Let G be a noncompact Abelian group.*

(i) *If $f \in L^p(G,A)$, $1 \leq p < \infty$, then*

$$\lim_{s \to +\infty} \|f + \tau_s f\|_{pA} = 2^{1/p} \|f\|_{pA}.$$

(ii) *If $f \in C_0(G,X)$, then*

$$\lim_{s \to +\infty} \|f + \tau_s f\|_{\infty A} = 2^{1/p} \|f\|_{\infty A}.$$

Proof. Suppose $g \in C_c(G,A)$ with compact support K. Since G is noncompact if $s \notin KK^{-1}$ then the support of g and $\tau_s g$ are disjoint. Consequently, on the one hand, as $s \notin KK^{-1}$, we have

$$\begin{aligned}\|g+\tau_s g\|_{pA} &= \left(\int_G \|g(t)+\tau_s g(t)\|_A^p dt\right)^{1/p} \\ &= \left(\int_K \|g(t)\|_A^p dt + \int_{K_s} \|\tau_s g(t)\|_A^p dt\right)^{1/p} \\ &= 2^{1/p}\|g\|_{pA}\end{aligned}$$

for $1 \le p < \infty$. Here denote K_s, the support of $\tau_s g$. While on the other hand,

$$\|g+\tau_s g\|_{\infty A} = \|g\|_{\infty A}.$$

Since the A-valued space $C_c(G,A)$ of continuous functions with compact support in G is dense $L^p(G,A)$, $1 \le p < \infty$. If $f \in L^p(G,A)$, and $\epsilon > 0$ choose $g \in C_c(G,A)$ such that $\|f-g\|_{pA} < \epsilon/4$. Let K be the support of g. Then, if $s \notin KK^{-1}$, we have

$$\begin{aligned}&\left|\|f+\tau_s f\|_{pA} - 2^{1/p}\|f\|_{pA}\right| \le \left|\|f+\tau_s f\|_{pA} - \|g+\tau_s g\|_{pA}\right| \\ &+ \left|\|g+\tau_s g\|_{pA} - 2^{1/p}\|g\|_{pA}\right| + \left|2^{1/p}\|g\|_{pA} - 2^{1/p}\|f\|_{pA}\right| \\ &\le \|f-g\|_{pA} + \|\tau_s f - \tau_s g\|_{pA} + 2^{1/p}\|f-g\|_{pA} \\ &< \epsilon/4 + \epsilon/4 + 2^{1/p}\epsilon/4 \le \epsilon.\end{aligned}$$

Therefore

$$\lim_{s\to\infty} \|f+\tau_s f\|_{pA} = 2^{1/p}\|f\|_{pA} \text{ for all } f \in L^p(G,A), 1 \le p < \infty.$$

The assertion for $f \in C_0(G,A)$ is deduced essentially in the same manner.

THEOREM 1. *Let G be a noncompact locally compact Abelian group and $1 < p < \infty$. Then the following statements are equivalent;*

(i) *$T \in Hom_{L^1(G,A)}(L^1(G,A) \cap L^p(G,A), L^1(G,A))$.*

(ii) *There exists a unique A-valued vector measure $\mu \in M(G,A)$ such that $Tf = f * \mu$ for each $f \in L^1(G,A) \cap L^p(G,A)$.*

Morever, the correspondence between T and μ defines an isometric algebra isomorphism of $Hom_{L^1(G,A)}(L^1(G,A)\cap L^p(G,A), L^1(G,A))$ onto $M(G,A)$.

Proof. $(ii) \Rightarrow (i)$: If $f \in L^1(G,A)$ and $\mu \in M(G,A)$, then it is known that the convolution

$$f * \mu(t) = \int_G f(ts^{-1})d\mu(s)$$

defines an element in $L^1(G,A)$[8]. Thus, for $f \in L^1(G,A) \cap L^p(G,A)$ and, $\mu \in M(G,A)$, the mapping

$$T : f \to f * \mu(\cdot) = \int_G \tau_s f(\cdot)d\mu(s)$$

defines a bounded linear map from $L^1(G,A)\cap L^p(G,A)$ to $L^1(G,A)$ and

$$\|Tf\|_{1A} = \|f * \mu\|_{1A} \leq \|\mu\|\|f\|_{1A} \leq \|\mu\|\|f\|_{1,pA}$$

implies

$$\|T\| \leq \|\mu\|.$$

Morever, for $f \in L^1(G,A) \cap L^p(G,A)$ and $g \in L^1(G,A)$

$$T(g * f) = (g * f) * \mu = g * Tf.$$

Hence $T \in Hom_{L^1(G,A)}(L^1(G,A) \cap L^p(G,A), L^1(G,A))$.

