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DERIVED LIMITS AND GROUPS OF PURE EXTENSIONS

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Abstract For a k-connected inverse system $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ of pointed topological spaces and pointed preserving weak fibrations, inducing epimorphic chain maps, over a directed set, we show that the homotopy group $\pi_k(\lim \mathfrak{X}, *)$ of the inverse limit is isomorphic to the integral homology group $H_k(\lim \mathfrak{X}; \mathbb{Z})$. Using the result of S. Mardešić, we prove that the group of pure extension $\mathrm{Pext}(\mathrm{colim} H^n(\mathfrak{X}), A)$ is isomorphic to the group of extension $\mathrm{Ext}(\Delta(\lambda), \mathrm{Hom}(H^n(\mathfrak{X}), A))$.

1. Introduction

It appears, in algebraic topology, very often that a certain cohomology expression can be descrived as a derived functor $\lim^n(-)$, $n \geq 0$ defined by J. E. Roos and G. Nöbeling independently and simultaneously. The first derived limit is an important algebraic tool in the computation of phantom maps. C. A. McGibbon [9] wrote a good book on the derived limits and phantom maps. C. A. McGibbon and R. Steiner [10] introduced some questions about the first derived limits of the inverse limits and phantom maps. Strong homology groups were defined by J. T. Lisica and S. Mardešić [3] in 1985. S. Mardešić [4,5] have proved that the strong homology group does not have compact supports and that there

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exists a paracompact space whose n-th derived limit is not trivial. Recently S. Mardešić and A. V. Prasolov [7] have constructed a structure theorem which shows a lot of information about the derived limits and strong homology groups of some inverse systems. Using the Növeling-Roos cohomology (derived limit in this paper), T. Watanabe [14] gave an elementary and concrete proof of the properties of derived functors on two categories.

Let $\mathfrak{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system of topological spaces X_λ and continuous maps $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda, \lambda \leq \lambda'$ over a directed set Λ and let $\mathbb{Z}$ be the set of all integers. In this paper, we show that if $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ is a k-connected inverse system of pointed topological spaces and pointed preserving weak fibrations, inducing epimorphic chain maps, over the directed set, then the homotopy group $\pi_k(\lim \mathfrak{X}, *)$ of the inverse limit is isomorphic to the integral homology group $H_k(\lim \mathfrak{X}; \mathbb{Z})$ (Theorem 2.5). Using the S. Mardešić's results about an extension functor and a derived functor, we give more detailed and concrete proof than his one. We also show that if the direct system $H^*(\mathfrak{X}) = (H^*(X_\lambda; \mathbb{Z}), p^*_{\lambda\lambda'}, \Lambda)$ (induced by $\mathfrak{X}$) consists of finitely generated cohomology groups, then the group of pure extension $\mathrm{Pext}(\mathrm{colim} H^n(\mathfrak{X}), A)$ of A by the colimit of $H^n(\mathfrak{X})$ is isomorphic to the group of extension $\mathrm{Ext}(\Delta(\lambda), \mathrm{Hom}(H^n(\mathfrak{X}), A))$ (Theorem 3.5), where $\Delta(\lambda) = (\Delta_\lambda, id_{\lambda\lambda'}, \Lambda)$ is an inverse system defined by $\Delta_\lambda = \mathbb{Z}$ and $id_{\lambda\lambda'}$ is an identity map on $\mathbb{Z}$.

2. Applications of the derived limit and the Hurewicz homomorphism

Let $\mathfrak{A} = (A_\lambda, a_{\lambda\lambda'}, \Lambda)$ be an inverse system of abelian groups A_λ and group homomorphisms $a_{\lambda\lambda'} : A_{\lambda'} \to A_\lambda$, $\lambda \leq \lambda'$ over the directed set Λ. Let $\Lambda^n, n \geq 0$ be the set of all increasing sequences $\bar{\lambda} = (\lambda_0, \lambda_1, \cdots, \lambda_n)$, $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n, \lambda_i \in \Lambda$. The sequence $\bar{\lambda}_j = (\lambda_0, \lambda_1, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_n) \in \Lambda^{n-1}$ is obtained from $\bar{\lambda}$ by deleting the j-th factor λ_j, $0 \leq j \leq n$.

