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SEVERAL PROPERTIES OF THE SUBCLASS OF G_k DESCRIBED BY SUBORDINATION

YOUNG OK PARK

Dept. of Mathematics,
Ewha Womans University, Seoul, 120-750, Korea.
E-mail: yok@math.snu.ac.kr.

Abstract In this paper we generalize the definition of strongly close-to-convex functions by using the functions $g(z)$ of bounded boundary rotation and investigate the distortion and rotation theorem, coefficient inequalities, invariance property and inclusion relation for the new class $G_k[A, B]$.

1. Introduction

Let C denote the class of normalized holomorphic functions $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which is close-to-convex and univalent in the open unit disk $E = \{z : |z| < 1\}$.

Kaplan[1] introduced first the concept of close-to-convex functions and H.Silverman[7] studied the generalization of Kaplan's class.

Also, Ch.pommerenke[5] considered the class R_β of strongly close-to-convex functions ; $f \in R_\beta$ if there exists a starlike function $h(z)$ such that

$$|\arg \frac{zf'(z)}{h(z)}| \leq \frac{\beta\pi}{2}, \quad (z \in E). \tag{1.2}$$

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For $k \geq 2$, we denote by V_k the set of functions f represented by (1.1) in E and satisfy $f'(z) \neq 0$ in E and

$$(1.3) \quad \int_0^{2\pi} |Re\{1 + \frac{zf''(z)}{f'(z)}\}| d\theta \leq k\pi \qquad (z = re^{i\theta}, 0 < r < 1).$$

This class of functions was introduced by Paatero in [3]. For $k = 2$, the class reduces to the class of normalized convex functions K in E and for $2 \leq k \leq 4$, V_k consists entirely of univalent functions. If $k = 4$,the class reduces to the class of close-to-convex functions C in E. Each class V_k with $k > 4$ contains non-univalent functions. The function

$$(1.4) \qquad g_k(z) = \frac{1}{k}[(\frac{1+z}{1-z})^{\frac{k}{2}} - 1] = z + \sum_{n=2}^{\infty} a_n(k) z^n$$

belongs to V_k and is extremal for a number of problems within the class.

And, for $2 \leq k \leq 4$ we define the class W_k as follows ; $f(z) \in V_k$ if and only if $zf'(z) \in W_k$.

Now we shall generalize the concept of strongly close-to-convex functions by replacing the denominater in (1.2) by a function $g(z)$ in the class V_k of bounded boundary rotation when $2 \leq k \leq 4$. We shall denote this class of functions by G_k. Moreover we shall define a new class $G_k[A, B]$ that is described by subordination.

DEFINITION 1. Let $f(z)$ be a holomorphic function in E with normalizations $f(0) = 0, f'(0) = 1$. $f(z)$ belongs to the class G_k if $f'(z) \neq 0$ in E and satisfies the condition

$$(1.5) \qquad |\arg \frac{f'(z)}{g'(z)}| \leq \frac{\pi}{2}, \quad (z \in E)$$

for some $g(z) \in V_k$ $\quad (2 \leq k \leq 4)$.

A holomorphic function $f(z)$ is said to be subordinate to a holomorphic function $h(z)$ (written $f \prec h$) if $f(z) = h[w(z)], z \in E$, for some regular function w with $w(0) = 0, |w(z)| < 1$ in E.

DEFINITION 2. Let $f(z)$ be a holomorphic function in E with $f(0) = 0, f'(0) = 1$. Let $G_k[A, B]$ denote the class of function $f(z)$ satisfying

$$\frac{f'(z)}{g'(z)} \prec \frac{1 + Az}{1 + Bz}, \quad (-1 \le B < A \le 1) \tag{1.6}$$

for some $g(z)$ in V_k, $(2 \le k \le 4)$.

2. Main Results for the Class $G_k[A, B]$

For A and B, $-1 \le B < A \le 1$, a function $p(z)$ which is analytic in E with $p(0) = 1$ is said to belong to the class $\mathcal{P}[A, B]$ if $p(z) \prec \frac{1+Az}{1+Bz}$.

