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ALGEBRAICITY OF PROPER HOLOMORPHIC MAPPINGS

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Abstract Suppose that Ω is a bounded domain with C^∞ smooth boundary in the plane whose associated Bergman kernel, exact Bergman kernel, or Szegő kernel function is an algebraic function. We shall prove that any proper holomorphic mapping of Ω onto the unit disc is algebraic.

1. Introduction

Suppose that Ω is a C^∞ smoothly bounded n-connected domain in the plane. Suppose that T is the complex unit tangent vector function on the boundary $b\Omega$ of Ω pointing in the direction of the standard orientation of the boundary. For instance, if $z(t)$ represents a boundary component of the boundary $b\Omega$, then $T(z(t))$ is equal to $\frac{z'(t)}{|z'(t)|}$. Let ds represent the differential of the arc length. $L^2(b\Omega)$ is the Hilbert space completion of $C^\infty(b\Omega)$ with respect to the inner product $< u, v >= \int_{b\Omega} u\bar{v}\, ds$ for $u, v \in C^\infty(b\Omega)$. Let $H^2(b\Omega)$ denote the classical Hardy space associated to Ω, i.e., the space of holomorphic functions on Ω with L^2 boundary values. Then the orthogonal projection denoted by P of $L^2(b\Omega)$ onto $H^2(b\Omega)$ is the Szegő projection.

The Szegő kernel function $S(z,\zeta)$ is the kernel for the Szegő projection P in the following sense: $(Ph)(\zeta) = \int_{b\Omega} S(\zeta, z)h(z)ds_z$

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for $h \in L^2(b\Omega)$. The Garabedian kernel $L(z,\zeta)$ is a kernel related to the Szegő kernel via

$$\overline{S(z,\zeta)} = -iL(z,\zeta)T(z), \quad z \in b\Omega,\ \zeta \in \Omega. \tag{1}$$

For $\zeta \in \Omega$, the function $L(z,\zeta)$ is a holomorphic function of z on $\Omega \setminus \{\zeta\}$ with a simple pole at ζ with residue $\frac{1}{2\pi}$. Let $B^2(\Omega)$ be the Bergman space, i.e., the space of holomorphic functions on Ω which are square integrable on Ω with respect to area measure dA. It is well known that the Bergman space $B^2(\Omega)$ is a closed subspace of the space $L^2(\Omega)$ of square integrable functions on Ω. Thus there is the orthogonal projection of $L^2(\Omega)$ onto $B^2(\Omega)$, called the Bergman projection. It is easy to see that the evaluation map $h \longmapsto h(\zeta)$ for a fixed ζ is a continuous linear functional on $B^2(\Omega)$. Hence the Riesz representation theorem implies that there is a unique holomorphic function ϕ_ζ in $B^2(\Omega)$ such that $h(\zeta) = < h, \phi_\zeta > = \int_\Omega h(z)\overline{\phi_\zeta(z)}dA$ for all $h \in B^2(\Omega)$. Now the function $B(\cdot,\cdot)$ on $\Omega \times \Omega$ defined by $B(\zeta,z) = \overline{\phi_\zeta(z)}$ for $\zeta, z \in \Omega$ is called the Bergman kernel function for Ω. Then the Bergman kernel function $B(\cdot,\cdot)$ has the reproducing property in the sense that $h(\zeta) = \int_\Omega h(z)B(\zeta,z)dA_z$.

Let $E^2(\Omega)$ denote the space of functions in the Bergman space $B^2(\Omega)$ which have single-valued indefinite integrals. Similary as the case of the Bermgan kernel, the exact Bergman kernel $E(z,\zeta)$ is the reproducing kernel for the space $E^2(\Omega)$, i.e., for any holomorphic function $h' \in E^2(\Omega)$, $h'(\zeta) = \int_\Omega E(\zeta,z)h'(z)dA_z$.

Now in this paper, we shall prove

THEOREM 1. *Suppose that Ω is a bounded domain with C^∞ smooth boundary in the plane whose associated Bergman kernel, exact Bergman kernel, or Szegő kernel is an algebraic function.*

Then any proper holomorphic mapping of Ω onto the unit disc must be algebraic.