We define n-cochain groups $C^n(\mathfrak{A})$ of $\mathfrak{A}$ by

$$C^n(\mathfrak{A}) = \prod_{\bar{\lambda} \in \Lambda^n} A_{\bar{\lambda}}, \; n \geq 0,$$

where $A_{\bar\lambda} = A_{\lambda_0}$.

Let $pr_{\bar\lambda} : C^n(\mathfrak{A}) \to A_{\bar\lambda}$ be a projection. If x is an element of $C^n(\mathfrak{A})$, then we denote the element $x_{\bar\lambda}$ of $A_{\bar\lambda}$ by

$$x_{\bar\lambda} = pr_{\bar\lambda}(x).$$

The coboundary operators $\delta^n : C^{n-1}(\mathfrak{A}) \to C^n(\mathfrak{A}), n \geq 1$ are defined by

$$(\delta^n x)_{\bar\lambda} = a_{\lambda_0\lambda_1}(x_{\bar\lambda_0}) + \sum_{j=1}^{n}(-1)^j x_{\bar\lambda_j},$$

where $x \in C^{n-1}(\mathfrak{A})$. For $n = 0$, we put $\delta^0 = 0 : 0 \to C^0(\mathfrak{A})$. Then we have a cochain complex

$$(C^*(\mathfrak{A}), \delta) : 0 \xrightarrow{\delta^0} C^0(\mathfrak{A}) \xrightarrow{\delta^1} C^1(\mathfrak{A}) \to \cdots$$
$$\to C^{n-1}(\mathfrak{A}) \xrightarrow{\delta^n} C^n(\mathfrak{A}) \to \cdots .$$

The n-th *derived limit* [11] $\lim^n \mathfrak{A}$ of $\mathfrak{A}$ is defined by

$$\lim{}^n \mathfrak{A} = \ker(\delta^{n+1})/\mathrm{im}(\delta^n).$$

We can see that $\lim^0 \mathfrak{A}$ is equal to the inverse limit $\lim \mathfrak{A}$ of the inverse system $\mathfrak{A}$.

Let $\mathfrak{D} = (D_\lambda, d_{\lambda\lambda'}, \Lambda)$ and $\mathfrak{E} = (E_\gamma, e_{\gamma\gamma'}, \Gamma)$ be inverse systems in any category $\mathfrak{C}$. We say that $s = \{\varphi, s_\gamma : \gamma \in \Gamma\} : \mathfrak{D} \to \mathfrak{E}$ is a *rigid system map* from $\mathfrak{D}$ to $\mathfrak{E}$ if $\varphi : \Gamma \to \Lambda$ is an increasing function, $s_\gamma : D_{\varphi(\gamma)} \to E_\gamma, \gamma \in \Gamma$ is a morphism in the category $\mathfrak{C}$ and for any $\gamma \leq \gamma'$ in Γ the following diagram

$$\begin{array}{ccc} D_{\varphi(\gamma)} & \xleftarrow{d_{\varphi(\gamma)\varphi(\gamma')}} & D_{\varphi(\gamma')} \\ {\scriptstyle s_\gamma}\downarrow & & \downarrow{\scriptstyle s_{\gamma'}} \\ E_\gamma & \xleftarrow{e_{\gamma\gamma'}} & E_{\gamma'} \end{array}$$

is commutative. We can make a category inv-$\mathfrak{C}$ of inverse systems in $\mathfrak{C}$ and rigid system maps. The rigid system map is called a *level system map* provided $\Gamma = \Lambda$ and φ is an identity map on Λ. It is easy to see that the category $\mathfrak{C}^\Lambda$ of the inverse systems and the level system maps is not full subcategory but subcategory of inv-$\mathfrak{C}$.

For a given pointed inverse system $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$, we obtain the following inverse systems

(1) $\pi_k(\mathfrak{X}, *) = (\pi_k(X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$;
(2) $H_k(\mathfrak{X}; \mathbb{Z}) = (H_k(X_\lambda; \mathbb{Z}), p_{\lambda\lambda'}, \Lambda)$

induced by $(\mathfrak{X}, *)$.

The well known Hurewicz homomorphism $h_\lambda : \pi_k(X_\lambda, *) \to H_k(X_\lambda; \mathbb{Z}), \lambda \in \Lambda$ induces a morphism (level system map) $h : \pi_k(\mathfrak{X}, *) \to H_k(\mathfrak{X}; \mathbb{Z})$ in the category Gr^Λ of inverse systems of groups and level system maps over Λ.

