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On the Fibrewise Cofibrations and Fibrewise Fibrations in a Quasitopos

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Quasitopos에서의 섬유 코화이버 함수와 섬유 화이버 함수에 대한 연구

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In this paper, we obtain that if ϕ: E → F is a fibrewise fibration then the postcomposition ϕ^* : CB(Y, E) → CB(Y, F) is a fibrewise fibration and if (X, A) is a closed fibrewise cofibration then the precomposition u* : CB(X, E) → CB(A, E) is also a fibrewise fibration.

본 논문에서는 ϕ : E → F가 섬유 화이버 함수일 때 후합성 ϕ^* : CB(Y, E) → CB(Y, F)도 섬유 화이버 함수이고, (X, A)가 닫힌 섬유 화이버 함수일 때 전합성 u* : CB(X, E) → CB(A, E) 도 섬유 화이버 함수임을 보인다.

Key words : Fibrewise cofibration, Fibrewise fibration, Fibrewise mapping cylinder, Adjoint

Ⅰ. Introduction

The results of [3] concerning the fibrewise cofibrations and the fibrewise fibrations are basic to the theory of fibrewise topology. But the results are sometimes restricted to special cases. The main reason is that the category of topological spaces is not a quasitopos. Thus, it is natural to consider the category which is a quasitopos. In view of this fact, Kim [4] obtained that if p : X → B, q : Y → B were fibrations then so was CB(X, Y) → B in a quasitopos. Also, Kim [5] obtained the equivalent conditions that a continuous function was a fibration.

In this paper, we obtain that if φ : E → F is a fibrewise fibration then the postcomposition φ^* : CB(Y, E) → CB(Y, F) is a fibrewise fibration and if (X, A) is a closed fibrewise cofibration then the precomposition u* : CB(X,

E) $\to C_B(A, E)$ is also a fibrewise fibration in a quasitopos.

Throughout this paper the category considered is a quasitopos.

II. Main Results

Definition 2.1

The fibrewise map $u : A \to X$, where A and X are fibrewise spaces over B, is a fibrewise cofibration if u has the following fibrewise homotopy extension property. Let $f : X \to E$ be a fibrewise map, where E is a fibrewise space, and let $g : A \to C_B(I\times B, E)$ be a fibrewise homotopy such that $\rho_o g = fu$. Then there exists a fibrewise homotopy $h : X \to C_B(I\times B, E)$ such that $\rho_o h = f$ and $hu = g$

$$\begin{array}{ccc} A & \xrightarrow{\ g\ } & C_B(I\times B, E) \\ {\scriptstyle u}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle \rho_o} \\ X & \xrightarrow[\ f\ ]{} & E \end{array}$$

An important special case is when A is a subspace of X. In that case we describe (X, A) as a fibrewise cofibred pair when the inclusion $A \to X$ is a fibrewise cofibration. For example (X, X) and (X, $\emptyset$) are fibrewise cofibred pairs, for all fibrewise spaces X.

Given a fibrewise map $u : A \to X$, where A and X are spaces over B, the fibrewise mapping cylinder $M = M_B(u)$ is defined to be the fibrewise push - out of the cotriad.

$$I\times A \xleftarrow{\ \sigma_o\ } A \xrightarrow{\ u\ } X.$$

In the case in which A is a subspace of X, with u the inclusion, we have a continuous fibrewise bijection.

$$M \longrightarrow (\{o\} \times X) \cup (I \times A).$$

For each fibrewise map $u : A \to X$ the fibrewise mapping cylinder $M = M_B(u)$ comes equipped with a fibrewise map $k : M \to I \times X$ which is derived from the diagram shown below.

$$\begin{array}{ccc} A & \xrightarrow{\ u\ } & X \\ {\scriptstyle \sigma_c}\downarrow & & \downarrow{\scriptstyle \sigma_o} \\ I \times A & \xrightarrow[\ 1 \times u\ ]{} & I \times X \end{array}$$

Proposition 2.2

The fibrewise map $u : A \to X$, where A and X are fibrewise spaces over B, is a fibrewise cofibration if and only if the fibrewise map k admits a left inverse $\ell : I \times X \to M$.

