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THE NORMALIZED HILBERT COEFFICIENTS IN C–M MODULES

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1. Introduction

Let R be a commutative Noetherian ring with identity, M an R-module of finite type, and $\mathfrak{a}$ an ideal of R such that the R-module $M/\mathfrak{a}M$ has finite length (or simply, $\mathfrak{L}(M/\mathfrak{a}M) < \infty$).

For $n \gg 0$, $\mathfrak{L}(M/\mathfrak{a}^n M)$ is given by $P_\mathfrak{a}(M, n)$, where $P_\mathfrak{a}(M, X)$ is a polynomial in X with a rational coefficients and is known as the *Hilbert -Samuel polynomial* of $\mathfrak{a}$ and M. The polynomial $P_\mathfrak{a}(M, X+1)$ may be written

$$e_0\binom{X+d}{d} - e_1\binom{X+d-1}{d-1} + \cdots + (-1)^{d-1}e_{d-1}\binom{X+1}{1} + (-1)^d e_d,$$

where d is its degree and the coefficients $e_i = e_i(\mathfrak{a}, M)$ are called the normalized Hilbert coefficients of $\mathfrak{a}$ and M.

Under the assumption that R is a Cohen-Macaulay (C-M for short) local ring, that M is the ring R itself, Northcott([5]), Narita([4]) and Malay([3]) developed lots of theory about the coefficients e_i. Fillmore([1]) gave the formula extracting the value of the coefficients e_i in a C-M module of dimension $d \geq 1$. The purpose of this paper is to find the formula on the value of the normalized Hilbert coefficients under the restricted condition of a regular sequence in a grade Module $\bar{M}$.

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2. The basic theorems on e_i

Throughout this paper all rings are commutative Noetherian ring with identity. All modules are unitary and of finite type(there fore Noetherian). The Krull dimension of a ring R(denoted by $dim(R)$) is the least upper bound on the length of chains of prime ideals in R.
If $M \neq (0)$ is an R-module, the dimension of M(denoted by $dim(M)$) is the dimension of the ring $R/Ann(M)$ where $Ann(M) = \{a \in R \,|\, aM = (0)\}$ is the annihilator of M. If $M = (0)$, we define $dim\,(M) = -\infty$.

Let $(R, \mathfrak{m})$ be a local ring and M an R-module, and $\mathfrak{q}$ a proper ideal of R. Then $\mathfrak{L}(M/\mathfrak{q}M) < \infty$ if and only if $\mathfrak{q} + Ann(M)$ is a defining ideal (i.e. an $\mathfrak{m}$-primary ideal) of R (in case $M \neq (0)$) or equals R (in case $M = (0)$) ([6], Proposition 7,p 194). Therefore there exists the Hilbert-Samuel polynomial of $\mathfrak{a}$ and M, $P_{\mathfrak{a}}(M, X)$([9],Theorem 3.3).
The well-known theorem of Krull-Chevalley-Samuel states that for an R-module $M \neq 0$ the following integers are equal([9],Theorem 3.7):

(i) $dim\,(M)$,
(ii) the degree of the polynomial $P_{\mathfrak{q}}(M, X)$ for any proper ideal $\mathfrak{q}$ such that $\mathfrak{L}(M/\mathfrak{q}M) < \infty$,
(iii) the least integer n such that there exist n elements $x_1, x_2, \ldots, x_n$ in $\mathfrak{m}$ for which $\mathfrak{L}(M/(x_1M + x_2M + \cdots + x_nM)) < \infty$.

Note that if the zero polynomial is assigned the degree $-\infty$, (i) and (ii) are equivalent when $M = (0)$.

DEFINITION 2.1([2]). If $(R, \mathfrak{m})$ is a local ring, an R-module M is called a C-M module if $M = (0)$ or if $M \neq (0)$ and $depth(M) = dim(M)$. In the general case M is a C-M module if $M_{\mathfrak{m}}$ (considered as an R_m module) is C-M for all $\mathfrak{m} \in Max(R)$. R is called a C-M ring if as an R-module R is C-M.