DEFINITION 2.1. A level system map $h : \pi_k(\mathfrak{X}, *) \to H_k(\mathfrak{X}; \mathbb{Z})$ in Gr^Λ is called the *Hurewicz level system map* of $(\mathfrak{X}, *)$.

A pointed inverse system $(\mathfrak{X}, *)$ is called *k-connected* if the induced inverse system $\pi_n(\mathfrak{X}, *)$ is trivial for $0 \leq n \leq k$.

PROPOSITION 2.2. *(Hurewicz isomorphism theorem) Let $(\mathfrak{X}, *)$ be a pointed k-connected inverse system. If $k \geq 1$, then we have the following facts:*

(1) $H_n(\mathfrak{X}; \mathbb{Z}) = 0, \ 1 \leq n < k+1$
(2) $h : \pi_{k+1}(\mathfrak{X}, *) \to H_{k+1}(\mathfrak{X}; \mathbb{Z})$ *is an isomorphism of inverse systems induced by* $(\mathfrak{X}, *)$.

Proof. See Theorem 2, section 4.1 of the second chapter in [8].

A pointed preserving map $t : (X, *) \to (Y, *)$ is called a *pointed preserving weak fibration* provided t has the homotopy lifting property with respect to the collection of cubes $\{I_n\}_{n \geq 0}$.

PROPOSITION 2.3. *Let $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ be an inverse system of pointed topological spaces and pointed preserving weak fibrations, Then the sequence*

$$0 \to \lim{}^1 \pi_{k+1}(\mathfrak{X}, *) \to \pi_k(\lim \mathfrak{X}, *) \to \lim \pi_k(\mathfrak{X}, *) \to 0$$

is exact for any $k \geq 0$.

Proof. See Theorem 1, section 7.1 of the second chapter in [8].

LEMMA 2.4. *Let* $\mathfrak{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be an inverse system of topological spaces and continuous maps inducing epimorphic chain maps* $p_{\lambda\lambda'\sharp} : C_\sharp(X_{\lambda'}; \mathbb{Z}) \to C_\sharp(X_\lambda; \mathbb{Z})$, $\lambda \leq \lambda'$. *Then the sequence*

$$0 \to \lim^1 H_{k+1}(\mathfrak{X}; \mathbb{Z}) \to H_k(\lim \mathfrak{X}; \mathbb{Z}) \to \lim H_k(\mathfrak{X}; \mathbb{Z}) \to 0$$

is also exact.

Proof. See Lemma 1 of [6] and Theorem 2 of [7].

THEOREM 2.5. *Let* $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ *be a* k*-connected inverse system of pointed topological spaces. If the bonding morphisms are weak fibrations inducing epimorphic chain maps* $p_{\lambda\lambda'\sharp} : C_\sharp(X_{\lambda'}; \mathbb{Z}) \to C_\sharp(X_\lambda; \mathbb{Z})$, $\lambda \leq \lambda'$, *then*

$$\pi_k(\lim \mathfrak{X}, *) \cong H_k(\lim \mathfrak{X}; \mathbb{Z}).$$

Proof. Considering an exact sequence of derived limits of homotopy groups and the Hurewicz isomorphism theorem, by Proposition 2.3 and Lemma 2.4 we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \lim^1 \pi_{k+1}(\mathfrak{X}, *) & \longrightarrow & \pi_k(\lim \mathfrak{X}, *) & \longrightarrow & \lim \pi_k(\mathfrak{X}, *) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \lim^1 H_{k+1}(\mathfrak{X}; \mathbb{Z}) & \longrightarrow & H_k(\lim \mathfrak{X}; \mathbb{Z}) & \longrightarrow & \lim H_k(\mathfrak{X}; \mathbb{Z}) & \longrightarrow & 0. \end{array}$$