THEOREM 2.1. *For* $f \in G_k[A, B]$, $|z| \le r < 1$,

$$\frac{1 - Ar}{1 - Br} \cdot \frac{(1 - r)^{\frac{1}{2}k-1}}{(1 + r)^{\frac{1}{2}k+1}} \le |f'(z)| \le \frac{1 + Ar}{1 + Br} \cdot \frac{(1 + r)^{\frac{1}{2}k-1}}{(1 - r)^{\frac{1}{2}k+1}}.$$

The bounds are sharp.

Proof. For $f \in G_k[A, B]$, there exists a $g \in V_k$ and $p \in \mathcal{P}[A, B]$ such that

$$f'(z) = g'(z)p(z). \tag{2.1}$$

Since $g \in V_k$ for $|z| \le r < 1$, in [4]

$$\frac{(1 - r)^{\frac{1}{2}k-1}}{(1 + r)^{\frac{1}{2}k+1}} \le |g'(z)| \le \frac{(1 + r)^{\frac{1}{2}k-1}}{(1 - r)^{\frac{1}{2}k+1}} \tag{2.2}$$

For $p \in \mathcal{P}[A, B], |z| \le r$, the univalence of $\frac{1+Az}{1+Bz}$ gives

$$\frac{1 - Ar}{1 - Br} \le |p(z)| \le \frac{1 + Ar}{1 + Br}. \tag{2.3}$$

The result follows immediately upon applying (2.2) and (2.3) to (2.1). Equality is obtained for $f \in G_k[A, B]$ satisfying

$$f'(z) = \frac{1 + Az}{1 + Bz} \cdot \frac{(1 + z)^{\frac{1}{2}k-1}}{(1 - z)^{\frac{1}{2}k+1}} \quad \text{and} \quad z = \pm r.$$

THEOREM 2.2. *For $f \in G_k[A, B]$, $|z| \le r < 1$,*

$$|\arg f'(z)| \le k \cdot \arcsin r + \arcsin \frac{(A-B)r}{1-ABr^2}.$$

Proof. For $f \in G_k[A, B]$, we have

$$|\arg f'(z)| \le |\arg g'(z)| + |\arg p(z)|, \tag{2.4}$$

where $g(z) \in V_k$ and $p(z) \in \mathcal{P}[A, B]$. Since $g(z) \in V_k$, we know [4] that for $|z| \le r < 1$,

$$|\arg g'(z)| \le k \cdot \arcsin r. \tag{2.5}$$

For $p \in \mathcal{P}[A, B]$, $p(|z| < r)$ is contained in the disk

$$|p(z) - \frac{1-ABr^2}{1-B^2r^2}| < \frac{(A-B)r}{1-B^2r^2}$$

from which it follows that

$$|\arg p(z)| \le \arcsin \frac{(A-B)r}{1-ABr^2}. \tag{2.6}$$

From (2.4), (2.5) and (2.6)

$$|\arg f'(z)| \le k \cdot \arcsin r + \arcsin \frac{(A-B)r}{1-ABr^2}.$$

LEMMA 1 [2]. *Let $f(z)$ be in V_k, $f(z) = z + b_2 z^2 + \cdots$. Then $|b_2| \le \frac{k}{2}$ and*

$$\begin{aligned} \max_{f \in V_k} |b_3 - b_2^2| &\le \frac{1}{12}[\frac{k^2-4}{2} + (k+2) + \frac{k^2-4}{4}] \\ &= \frac{1}{12}[\frac{3}{4}(k^2-4) + (k+2)], \quad if \quad 2 \le k \le 4. \end{aligned}$$