A function $A(z)$ of N complex variables $z = (z_1, z_2, \cdots, z_N)$ is called algebraic if there is a nonnegative integer k such that there are polynomials $a_0(z), a_1(z), \cdots, a_k(z)$ in $z = (z_1, z_2, \cdots, z_N)$ with $a_k(z) \not\equiv 0$ which satisfy the algebraic equation

$$a_k(z)A(z)^k + a_{k-1}A(z)^{k-1} + \cdots + a_0(z) = 0.$$

Rational functions are obviously algebraic. A function $f(z, \zeta)$ of two complex variables z and ζ on a product domain $\Omega_1 \times \Omega_2$ is algebraic if and only if it is algebraic in z for each fixed $\zeta \in \Omega_2$ and algebraic in ζ for each fixed $z \in \Omega_1$ (see [3, pages 202-203]).

Let Ω be a bounded domain in the plane with C^∞ smooth boundary $b\Omega$. We shall say that the Szeg ő kernel function $S(z, \zeta)$ associated to a domain Ω is algebraic if it can be written as $A(z, \bar{\zeta})$ where A is a holomorphic algebraic function of two variables z and ζ. Since the Bergman kernel is hermitian, the facts above imply that $S(z, \zeta)$ is algebraic if and only if, for each fixed $\zeta_0 \in \Omega$, the function $S(z, \zeta_0)$ is a algebraic function of z.

Now assume that the Szegő kernel $S_\Omega(z, \zeta)$ associated to Ω is algebraic. Suppose f is a proper holomorphic mapping of Ω onto the unit disc D. Then there is a positive integer m such f is an m-to-one mapping of Ω onto D. And there are discrete sets $\widetilde{\Omega} \subset \Omega$ and $\widetilde{D} \subset D$ such that f is an m-to- one covering map of $\Omega - \widetilde{\Omega}$ onto $D - \widetilde{D}$. In fact, we can take $\widetilde{D} = f(\{z \in \Omega : f'(z) = 0\})$ and $\widetilde{\Omega} = f^{-1}(\widetilde{D})$ (see [8]). For every point $t_0 \in D - \widetilde{D}$, there are $\epsilon > 0$ and m holomorphic functions $F_1, F_2, \cdots, F_m$ on the ϵ-ball $D_\epsilon(t_0)$ about t_0 such that each F_μ maps $D_\epsilon(t_0)$ into $\Omega - \widetilde{\Omega}$ such that $f(F_\mu(t)) = t$, i.e., F_μ's are the local inverses to f.

It is known (see [7]) that the Szegő kernel function S_Ω and S_D associated to Ω and D, respectively, transform via

$$\text{(A)} \qquad f'(z)S_D(f(z), t)^2 = \sum_{\mu=1}^{m} S_\Omega(z, F_\mu(t))^2 \overline{F'_\mu(t)}.$$

Notice that the function on the right hand side of the above transformation formula is globally well defined on $D - \widetilde{D}$ because it is symmetric in the F_μ's. In fact, the function is holomorphic in ζ and antiholomorphic in ζ for $(z, \zeta) \in \Omega \times (D - \widetilde{D})$. It is easy to show that the Szegő kernel function associated to the unit disc D is

$$S_D(s,t) = \frac{1}{2\pi(1 - s\,\overline{t})}.$$

We may assume that $0 \notin \widetilde{D}$ so that the transformation formula makes sense for $t = 0$, by taking a Möbius transformation.

Inserting $t = 0$ into the transformation formula, we obtain

$$\text{(B)} \qquad \frac{1}{2\pi} f'(z) = \sum_{\mu=1}^{m} S_\Omega(z, F_\mu(0))^2\, \overline{F'_\mu(0)}.$$

Suppose that g and h are algebraic functions of one variable. Then there exist nonnegatives integers k and l such that there are polynomials $a_0(z), a_1(z), \cdots, a_k(z)\,; b_0(z), b_1(z), \cdots, b_l(z)$ in z such that

$$a_k(z)g(z)^k + a_{k-1}(z)g(z)^{k-1} + \cdots + a_0(z) = 0$$
$$b_l(z)h(z)^l + b_{l-1}(z)h(z)^{l-1} + \cdots + b_0(z) = 0$$

where $a_k(z), b_l(z) \not\equiv 0$.