Since the inverse system $(\mathfrak{X}, *)$ is k-connected, by Proposition 2.2 we have

$$\pi_{k+1}(\mathfrak{X}, *) \cong H_{k+1}(\mathfrak{X}; \mathbb{Z})$$

and

$$H_n(\mathfrak{X}; \mathbb{Z}) = 0$$

for $1 \leq n \leq k$. Therefore we have

$$\begin{aligned} \pi_k(\lim \mathfrak{X}, *) &\cong \lim^1 \pi_{k+1}(\mathfrak{X}, *) \quad (\pi_k(\mathfrak{X}, *) \text{ is trivial}) \\ &\cong \lim^1 H_{k+1}(\mathfrak{X}; \mathbb{Z}) \quad (\text{Hurewicz isomorphism theorem}) \\ &\cong H_k(\lim \mathfrak{X}; \mathbb{Z}). \end{aligned}$$

An inverse system $\mathfrak{A} = (A_\lambda, a_{\lambda\lambda'}, \Lambda)$ has the *Mittag-Leffler property* if every $\lambda \in \Lambda$ admits a $\lambda' \in \Lambda$, $\lambda' \geq \lambda$ such that

$$a_{\lambda\lambda'}(A_{\lambda'}) = a_{\lambda\lambda''}(A_{\lambda''})$$

for any $\lambda'' \geq \lambda'$.

PROPOSITION 2.6. *If the inverse system $\mathfrak{A} = (A_\lambda, a_{\lambda\lambda}, \Lambda)$ has the Mittag-Leffler property, then*

$$\lim{}^1 \mathfrak{A} = 0.$$

Proof. See Theorem 10, section 6.2 of the second chapter in [8].

COROLLARY 2.7. *Let $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ be a k-connect ed inverse system of pointed topological spaces. If the bonding morphisms are weak fibrations inducing epimorphic chain maps $p_{\lambda\lambda'\sharp} : C_\sharp(X_{\lambda'}; \mathbb{Z}) \to C_\sharp(X_\lambda; \mathbb{Z}), \lambda \leq \lambda'$ and if $\pi_{k+1}(\mathfrak{X}, *)$ has the Mittag-Leffler property, then*

$$H_k(\lim \mathfrak{X}; \mathbb{Z}) = 0.$$

Proof. By Proposition 2.3, Theorem 2.5 and Proposition 2.6, we have

$$\begin{aligned} H_k(\lim \mathfrak{X}; \mathbb{Z}) &\cong \pi_k(\lim \mathfrak{X}, *) \\ &\cong \lim{}^1 \pi_{k+1}(\mathfrak{X}, *) \\ &= 0. \end{aligned}$$

Let HPol and HPol$_*$ be homotopy category and pointed homotopy category of polyhedra respectively. And let $p : X \to \mathfrak{X}$ be an HPol-expansion. The *Čech homology group* $\check{H}_k(X; A)$ of X with coefficients in an abelian group A is defined by

$$\check{H}_k(X; A) = \lim[H_k(\mathfrak{X}; A)].$$

where [ ] means the equivalence class of inverse systems.

Let $p : (X, *) \to (\mathfrak{X}, *)$ be an HPol$_*$-expansion. The k-th *shape group* $\check{\pi}_k(X, *)$ is defined by

$$\check{\pi}_k(X, *) = \lim[\pi_k(\mathfrak{X}, *)].$$

COROLLARY 2.8. *Let* $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ *be a* k*-connected inverse system of pointed topological spaces and* $p : (X, *) \to (\mathfrak{X}, *)$ *an* $HPol_*$*-expansion of* $(X, *)$*, then*

$$\check{\pi}_{k+1}(X, *) \cong \check{H}_{k+1}(X; \mathbb{Z}).$$

Proof. By the Hurewicz isomorphism theorem, we have

$$\begin{aligned}\check{\pi}_{k+1}(X, *) &\cong \lim[\pi_{k+1}(\mathfrak{X}, *)]\\ &\cong \lim[H_{k+1}(\mathfrak{X}; \mathbb{Z})]\\ &\cong \check{H}_{k+1}(X; \mathbb{Z}).\end{aligned}$$