THEOREM 2.3. *For* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in G_k[A, B]$

$$|a_2| \leq \frac{k}{2} + \frac{A-B}{2}$$
$$|a_3| \leq \frac{1}{12} \cdot \frac{(5k-2)(3k+2)}{4} + \frac{(A-B)(k+1)}{3}, \quad \textit{if } 2 \leq k \leq 4.$$

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in G_k[A, B]$, there exists a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_k$ and a Schwarz function $w(z) = \sum_{n=1}^{\infty} \gamma_n z^n$ such that $\frac{f'(z)}{g'(z)} = \frac{1+Aw(z)}{1+Bw(z)}$, $z \in E$. Comparing series expansions, we see
(2.7)
$$a_2 = b_2 + \frac{A-B}{2}\gamma_1, a_3 = b_3 + \frac{2}{3}(A-B)b_2\gamma_1 + \frac{(A-B)}{3}(\gamma_2 - B\gamma_1^2).$$

The bounds for
$$\begin{aligned} |a_2| &= |b_2 + \frac{A-B}{2}\gamma_1| \\ &\leq |b_2| + |\frac{A-B}{2}||\gamma_1| \\ &\leq \frac{k}{2} + \frac{A-B}{2} \end{aligned}$$

follows from the above Lemma 1. The lemma to (2.7), we have

$$\begin{aligned} |a_3| &\leq |b_2|^2 + \frac{1}{12} \cdot \frac{(3k-2)(k+2)}{4} \\ &\quad + \frac{2(A-B)}{3}|b_2| + \frac{(A-B)}{3}\max(1, |B|) \\ &\leq \frac{k^2}{4} + \frac{1}{12} \cdot \frac{3k^2+4k-4}{4} + \frac{2(A-B)}{3} \cdot \frac{k}{2} + \frac{A-B}{3} \\ &= \frac{1}{12} \cdot \frac{(5k-2)(3k+2)}{4} + \frac{(A-B)(k+1)}{3}, \quad \text{if } 2 \leq k \leq 4. \end{aligned}$$

The *convolution* of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$.

LEMMA 2. *If $\phi \in K$ and $g \in V_k$, $2 \leq k \leq 4$, then for each function F, analytic in E with $F(0) = 1$, the image of E under $\frac{\phi * Fg'}{\phi * g'}$ is a subset of the convex hull of $F(E)$.*

Proof. If g is in V_k, $2 \leq k \leq 4$ then we have defined that $zg'(z)$ is in W_k. Due to Ruscheweyh and Sheil-Small[6] it is easy to see that

$$\frac{\phi * Fg'}{\phi * g'} = \frac{z(\phi * Fg')}{z(\phi * g')} = \frac{\phi * F(zg')}{\phi * (zg')}$$

is a subset of the convex hull of $F(E)$.

THEOREM 2.4. *If $f \in G_k[A, B]$, then so is $f * \varphi$ for any function $\varphi(z) = z + \cdots$, analytic and convex in E.*

Proof. We have $f \in G_k[A, B]$ if and only if, for $z \in E$, there is a function $g \in V_k$ such that

$$F = \frac{f'}{g'} \prec \frac{1 + Az}{1 + Bz} = H, \quad -1 \leq B < A \leq 1.$$

Since H is convex, an application of Lemma 2 yields

$$\frac{(\varphi * f)'}{(\varphi * g)'} = \frac{\varphi * Fg'}{\varphi * g'} \prec H,$$

so that $\varphi * f \in G_k[A, B]$.

COROLLARY 1. *For $f \in G_k[A, B]$, $2 \leq k \leq 4$, let*

$$F_1(z) = \int_0^z \frac{f(t)}{t} dt$$
$$F_2(z) = \frac{2}{z} \int_0^z f(t) dt$$

then $F_1(z), F_2(z)$ are also in $G_k[A, B]$.

Proof. If $F_i(z) = f * \phi_i, i = 1, 2,$ where $\phi_1(z) = -\log(1 - z)$ and $\phi_2(z) = \frac{-2[z + \log(1-z)]}{z}$ then trivially $\phi_i, i = 1, 2$ is convex and from *theorem*2.4 $F_i(z) \in G_k[A, B], i = 1, 2$.