Consider two algebraic equations in w:

$$\text{(C)} \qquad \mathcal{G}(z,w) \equiv a_k(z)w^k + a_{k-1}w^{k-1} + \cdots + a_0(z) = 0$$

$$\text{(D)} \qquad \mathcal{H}(z,w) \equiv b_l(z)w^k + b_{l-1}w^{l-1} + \cdots + b_o(z) = 0.$$

There are l solutions $h_1(z), h_2(z), \cdots, h_l(z)$ of (D) with $h_1(z) = h(z)$ such that $\mathcal{H}(z, h_\nu(z)) = 0$, $\nu = 1, 2, \cdots, l$. If for each $\nu = 1, 2, \cdots, l$ we substitute $(g + h_\nu) - h_\nu$ for w in (C) and expand it out, we get

$$\text{(E)} \quad c_k(z,h_\nu)(g+h_\nu)^k + c_{k-1}(z,h_\nu)(g+h_\nu)^{k-1} + \cdots + c_0(z,h_\nu) = 0$$

where $c_\mu(z,h_\nu)$'s are polynomials in z and h_ν, because $g = (g + h_\nu) - h_\nu$ satisfies $\mathcal{G}(z,g) = 0$. Let us consider the product

(F)

$$\mathcal{K}(z,w) = \prod_{\nu=1}^{l} \left[c_k(z,h_\nu)w^k + c_{k-1}(z,h_\nu)w^{k-1} + \cdots + c_0(z,h_\nu)\right].$$

Observe that if we expand the product (F) with respect to w, its coefficients of are symmetric functions in $h_1, h_2, \cdots, h_l$. It follows from the fundamental theorem on symmetric functions that the coefficients of $\mathcal{K}(z,w)$ are rational functions of elementary symmetric functions of $h_1, h_2, \cdots, h_l$. Now since two polynomial equations in w

$$\prod_{\nu=1}^{l}(w-h_\nu)=0 \quad \text{and}$$

$$\frac{\mathcal{H}(z,w)}{b_l(z)}=w^l+\frac{b_{l-1}(z)}{b_l(z)}w^{l-1}+\cdots+\frac{b_0(z)}{b_l(z)}=0$$

have the same l solutions $h_1, h_2, \cdots, h_l$ and they are monic polynomials in w, they are actually identically equal. However, the elementary symmetric functions of $h_1, h_2, \cdots, h_l$ are the coefficients of the product $\prod_{\nu=1}^{l}(w-h_\nu)$ and hence of $\dfrac{\mathcal{H}(z,w)}{b_l(z)}$, so they are rational functions in z. Thus the coefficients of $\mathcal{K}(z,w)$ are rational functions in z. Notice from (E) that the function $g+h$ is a solution of the equation $\mathcal{K}(z,w)$ for $w=g+h$, i.e., $\mathcal{K}(z,g(z)+h(z))=0$. Therefore, by multiplying the equation (F) by a common multiple of the denominators of the coefficients of $\mathcal{K}(z,w)$ and by using the fact that $g+h$ is a solution of (F), we can get polynomials $d_0(z), d_1(z), \cdots, d_{k+l}(z)$ is z such that

$$\begin{aligned}d_{k+l}(z)\left(g(z)+h(z)\right)^{k+l}+d_{k+l-1}(z)\left(g(z)+h(z)\right)^{k+l-1}\\ +\cdots+d_0(z)=0,\end{aligned}$$

which implies that the function $f(z)+g(z)$ is algebraic.

Hence it follows from (B) that since each $S_\Omega(z,F_\mu(0))$ is algebraic, f' is a algebraic function.

Now we want to differentiate the transformation formula (A) with respect to $\bar{t}$. It follows from

$$S_D(f(z),t)=\frac{1}{2\pi\left(1-f(z)\bar{t}\right)},$$

that

$$\frac{f(z)f'(z)}{2\pi^2\left(1-f(z)\bar{t}\right)^3}=\sum_{\mu=1}^{m}2S_\Omega(z,F_\mu(t))\frac{\partial}{\partial\bar{t}}S_\Omega(z,F_\mu(t))\overline{F'_\mu(t)}^2$$
$$+\sum_{\mu=1}^{m}S_\Omega(z,F_\mu(t))^2\overline{F''_\mu(t)}.$$

Letting $t=0$, we obtain

$$\frac{1}{2\pi^2}f(z)f'(z)=\sum_{\mu=1}^{m}2S_\Omega(z,F_\mu(0))\frac{\partial}{\partial\bar{t}}S_\Omega(z,F_\mu(0))\overline{F'_\mu(0)}^2$$
$$\text{(G)}\qquad +\sum_{\mu=1}^{m}S_\Omega(z,F_\mu(0))\overline{F''_\mu(0)}.$$

It can be shown that the derivative of an algebraic function is algebraic. In fact, suppose that g is algebraic with polynomials $a_0(z),a_1(z),\cdots,a_k(z)$ such that

$$a_k(z)g(z)^k+a_{k-1}(z)g(z)^{k-1}+\cdots+a_0(z)=0.$$

Differentiating the above equation with respect to z, we have

$$g'(z)=-\frac{a'_k(z)g(z)^k+a'_{k-1}(z)g(z)^{k-1}+\cdots+a'_0(z)}{ka_k(z)g(z)^{k-1}+(k-1)a_{k-1}(z)g(z)^{k-2}+\cdots+a_1(z)}$$

which is algebraic. Therefore, from (C), ff' is algebraic and since f' is algebraic, f is an algebraic function.