3. An isomorphism between a pure extension and an extension functor

Let $\Delta(\lambda) = (\Delta_\lambda, id_{\lambda\lambda'}, \Lambda)$ be an inverse system defined by $\Delta_\lambda = \mathbb{Z}$ and $id_{\lambda\lambda'}$ is an identity map on $\mathbb{Z}$. Consider a free abelian group

$$P^n = \bigoplus_{\lambda_0 \leq \cdots \leq \lambda_n} \mathbb{Z}$$

whose basis is formed by elements $< \lambda_0, \cdots, \lambda_n >$ corresponding to $\lambda_0 \leq \cdots \leq \lambda_n$ in Λ. One can define P^n_λ as the subgroup of P^n by

$$P^n_\lambda = \bigoplus_{\lambda \leq \lambda_0 \leq \cdots \leq \lambda_n} \mathbb{Z}$$

and $i_{\lambda\lambda'} : P^n_{\lambda'} \to P^n_\lambda, \lambda \leq \lambda'$ as the natural inclusion. Then $\mathfrak{P}^n = (P^n_\lambda, i_{\lambda\lambda'}, \Lambda)$ is a clearly inverse system of free groups and inclusions over Λ.

B. L. Osofsky [12] and S. Mardešić [5] have considered the morphisms $e : \mathfrak{P}^0 \to \Delta(\lambda)$ and $d^{n-1} : \mathfrak{P}^n \to \mathfrak{P}^{n-1}$ defined as follows: For each λ_0 and $\lambda_0 \leq \cdots \leq \lambda_n$ in Λ,

$$e_{\lambda_0} < \lambda_0 > = 1$$

$$d^{n-1}_\lambda < \lambda_0, \cdots, \lambda_n > = \sum_{j=0}^{n} (-1)^j < \lambda_0, \cdots, \hat{\lambda}_j, \cdots, \lambda_n >,$$

where $e_{\lambda_0} = e|_{P^0_{\lambda_0}}$, $d^{n-1}_\lambda = d^{n-1}|_{P^n_\lambda}$ and $\hat{\lambda}_j$ means the deletion of λ_j.

PROPOSITION 3.1. *The inverse systems* $\Delta(\lambda), \mathfrak{P}^n$ *and the morphisms* $e, d^n, n \geq 0$ *form a standard projective resolution*

$$0 \leftarrow \Delta(\lambda) \overset{e}{\leftarrow} \mathfrak{P}^0 \overset{d^0}{\longleftarrow} \mathfrak{P}^1 \leftarrow \cdots \leftarrow \mathfrak{P}^{n-1} \overset{d^{n-1}}{\longleftarrow} \mathfrak{P}^n \leftarrow \cdots$$

of $\Delta(\lambda)$.

Proof. See Lemma 7 of [5].

For any inverse system $\mathfrak{A}$ of abelian groups, let $L(\mathfrak{A})$ be a cochain complex

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathfrak{P}^0, \mathfrak{A}) \to \mathrm{Hom}(\mathfrak{P}^1, \mathfrak{A}) \to \cdots \\ \to \mathrm{Hom}(\mathfrak{P}^{n-1}, \mathfrak{A}) \to \mathrm{Hom}(\mathfrak{P}^n, \mathfrak{A}) \to \cdots \end{aligned}$$

induced by the standard projective resolution of $\Delta(\lambda)$. A map $\Phi^n_{\mathfrak{A}} : \mathrm{Hom}(\mathfrak{P}^n, \mathfrak{A}) \to C^n(\mathfrak{A})$ is defined as follows: If $f \in \mathrm{Hom}(\mathfrak{P}^n, \mathfrak{A})$ is given by the homomorphisms $f_\lambda : P^n_\lambda \to A_\lambda, \lambda \in \Lambda$, then $\Phi^n_{\mathfrak{A}}(f)$ is the n-cochain

$$\begin{aligned} x &= (\cdots, x_{(\lambda_0, \cdots, \lambda_n)}, \cdots) \\ &= (\cdots, f_{\lambda_0}(< \lambda_0, \cdots, \lambda_n >), \cdots), \end{aligned}$$

where $x_{(\lambda_0, \cdots, \lambda_n)} \in A_{\lambda_0}$ and $< \lambda_0, \cdots, \lambda_n >, \lambda \leq \lambda_0$ is a basis of P^n_λ. That $\Phi^n_{\mathfrak{A}}$ is an epimorphism was proved by S. Mardešić [5]. We will prove a full detail of the fact that $\Phi_{\mathfrak{A}}$ is a cochain map and that the extension functor $\mathrm{Ext}^n(\Delta(\lambda), -)$ is naturally equivalent to the derived functor $\lim^n(-)$.