Fillmore ([1]) used the superficial elements to get the value of e_i. His construction is as follows : let M be an C-M R-module of dimension $d \geq 1$ and $\mathfrak{q}$ a proper ideal of R such that $\mathfrak{L}(M/\mathfrak{q}M) < \infty$. Then there is an M-regular sequence $(x_j)_{1 \leq j \leq d}$ of elements of

$\mathfrak{q}$ such that for $j = 1, 2, \ldots, d$, x_j is $(M/\sum_{i=1}^{j-1} x_i M)$-superficial of order 1 for $\mathfrak{q}$. Define integers

$$A_n = \mathfrak{L}(M/\sum_{i=1}^{d} x_i M) - \mathfrak{L}(M/\mathfrak{q}^{n+1}M + \sum_{i=1}^{d} x_i M),$$

$$B_{jn} = \mathfrak{L}(\mathfrak{q}^{n+1}(M/\sum_{i=1}^{j-1} x_i M) : Rx_j/\mathfrak{q}^n(M/\sum_{i=1}^{j-1} x_i M)).$$

Then these integers are finite and non-negative for $j = 0, 1, \ldots, d$ and $n = 0, 1, \ldots$. For $r \geq 1$, let

$$A_n{}^{(r)} = \sum_{k=0}^{n} \binom{n-k+r-1}{r-1} A_k.$$

Proposition 2.2 ([1]). *Under the same assumption and notations as above, we have*

$$e_0(\mathfrak{q}, M) = \mathfrak{L}(M/(x_1M + x_2M + \cdots + x_dM))$$

and

$$e_i(\mathfrak{q}, M) = \sum_{k=i-1}^{\infty} \binom{k}{i-1} A_k + \sum_{j=d-i+1}^{d} \sum_{k=i-1+j-d}^{\infty} (-1)^{d-j} \binom{k}{i-1+j-d} B_{jk}$$

for i=1,2, ... ,d.

In following Corollary, we got the relation between e_1 and e_2.

Corollary 2.3. *Let $(R, \mathfrak{m})$ be a local ring and let M be a C-M R-module of dimension $d \geq 2$. If $\mathfrak{a}$ is a proper ideal of R such that $\mathfrak{L}(M/\mathfrak{a}M) < \infty$, Then $e_2(\mathfrak{a}, M) \leq \frac{1}{2}e_1(\mathfrak{a}, M)(e_1(\mathfrak{a}, M) - 1)$.*

Proof. In virtue of [[1],Lemma 4.1], it suffices to prove the theorem in the case that $d = 2$. By Proposition 2.2, $e_1(\mathfrak{a}, M) = \sum_{k=0}^{\infty}(A_k + B_{2k})$ and

$$e_2(\mathfrak{a}, M) = \sum_{k=1}^{\infty} kA_k + \sum_{k=1}^{\infty} kB_{2k} - \sum_{k=0}^{\infty} B_{1k} \leq \sum_{k=1}^{\infty} k(A_k + B_{2k})$$

Let l be the least non-negative integer such that $A_n+B_{2n}=0$ for all $n>l$. If $A_m+B_{2m}=0$ for some $m\geq 0$, then $A_n+B_{2n}=0$ for all $n\geq m$. Thus $A_n+B_{2n}\geq 1$ whenever $0\leq n\leq l$ and so if $1\leq k\leq l+1$, then

$$e_1(\mathfrak{a},M)-k=\sum_{n=0}^{k-1}(A_n+B_{2n}-1)+\sum_{n=k}^{l}(A_n+B_{2n})\geq\sum_{n=k}^{l}(A_n+B_{2n})$$

Therefore

$$\begin{aligned}e_2(\mathfrak{a},M)=\sum_{n=1}^{l}n(A_n+B_{2n})&=\sum_{k=1}^{l}\sum_{n=k}^{l}(A_n+B_{2n})\\&\leq\sum_{k=1}^{l}(e_1(\mathfrak{a},M)-k)\end{aligned}$$

But $e_1(\mathfrak{a},M)=\sum_{n=0}^{l}(A_n+B_{2n})\geq l+1$. Therefore

$$e_2(\mathfrak{a},M)\leq\sum_{k=1}^{e_1(\mathfrak{a},M)-1}(e_1(\mathfrak{a},M)-k)=\frac{1}{2}e_1(\mathfrak{a},M)(e_1(\mathfrak{a},M)-1).$$

We shall say that $\mathfrak{a}$ is *M-parametric* if there exists a system of parameters for M which generate an ideal $\mathfrak{a}_1$ such that $\mathfrak{a}M=\mathfrak{a}_1M$.