$(i) \Rightarrow (ii)$: Suppose that $T \in Hom_{L^1(G,A)}(L^1(G,A)\cap L^p(G,A), L^1(G,A))$. Then, for each $f \in L^1(G,A) \cap L^p(G,A)$, we have

$$\|Tf\|_{1A} \leq \|T\|(\|f\|_{1A} + \|Tf\|_{pA}).$$

Combining this estimate with Lemma 1 (a) we deduce that

$$\begin{aligned} 2\|Tf\|_{1A} &= \lim_{t\to+\infty} \|Tf + \tau_s Tf\|_{1A} = \lim_{s\to+\infty} \|T(f + \tau_s f)\|_{1A} \\ &\leq \lim_{s\to+\infty} \|T\|(\|f + \tau_s f\|_{1A} + \|f + \tau_s f\|_{pA}) \\ &= \|T\|(2\|f\|_{1A} + 2^{1/p}\|f\|_{pA}) \end{aligned}$$

for each $f \in L^1(G,A) \cap L^p(G,A)$. Thus

$$\|Tf\|_{1A} \leq \|T\|(\|f\|_{1A} + 2^{\frac{1}{p}-1}\|Tf\|_{pA}), (f \in L^1(G,A) \cap L^p(G,A)).$$

Repeating this process n times we see that

$$\|Tf\|_{1A} \leq \|T\|(\|f\|_{1A} + 2^{n(\frac{1}{p}-1)}\|Tf\|_{pA}),$$
$$(f \in L^1(G,A) \cap L^p(G,A)).$$

Since $p > 1$ we have $\lim_{n\to\infty} 2^{n(\frac{1}{p}-1)} = 0$, so we conclude that

$$\|Tf\|_{1A} \leq \|T\|\|f\|_{1A}, (f \in L^1(G,A) \cap L^p(G,A)).$$

Hence T defines a continuous linear transformation from $L^1(G,A)$ $\cap L^p(G,A)$ considered as a subspace of $L^1(G,A)$ to $L^1(G,A)$ which commutes with translations. Thus, since $L^1(G,A) \cap L^p(G,A)$ is norm dense $L^1(G,A)$, T determines a unique element $T^{/}$ of $Hom_{L^1(G,A)}(L^1(G,A) \cap L^p(G,A), L^1(G,A))$ and $\|T^{/}\| \leq \|T\|$. Since $Hom_{L^1(G,A)}(L^1(G,A) \cap L^p(G,A)) = M(G,A)$, there exists a unique $\mu \in M(G,A)$ such that $T^{/}f = f * \mu$ for each $f \in L^1(G,A)$ and $\|\mu\| = \|T^{/}\|$. Consequently, $Tf = f * \mu$ for each $f \in L^1(G,A) \cap L^p(G,A)$ and $\|\mu\| \leq \|T\|$. Therefore (i) and (ii) are equivalent

It is evident that the correspondence between T and μ defines an isometric algebra isomorphism from $Hom_{L^1(G,A)}(L^1(G,A) \cap L^p(G,A), L^1(G,A))$ onto $M(G,A)$.

Utilizing the second portion of Lemma 1 and Lai[5,Thm 1] we can prove by essentially the same arguments as those just given the analogous result for $Hom_{L^1(G,A)}(L^1(G,A) \cap C_0(G,A), L^1(G,A))$.

THEOREM 2. *Let G be a noncompact locally compact Abelian group. Then followings are equivalent*

(1) $T \in Hom_{L^1(G,A)}(L^1(G,A) \cap C_0(G,A), L^1(G,A))$.
(2) *There exists a unique measure $\mu \in M(G,A)$ such that $Tf = f * \mu$ for each $f \in L^1(G,A) \cap C_0(G,A)$.*

Morever the correspondence between T and μ defines an isometric algebra isomorphism of $Hom_{L^1(G,A)}(L^1(G,A) \cap C_0(G,A), L^1(G,A))$ onto $M(G,A)$.

If $A = \mathbb{C}$, the complex field, then we obtain the theorem 3.5.1 and the theorem 3.5.2 of [7].

References

1. N. Dinculeanu, *Integration on Locally Compact Spaces*, Noordhoff International Publishing, 1974.
2. N. Dinculeanu, *Vector Measures*, Pergamon, Oxford, 1967.
3. A. Hausner, *Group algebras of vector valued functions*, Bull. Amer. Math. Soc. **62** (1956), 383.
4. G. P. Johnson, *Spaces of functions with values in a Banach algbra*, Trans. Amer. Math. Soc. **92** (1959), 411–429.
5. H. C. Lai, *Multipliers of Banach-valued function spaces*, J. Austral Math. Soc. **39, series A** (1985), 51–62.
6. H. C. Lai and T. K. Chang, *Multipliers and translation invariant operators*, Tohuku Math. J. **41** (1989), 31–41.
7. R. Larsen, *An Introduction to the Theory of Multipliers*, Springer-Verlag, New York, Heidelberg, 1971.
8. U. B. Tewari, M. Dutta and D. P. Vaidya, *Multipliers of group algbra of vector-valued functions*, Proc. Amer. Math. Soc. **81** (1981), 223–229.
9. E. Thomas, *The Lebesgue-Nikodym Theorem for vector-valued Radon Measures*, Memoirs, Amer. Math. Soc., No. 139 (1974).