COROLLARY 2. *If* $f \in G_k[A,B]$, $2 \le k \le 4$, *then so is*

$$\frac{1+\gamma}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt, \qquad Re\,\gamma > 0.$$

Proof. We may write

$$\frac{1+\gamma}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt = f * \sum_{n=1}^{\infty}\frac{1+\gamma}{n+\gamma}z^n.$$

Since $\sum_{n=1}^{\infty}\frac{1+\gamma}{n+\gamma}z^n$ was shown to be convex by Ruscheweyh[6] the result follows from Theorem 2.4.

COROLLARY 3. *If* $f \in G_k[A,B]$, $2 \le k \le 4$, *then so is*

$$\int_0^z \frac{f(\zeta)-f(x\zeta)}{\zeta - x\zeta}d\zeta, \quad |x| \le 1, x \neq 1.$$

Proof. We may write

$$\int_0^z \frac{f(\zeta)-f(x\zeta)}{\zeta - x\zeta}d\zeta = (f * h)(z),$$

where $h(z) = \sum_{n=1}^{\infty}\frac{1-x^n}{(1-x)n}z^n = \frac{1}{1-x}\log[\frac{1-xz}{1-z}], |x| \le 1, x \neq 1$. Since h is clearly convex, the result follows from theorem 2.4.

THEOREM 2.5.

$$G_k[C,D] \subset G_k[A,B]$$

if and only if

$$|AD - BC| \le (A-B) - (C-D),$$

$$\textit{where} \quad -1 < B < A \le 1, \quad -1 < D < C \le 1.$$

Proof. Since $|z| = 1$ is mapped by $\frac{(1+Az)}{(1+Bz)}$, $(-1 < B < A \le 1)$ onto a circle centered at $\frac{1-AB}{1-B^2}$ with radius $\frac{A-B}{1-B^2}$, we have

$$G_k[C,D] \subset G_k[A,B]$$

if and only if

$$\{w : |w - \frac{1-CD}{1-D^2}| < \frac{C-D}{1-D^2}\} \subset \{w : |w - \frac{1-AB}{1-B^2}| < \frac{A-B}{1-B^2}\},$$

which is equivalent to the inequalities

$$\frac{1-A}{1-B} \leq \frac{1-C}{1-D} \text{ and } \frac{1+C}{1+D} \leq \frac{1+A}{1+B}$$

i.e.

$$|AD - BC| \leq (A-B) - (C-D).$$

THEOREM 2.6.

$$f \in G_k[A,B]$$

if and only if for all z in E, $\phi(z) \in V_k, 2 \leq k \leq 4$ and all $\zeta, |\zeta| = 1$,

$$\frac{1}{z}[(f * \frac{(1+B\zeta)z}{(1-z)^2}) - (\phi * \frac{(1+A\zeta)z}{(1-z)^2})] \neq 0.$$

Proof. A function f is in $G_k[A,B]$ if and only if there is a $\phi \in V_k$ such that

$$\frac{f'(z)}{\phi'(z)} \neq \frac{1+A\zeta}{1+B\zeta} \quad \text{for } z \in E \text{ and } |\zeta| = 1,$$

which is equivalent to

$$\begin{aligned}
&(1+B\zeta)f' - (1+A\zeta)\phi' \\
&\quad = \frac{1}{z}[(1+B\zeta)zf' - (1+A\zeta)z\phi'] \\
&\quad = \frac{1}{z}[(1+B\zeta)(f * \frac{z}{(1-z)^2}) - (1+A\zeta)(\phi * \frac{z}{(1-z)^2})] \\
&\quad = \frac{1}{z}[(f * \frac{(1+B\zeta)z}{(1-z)^2}) - (\phi * \frac{(1+A\zeta)z}{(1-z)^2})] \\
&\quad \neq 0.
\end{aligned}$$

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