COROLLARY 2.4. *Let $(R,\mathfrak{m})$ be a local ring, M a C-M R-module of dimension $d\geq 2$ and $\mathfrak{q}$ a proper ideal such that $\mathfrak{L}(M/\mathfrak{q}M)<\infty$. If $\mathfrak{a}$ is M-parametric and $\mathfrak{q}^2M=\mathfrak{q}\mathfrak{a}M$, then $e_i(\mathfrak{q},M)=0$ for each $2\leq i\leq d$.*

Proof. Let $x_1,x_2,\ldots,x_d$ be a system of parameters for M such that
$\mathfrak{a}M=(x_1,x_2,\ldots,x_d)M$. Since
$\mathfrak{q}^2(M/\sum_{i=1}^{j-1}x_iM)=\mathfrak{q}(x_1,\cdots,x_d)(M/\sum_{i=1}^{j-1}x_iM)$, [[1](Lemma 5.8)]
implies that $B_{jn}=0$ for all $n\geq 1$ and $1\leq j\leq d$.

Further $\mathfrak{q}^{n+1}M = \mathfrak{q}^n(Rx_1 + Rx_2 + \cdots + Rx_d)M \subseteq (Rx_1 + Rx_2 + \cdots + Rx_d)M$ for all $n \geq 1$. Therefore $A_n = 0$ for all $n \geq 1$. Therefore by Proposition 2.2, $e_i(\mathfrak{q}, M) = 0$ for all $2 \leq i \leq d$. This completes the proof.

3. Main Theorem

Let $(R, \mathfrak{m})$ be a local ring, M an R-module, and $\mathfrak{q}$ an ideal of R. Let us consider the direct sum

$$\bar{R} = \sum_{n=0}^{\infty} \mathfrak{q}^n/\mathfrak{q}^{n+1}, \quad \bar{M} = \sum_{n=0}^{\infty} \mathfrak{q}^n M/\mathfrak{q}^{n+1}M,$$

where $\bar{R}_n = \mathfrak{q}^n/\mathfrak{q}^{n+1}$ or $\bar{M}_n = \mathfrak{q}^n M/\mathfrak{q}^{n+1}M$ is considered as homogeneous elements of degree n. Then $\bar{R}$ is a grade ring and $\bar{M}$ is a graded $\bar{R}$-module under the usual operations([8], II,p,248). The leading form $\bar{M}$ of an element m of M is defined to be $m \, mod \, \mathfrak{q}^{n+1}M$ if $m \in \mathfrak{q}^n M - \mathfrak{q}^{n+1}M$ and 0 if $m = 0$. The leading submodule $\bar{N}$ of a submodule N of M is defined by

$$\bar{N} = \sum_{n=0}^{\infty} \bar{N}_n$$

where

$$\bar{N}_n = (N + \mathfrak{q}^{n+1}M) \cap \mathfrak{q}^n M)/\mathfrak{q}^{n+1}M = ((N \cap \mathfrak{q}^n M) + \mathfrak{q}^{n+1}M)/\mathfrak{q}^{n+1}M$$

The purpose of this section is to establish Theorem 3.1.

THEOREM 3.1. *Let $(R, \mathfrak{m})$ be a local ring, M a C-M R-module of dimension $r \geq 0$, and $\mathfrak{q}$ a proper ideal of R such that $\mathfrak{L}(M/\mathfrak{q}M) < \infty$ and $\mathfrak{q}$ M-parametric. Let $(x_j)_{1 \leq j \leq s}$, $s \leq r$, be a sequence of elements from $\mathfrak{m}$ such that the sequence of leading forms $(\bar{x}_j)_{1 \leq j \leq s}$ is $\bar{M}$-regular. Then*

(i) The sequence $(x_j)_{1 \leq j \leq s}$ is M-regular.