LEMMA 3.2. *For every inverse system* $\mathfrak{A} = (A_\lambda, a_{\lambda\lambda'}, \Lambda)$ *of abelian groups and for each* $n \geq 0$, *there exists a natural isomorphism*

$$\Phi^n_{\mathfrak{A}*} : Ext^n(\Delta(\lambda), \mathfrak{A}) \overset{\cong}{\longrightarrow} \lim{}^n \mathfrak{A}.$$

Proof. It is easy to check that $\Phi^n_{\mathfrak{A}}$ is a monomorphism. Thus, in order to complete the proof of this Lemma, it remains to show that $\Phi_{\mathfrak{A}}$ is a cochain map and $\mathrm{Ext}^n(\Delta(\lambda), -)$ is naturally equivalent to $\lim^n(-)$.

For each $\bar{\lambda} = (\lambda_0, \cdots, \lambda_n) \in \Lambda^n$ and $f \in \operatorname{Hom}(\mathfrak{P}^{n-1}, \mathfrak{A})$, we have

$$
\begin{aligned}
&(\delta \circ \Phi_{\mathfrak{A}}^{n-1}(f))_{\bar{\lambda}} \\
&= a_{\lambda_0\lambda_1}(\Phi_{\mathfrak{A}}^{n-1}(f))_{\bar{\lambda}_0} + \sum_{j=1}^{n}(-1)^j(\Phi_{\mathfrak{A}}^{n-1}(f))_{\bar{\lambda}_j} \\
&= a_{\lambda_0\lambda_1}(f_{\lambda_1}(< \lambda_1, \cdots, \lambda_n >) \\
&\qquad + \sum_{j=1}^{n}(-1)^j f_{\lambda_0}(< \lambda_0, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_n >) \\
&= a_{\lambda_0\lambda_1}(f_{\lambda_1}(< \lambda_1, \cdots, \lambda_n >) \\
&\qquad + f_{\lambda_0}(\sum_{j=1}^{n}(-1)^j(< \lambda_0, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_n >)) \\
&= f_{\lambda_0}(i_{\lambda_0\lambda_1}(< \lambda_1, \cdots, \lambda_n >) \\
&\qquad + f_{\lambda_0}(\sum_{j=1}^{n}(-1)^j(< \lambda_0, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_n >)) \\
&\qquad (f \text{ is a level system map}) \\
&= f_{\lambda_0}(< \lambda_1, \cdots, \lambda_n >) \\
&\qquad + f_{\lambda_0}(\sum_{j=1}^{n}(-1)^j(< \lambda_0, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_n >)) \\
&\qquad (i_{\lambda_0\lambda_1} \text{ is an inclusion map}) \\
&= f_{\lambda_0}(\sum_{j=0}^{n}(-1)^j < \lambda_0, \cdots, \hat{\lambda}_j, \cdots, \lambda_n >) \\
&= f_{\lambda_0}(d_{\lambda}^{n-1} < \lambda_0, \cdots, \lambda_n >) \\
&= (f \circ d^{n-1})_{\lambda}(< \lambda_0, \cdots, \lambda_n >) \\
&= (\Phi_{\mathfrak{A}}^{n} \circ \operatorname{Hom}(d^{n-1}, 1_{\mathfrak{A}})(f))_{\bar{\lambda}}
\end{aligned}
$$

which shows that $\Phi_{\mathfrak{A}}$ is a cochain map. Thus it induces a homomorphism $\Phi_{\mathfrak{A}*}^{n} : \operatorname{Ext}^n(\Delta(\lambda), \mathfrak{A}) \to \lim^n \mathfrak{A}$ induced by $\Phi_{\mathfrak{A}}^{n}$.