For the case of Bergman kernel, we can use the same method as the Szegő kernel to prove the above statement. I would also like to mention that the proof of the Bermgan kernel case is an easy corollary of [1](see also [2]).

On the other hand, it follows from (1) that the Garabedian kernel $L(z,\zeta)$ is algebraic if Szegő kernel is. I proved in [5] that the exact Bergman kernel $E(z,\zeta)$ is related to the Ahlfors map $f_\zeta = \dfrac{S(\cdot,\zeta)}{L(\cdot,\zeta)}$ via the identity

$$E(z,\zeta) = 2S(\zeta,\zeta)f_\zeta'(z) + \sum_{j,k=1}^{n-1} c_{jk}(\zeta)\int_{\gamma_k} S(\zeta,w)\frac{\partial}{\partial z}\frac{S(z,w)}{L(z,\zeta)}d\bar{w}$$

where $c_{jk}(\zeta), j,k = 1,\cdots,n-1$ are constants depending on ζ. Therefore we have completed the proof of Theorem 1.

REMARK 1. The version of Theorem 1 for the Bergman kernel with the same method as above can be actually proved even in the case of $\mathbb{C}^N$ when Ω and the unit disc are replaced by a bounded domain and a bounded circular domain, respectively.

REMARK 2. It is interesting to observe that the formula (B) implies that the derivative of any proper holomorphic map is related to the Szegő kernel. In fact, I (see [4, 5, 6]) found explicit formulas between the Bergman kernel and the derivative f_ζ' of the Ahlfors map f_ζ and between the exact Bergman kernel and the function f_ζ'.

For the simply connected case, we can easily classify the domains whose Bergman or Szegő kernels are algebraic functions. It follows from (A) and (H) that if f is a biholomorphic mapping between Ω and the unit disc, the Bergman kernel and the Szegő kernel transform under f via

$$B_\Omega(z,\zeta) = f'(z)B_D(f(z),f(\zeta))\overline{f'(\zeta)}$$
$$S_\Omega(z,\zeta) = \sqrt{f'(z)}S_D(f(z),f(\zeta))\overline{\sqrt{f'(\zeta)}}.$$

Since

$$B_D(f(z),f(\zeta)) = \frac{1}{\pi(1-f(z)\overline{f(\zeta)})^2} \quad \text{and}$$
$$S_D(f(z),f(\zeta)) = \frac{1}{2\pi(1-f(z)\overline{f(\zeta)})},$$

we have simple expressions

$$B_\Omega(z,\zeta) = \frac{f'(z)\overline{f'(\zeta)}}{\pi(1-f(z)\overline{f(\zeta)})^2}$$
$$S_\Omega(z,\zeta) = \frac{\sqrt{f'(z)}\overline{\sqrt{f'(\zeta)}}}{2\pi(1-f(z)\overline{f(\zeta)})}.$$

Hence if $f(a) = 0$ then we have $B_\Omega(z,a) = \left(\overline{f'(a)}\big/\pi\right) f'(z)$ and $S_\Omega(z,a) = \left(\overline{\sqrt{f'(a)}}\big/2\pi\right)\sqrt{f'(z)}$. If we differentiate the transformation formulas with respect to $\bar{\zeta}$ and let $\zeta = a$, we get

$$\frac{\partial B_\Omega}{\partial\bar{\zeta}}(z,a) = \frac{f'(z)}{\pi}\left(\overline{f''(a)} + 2\overline{f'(a)}^2 f(z)\right)$$
$$\frac{\partial S_\Omega}{\partial\bar{\zeta}}(z,a) = \frac{\sqrt{f'(z)}}{4\pi\overline{\sqrt{f'(a)}}}\left(\overline{f''(a)} + 2\overline{f'(a)}^{3/2} f(z)\right).$$

Since there is always the Riemann mapping function f_a associated to the pair (Ω, a) such that f_a is biholomorphic from Ω onto the unit disc and $f_a(a) = 0$, we have proved

THEOREM 2. *Suppose that Ω is a bounded simply connected domain with C^∞ smooth boundary. The Bergman kernel function or the Szegő kernel function associated to Ω is algebraic if and only if there exists an algebraic function which maps Ω onto the unit disc biholomorphically.*

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