(ii) For $0 \leq i \leq r - s$,

$$e_i(\mathfrak{q}, M/(x_1M + x_2M + \cdots + x_sM)) = \mathfrak{L}(M/\mathfrak{q}M) \sum \binom{l_1}{\lambda_1 + 1} \cdots \binom{l_s}{\lambda_s + 1}$$

$$\textit{if } 0 \le i \le \sum_{j=1}^{s}(l_j - 1)$$

and

$$e_i(\mathfrak{q}, M/(x_1M + x_2M + \cdots + x_sM)) = 0 \ \textit{otherwise.}$$

where l_j is the degree of $\bar{x}_j$ in $\bar{M}$ and the sum is taken over $\lambda_1, \lambda_2, \ldots, \lambda_s$ for which $\lambda_1+\lambda_2+\cdots+\lambda_s = i$ and $0 \le \lambda_j \le l_j - 1$ for $j = 1, 2, \ldots, s$.

Proof. We note that the cases $r = 0$ and $r = s$ are trivial. So we can assume $r > 0$ and $s < r$. Suppose that l_i is the degree of $\bar{x}_i$ and $(\bar{x}_i)_{1\le i\le s}$ is a $\bar{M}$-regular sequence. Since x_1 is not a zero-divisor for $\bar{M}$, we have R-isomorphisms

$$(\bar{x}\bar{M})_n = \begin{cases} \bar{M}_{n-l_1} & , \ n \ge l_1 \\ (0) & , \ n < l_1. \end{cases}$$

Therefore we have

$$\mathfrak{L}(\bar{M}_n) - \mathfrak{L}(\bar{M}_n/(\bar{x}\bar{M})_n) = \begin{cases} \mathfrak{L}(\bar{M}_{n-l_1}) & , \ n \ge l_1 \\ 0 & , \ n < l_1. \end{cases}$$

If t denotes an indeterminant, then

$$\sum_{n=0}^{\infty} \mathfrak{L}(\bar{M}_n/(\bar{x}\bar{M})_n)t^n = (1 - t^{l_1}) \sum_{n=0}^{\infty} \mathfrak{L}(\bar{M}_n)t^n.$$

By induction we have

$$\sum_{n=0}^{\infty} \mathfrak{L}(\bar{M}_n/(\bar{x}_1\bar{M}+\cdots+\bar{x}_s\bar{M})_n)t^n = (1-t^{l_1})(1-t^{l_2})\cdots(1-t^{l_s}) \sum_{n=0}^{\infty} \mathfrak{L}(\bar{M}_n)t^n$$

Put $\mathfrak{a} = Rx_1 + Rx_2 + \cdots + Rx_s$. Since $(\bar{x}_i)_{1\le i\le s}$ is $\bar{M}$-regular, $\bar{x}_1\bar{M} + \bar{x}_2\bar{M} + \cdots + \bar{x}_s\bar{M} = \overline{\mathfrak{a}M}$. From the R-isomorphisms

$$\begin{aligned} \bar{M}_n/\overline{(\mathfrak{a}M)}_n &= (\mathfrak{q}^nM/((\mathfrak{a}M + \mathfrak{q}^{n+1}M) \cap \mathfrak{q}^nM) \\ &= (\mathfrak{a}M + \mathfrak{q}^nM)/(\mathfrak{a}M + \mathfrak{q}^{n+1}M). \end{aligned}$$

We obtain

$$\sum_{n=0}^{\infty}\mathfrak{L}(\mathfrak{a}M+\mathfrak{q}^nM/\mathfrak{a}M+\mathfrak{q}^{n+1}M)t^n = (1-t^{l_1})(1-t^{l_2})\cdots(1-t^{l_s})\sum_{n=0}^{\infty}\mathfrak{L}(\mathfrak{q}^nM/\mathfrak{q}^{n+1}M)t^n. \tag{3.1}$$