Let $l : \mathfrak{A} \to \mathfrak{B} = (B_\lambda, b_{\lambda\lambda'}, \Lambda)$ be a level system map. Considering the definition of the cochain map $\Phi_{\mathfrak{A}}^{n}$, for any $[f] \in$

$\mathrm{Ext}^n(\Delta(\lambda), \mathfrak{A})$ we have

$$\begin{aligned}\lim{}^n(l) \circ \Phi^n_{\mathfrak{A}*}([f]) &= \lim{}^n(l) \circ \Phi^n_{\mathfrak{A}*}([(\cdots, f_{\lambda_0}, \cdots)]) \\ &= \lim{}^n(l)([(\cdots, f_{\lambda_0} < \lambda_0, \cdots, \lambda_n >, \cdots)]) \\ &= [(\cdots, l_{\lambda_0} \circ f_{\lambda_0} < \lambda_0, \cdots, \lambda_n >, \cdots)] \\ &= \Phi^n_{\mathfrak{B}*}([(\cdots, l_{\lambda_0} \circ f_{\lambda_0}, \cdots)]) \\ &= \Phi^n_{\mathfrak{B}*} \circ \mathrm{Ext}(1_{\Delta(\lambda)}, l)([(\cdots, f_{\lambda_0}, \cdots)]) \\ &= \Phi^n_{\mathfrak{B}*} \circ \mathrm{Ext}(1_{\Delta(\lambda)}, l)([f])\end{aligned}$$

which shows the required proof of the natural equivalence between $\mathrm{Ext}(\Delta(\lambda), -)$ and $\lim^n(-)$.

A subgroup S of T is called a *pure subgroup* if

$$S \cap nT = nS$$

for every integer n. An exact sequence

$$0 \to U \xrightarrow{u} V \xrightarrow{v} W \to 0$$

is said to be *pure exact* [1] if $\mathrm{im}(u)$ is a pure subgroup of V. For abelian groups U and W, let $\mathrm{Pext}(W, U)$ denote the group of pure extension, i.e., the subgroup of $\mathrm{Ext}(W, U)$ whose elements correspond to the classes of pure exact sequences.

Let $\mathfrak{G} = (G_\lambda, g_{\lambda\lambda'}, \Lambda)$ be a direct system of abelian groups G_λ and group homomorphisms $g_{\lambda\lambda'} : G_\lambda \to G_{\lambda'}, \lambda \leq \lambda'$ over Λ. Then we obtain the following inverse systems

(1) $\mathrm{Hom}(\mathfrak{G}, A) = (\mathrm{Hom}(G_\lambda, A), \bar{g}_{\lambda\lambda'}, \Lambda)$
(2) $\mathrm{Pext}(\mathfrak{G}, A) = (\mathrm{Pext}(G_\lambda, A), \bar{g}_{\lambda\lambda'}, \Lambda)$

induced by $\mathfrak{G}$. We denote a colimit $\mathrm{colim}\mathfrak{G}$ of $\mathfrak{G}$ by the direct limit of $\mathfrak{G}$.

LEMMA 3.3. *For any abelian group A and a direct system $\mathfrak{G} = (G_\lambda, g_{\lambda\lambda'}, \Lambda)$ of abelian groups, there exists an exact sequence*

$$0 \to \lim{}^1 Hom(\mathfrak{G}, A) \to Pext(colim\mathfrak{G}, A) \to \lim Pext(\mathfrak{G}, A) \\ \to \lim{}^2 Hom(\mathfrak{G}, A) \to 0.$$

Proof. See Proposition 1.4 of [2] or Proposition 26 of [7].

LEMMA 3.4. *Let $\mathfrak{G} = (G_\lambda, g_{\lambda\lambda'}, \Lambda)$ be a direct system of finitely generated abelian groups over Λ. Then, for any abelian group A,*

(1) $\lim^p Hom(\mathfrak{G}, A) = 0$ *for all* $p \geq 2$;
(2) $\lim^p Ext(\mathfrak{G}, A) = 0$ *for all* $p \geq 1$.

Proof. See Corollary 1.5 of [2].

THEOREM 3.5. *Let $H^*(\mathfrak{X}) = (H^*(X_\lambda; \mathbb{Z}), p^*_{\lambda\lambda}, \Lambda)$ be a direct system, induced by the inverse system $\mathfrak{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, of finitely generated cohomology groups $H^*(X_\lambda; \mathbb{Z})$, $\lambda \in \Lambda$. Then we have*

(1) $\lim^1 H_n(\mathfrak{X}; A) \cong \lim^1 Hom(H^n(\mathfrak{X}), A)$
(2) $\lim^p H_n(\mathfrak{X}; A) = 0$ *for all* $p \geq 2$
(3) $Pext(colim H^n(\mathfrak{X}), A) \cong Ext(\Delta(\lambda), Hom(H^n(\mathfrak{X}), A))$

for any abelian group A.

Proof. From the given inverse system $\mathfrak{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, we have the following induced inverse systems

(1) $\mathrm{Ext}(H^n(\mathfrak{X}), A) = (\mathrm{Ext}(H^n(X_\lambda; \mathbb{Z}), A), \bar{p}_{\lambda\lambda'}, \Lambda)$;
(2) $H_n(\mathfrak{X}; A) = (H_n(X_\lambda; A), p_{\lambda\lambda'*}, \Lambda)$;
(3) $\mathrm{Hom}(H^n(\mathfrak{X}), A) = (\mathrm{Hom}(H^n(X_\lambda; \mathbb{Z}), A), \tilde{p}_{\lambda\lambda'}, \Lambda)$.

Applying the universal coefficient theorem for cohomology [13], we have an exact sequence

$$0 \to \mathrm{Ext}(H^{n+1}(\mathfrak{X}), A) \to H_n(\mathfrak{X}; A) \to \mathrm{Hom}(H^n(\mathfrak{X}), A) \to 0$$

of inverse systems which induces a long exact sequence

(*)

$$\begin{aligned} 0 \to \lim\mathrm{Ext}(H^{n+1}(\mathfrak{X}), A) &\to \lim H_n(\mathfrak{X}; A) \to \lim \mathrm{Hom}(H^n(\mathfrak{X}), A) \\ &\to \lim^1\mathrm{Ext}(H^{n+1}(\mathfrak{X}), A) \to \lim^1 H_n(\mathfrak{X}; A) \\ &\to \lim^1\mathrm{Hom}(H^n(\mathfrak{X}), A) \to \quad \cdots \\ &\to \lim^r\mathrm{Ext}(H^{n+1}(\mathfrak{X}), A) \to \lim^r H_n(\mathfrak{X}; A) \\ &\to \lim^r\mathrm{Hom}(H^n(\mathfrak{X}), A) \to \quad \cdots \end{aligned}$$

of derived limits. Since the direct system $H^*(\mathfrak{X}) = (H^*(X_\lambda; \mathbb{Z}), p^*_{\lambda\lambda}, \Lambda)$ induced by $\mathfrak{X}$ consists of finitely generated cohomology groups, by Lemma 3.4 we have

$$\lim^p\mathrm{Ext}(H^{n+1}(\mathfrak{X}), A) = 0 \text{ for all } p \geq 1$$

and

$$\lim^p \mathrm{Hom}(H^n(\mathfrak{X}), A) = 0 \text{ for all } p \geq 2.$$

Thus we have

$$\lim^1 H_n(\mathfrak{X}; A) \cong \lim^1 \mathrm{Hom}(H^n(\mathfrak{X}), A)$$

and

$$\lim^p H_n(\mathfrak{X}; A) = 0 \text{ for all } p \geq 2.$$

Since every finitely generated abelian group is pure projective, we have

$$\lim \mathrm{Pext}(H^n(\mathfrak{X}), A) = 0.$$

Thus, by Lemma 3.2 and 3.3, we obtain

$$\mathrm{Pext}(\mathrm{colim} H^n(\mathfrak{X}), A) \cong \lim^1 \mathrm{Hom}(H^n(\mathfrak{X}), A)$$

which is isomorphic to $\mathrm{Ext}(\Delta(\lambda), \mathrm{Hom}(H^n(\mathfrak{X}), A))$.

COROLLARY 3.6. *In addition to the assumption of Theorem 3.5, if A is an injective $\mathbb{Z}$-module and $p : X \to \mathfrak{X}$ is an HPol-expansion, then*

$$\check{H}_n(X; A) \cong \lim \mathit{Hom}(H^n(\mathfrak{X}), A).$$

Proof. By the long exact sequence $(*)$ in the proof of Theorem 3.5, the sequence

$$\begin{aligned} 0 &\to \lim \mathrm{Ext}(H^{n+1}(\mathfrak{X}), A) \to \lim H_n(\mathfrak{X}; A) \\ &\to \lim \mathrm{Hom}(H^n(\mathfrak{X}), A) \to 0 \end{aligned}$$

is exact. Since the first term is trivial and the second term, by definition, is Čech homology group $\check{H}_n(X; A)$, we obtain the result.

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