Put $M' = M/\mathfrak{a}M$. Then

$$\begin{aligned}\sum_{n=0}^{\infty}&\mathfrak{L}(\mathfrak{a}M+\mathfrak{q}^nM/\mathfrak{a}M+\mathfrak{q}^{n+1}M)t^n\\ &=\sum_{n=0}^{\infty}(\mathfrak{L}(M'/\mathfrak{q}^{n+1}M')-\mathfrak{L}(M'/\mathfrak{q}^nM'))t^n\\ &=(1-t)\sum_{n=0}^{\infty}\mathfrak{L}(M'/\mathfrak{q}^{n+1}M')t^n.\end{aligned}$$

Similarly, $\sum_{n=0}^{\infty}\mathfrak{L}(\mathfrak{q}^nM/\mathfrak{q}^{n+1}M)t^n = (1-t)\sum_{n=0}^{\infty}\mathfrak{L}(M/\mathfrak{q}^{n+1}M)t^n$. From (3.1),

$$\sum_{n=0}^{\infty}\mathfrak{L}(M'/\mathfrak{q}^{n+1}M')t^n = (1-t^{l_1})(1-t^{l_2})\cdots(1-t^{l_s})\sum_{n=0}^{\infty}\mathfrak{L}(M/\mathfrak{q}^{n+1}M)t^n.$$

Since M is a C-M module of dimension $r > 0$ and $\mathfrak{q}$ is M-parametric, by [[1] Theorem 2.14],

$$\sum_{n=0}^{\infty}\mathfrak{L}(M/\mathfrak{q}^{n+1}M)t^n = \mathfrak{L}(M/\mathfrak{q}M)\sum_{n=0}^{\infty}\binom{n+r}{r}t^n = \mathfrak{L}(M/\mathfrak{q}M)(1-t)^{-r-1}.$$

Hence

$$\sum_{n=0}^{\infty}\mathfrak{L}(M'/\mathfrak{q}^{n+1}M')t^n = \mathfrak{L}(M/\mathfrak{q}M)(1-t^{l_1})(1-t^{l_2})\cdots(1-t^{l_s})(1-t)^{-r-1}.$$

If $P_{\mathfrak{q}}(M', X+1) = e_0\binom{X+d}{d} - e_1\binom{X+d-1}{d-1} + \cdots + (-1)^d e_d$, then

$$\sum_{n=0}^{\infty}P_{\mathfrak{q}}(M', n+1)t^n = \frac{e_0}{(1-t)^{d+1}} - \frac{e_1}{(1-t)^d} + \cdots + (-1)^d\frac{e_d}{(1-t)}.$$

Therefore we have
(3.2)

$$\frac{e_0}{(1-t)^{d+1}} - \frac{e_1}{(1-t)^d} + \cdots + (-1)^{d-1}\frac{e_{d-1}}{(1-t)^2} + (-1)^d \frac{e_d}{(1-t)}.$$
$$= \mathfrak{L}(M/\mathfrak{q}M)(1-t^{l_1})(1-t^{l_2})\cdots(1-t^{l_s})(1-t)^{-r-1} + Q(t)$$

where $d = dim(M')$, $e_i = e_i(\mathfrak{q}, M')$, and $Q(t)$ is a polynomial in t with integral coefficients. Multiplying (3.2) by $(1-t)^{r+1}$ we find that $r - d \geq s$ since each $l_i \geq 1$ implies the right hand side is divisible by $(1-t)^s$. But $d \geq r - s$ holds in general, so we have $d = r - s$. Thus $(x_j)_{1 \leq j \leq s}$ is an M-regular sequence. Thus (i) is proved. If we divide out this factor $(1-t)^s$, we obtain
(3.3)

$$e_0 - (1-t)e_1 + \cdots + (-1)^{d-1}(1-t)^{d-1}e_{d-1} + (-1)^d(1-t)^d e_d$$
$$= \mathfrak{L}(M/\mathfrak{q}M)\frac{(1-t^{l_1})}{1-t}\frac{(1-t^{l_2})}{1-t}\cdots\frac{(1-t^{l_s})}{1-t} + (1-t)^{d+1}Q(t)$$

From

$$t^{l_j} = (1-(1-t))^{l_j} = \sum_{\lambda_j=0}^{l_j}\binom{l_j}{\lambda_j}(-1)^{\lambda_j}(1-t)^{\lambda_j}$$
$$= 1 - \sum_{\lambda_j=0}^{l_{j-1}}\binom{l_j}{\lambda_j+1}(-1)^{\lambda_j}(1-t)^{\lambda_j+1},$$

we have

$$\frac{1-t^{l_j}}{1-t} = \sum_{\lambda_j=0}^{l_j-1}\binom{l_j}{\lambda_j+1}(-1)^{\lambda_j}(1-t)^{\lambda_j}. \tag{3.4}$$

Substitute (3.4) for $j = 1, 2, \ldots, s$ into (3.3) and compare coefficients of the powers of $1-t$ to obtain the e_i. Then (ii) is proved.

A local ring $(R, \mathfrak{m})$ is called a *complete intersection* if it can be written as $A/(Ax_1 + Ax_2 + \cdots + Ax_s)$ where A is a regular local ring , the x_j are elements of the maximal ideal $\mathfrak{n}$ of A, and $s = dim(A) - dim(R)$. It follows that $(x_j)_{1 \leq j \leq s}$ is part of a system of parameters for A and that a complete intersection is a C-M local ring.

COROLLARY 3,2. *Let $(A, \mathfrak{n})$ be a regular local ring and $R = A/Ax_1+Ax_2+\cdots+Ax_s$ and $s = dim(A)-dim(R)$. If the sequence $(\bar{x}_j)_{1\le j\le d}$ of leading forms in the form ring $\bar{R}$ of A with respect to $\mathfrak{n}$ is $\bar{R}$-regular, then for $0 \le i \le dim(R)$,*

$$e_i(\mathfrak{m}, R) = \sum \binom{l_1}{\lambda_1+1} \cdots \binom{l_s}{\lambda_s+1} \qquad if\ 0 \le i \le \sum_{j=1}^{s}(l_j-1)$$

and

$$e_i(\mathfrak{m}, R) = 0 \qquad otherwise$$

where $\mathfrak{m}$ is the maximal ideal of R, l_j is the degree of $\bar{x}_j$ in $\bar{R}$, and the sum is as described in the theorem 3.1.

Proof. From isomorphisms $A/\mathfrak{n}^n \cong (A/Ax_1+Ax_2+\cdots+Ax_s)/((\mathfrak{n}^n+Ax_1+Ax_2+\cdots+Ax_s)/(Ax_1+Ax_2+\cdots+Ax_s)) \cong R/\mathfrak{m}^n$, $e_i(\mathfrak{m}, R) = e_i(\mathfrak{n}, A)$ for $0 \le i \le dim(R)$. Furthermore, $\mathfrak{L}(A/\mathfrak{n}) = 1$. Therefore our assertion follows from Theorem 3.1.

EXAMPLE 3.3. Let $A = K[X_1.X_2, X_3]_{(X_1,X_2,X_3)}$ where K is a field. Then A is a regular local ring of dimension 3 with the maximal ideal $\mathfrak{n} = AX_1 + AX_2 + AX_3$. Since $\bar{R} = \sum_{n=0}^{\infty} \mathfrak{n}^n/\mathfrak{n}^{n+1} \cong (A/\mathfrak{n})[X, Y, Z]$, the sequence $\{X, Y, Z\}$ is a $\bar{R}$-regular. Therefore by Corollary 3.2, for $0 \le j \le 3$, and for $0 \le i \le 3-j$

$$e_0(\mathfrak{m}_j, R_j) = 1$$
$$e_i(\mathfrak{m}_j, R_j) = 0, \text{ otherwise}$$

where $R_j = A/AX_1 + \cdots + AX_j$ and m_j is the maximal ideal of R_j.

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