Proof. Suppose that ℓ exists. Given a space E over B and fibrewise maps $f : X \to E$, $g : A \to C_B(I\times B, E)$, we consider the triad.

$$I \times A \xrightarrow{\ \hat{g}\ } E \xleftarrow{\ f\ } X,$$

where $\hat{g}$ is the adjoint of g. Precompositionof ℓ with the fibrewise push-out $M \to E$ of the triad yields a fibrewise map $I \times X \to E$, of which the adjoint $X \to C_B(I\times B, E)$ provides the required fibrewise homotopy.

Conversely suppose that u is a fibrewise cofibration. Take E to be the fibrewise mapping cylinder $M = M_B(u)$ and consider the following diagram.

$$\begin{array}{ccc} A & \xrightarrow{\ \hat{H}\ } & C_B(I\times B, M) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \rho_o} \\ X & \xrightarrow[\ j\ ]{} & M \end{array}$$

The adjoint of the fibrewise homotopy $X \to C_B(I \times B, M)$ is the required left inverse of k.

Corollary 2.3

Let $u : A \to X$ be a fibrewise cofibration, where A and X are fibrewise spaces over B.

Then u : A $\to$ X is a fibrewise cofibration over B' for each space B' and map B $\to$ B'.

From now on we shall concentrate on fibrewise pair (X, A) with the property that the fibrewise mapping cylinder is isomorphic to ({o} $\times$ X) $\cup$ (I $\times$ A). This fibrewise pair (X, A) is called a closed fibrewise pair. Then (2.2) yields.

Proposition 2.4

Let (X, A) be a closed fibrewise pair over B. Then (X, A) is fibrewise cofibred if and only if ({o} $\times$ X) $\cup$ (I $\times$ A) is a fibrewise retract of I $\times$ X.

Corollary 2.5

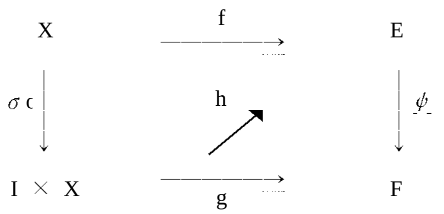
Let (X, A) be a closed fibrewise pair over B. If (X, A) is fibrewise cofibred then so is

$T \times_B (X, A) = (T \times_B X, T \times_B A)$

for all fibrewise spaces T.

Definition 2.6

The fibrewise map φ : E $\to$ F, where E and F are fibrewise spaces over B, is a fibrewise fibration if φ has the following property for all fibrewise spaces X. Let f : X $\to$ E be a fibrewise map and let g : I $\times$ X $\to$ F be a fibrewise homotopy such that $g \sigma o = \varphi f$. Then there exists a fibrewise homotopy h : I $\times$ X $\to$ E such that $h \sigma o = f$ and $\varphi h = g$.



Note that is φ is a fibration, in the ordinary sense, then φ is a fibrewise fibration. This is, of course, the opposite to the situation which occurs in the case of cofibrations. Also note that the projection p : E $\to$ B is always a fibrewise fibration for any fibrewise space E, thus there exist fibrewise maps which are fibrewise fibrations but not fibrations in the ordinary sense.

Proposition 2.7

Let φ: E $\to$ F be a fibrewise fibration, where E and F are fibrewise spaces over B. Then the postcomposition function

$\varphi^* : C_B(Y, E) \to C_B(Y, F)$

is a fibrewise fibration for all fibrewise space Y.

Proof. Given a space X over B and fibrewise maps f : X $\to$ $C_B(Y, E)$, g : I $\times$ X $\to$ $C_B(Y, F)$, $\varphi \widehat{f} = \widehat{g} \sigma o$, where $\widehat{f}$, $\widehat{g}$ are adjoints of f, g respectively. Since φ is a fibrewise fibration, there is a fibrewise homotopy $\widehat{h}$: I $\times$ X $\to$ E such that $\widehat{h} \sigma o = \widehat{f}$ and $\varphi \widehat{h} = \widehat{g}$. The adjoint h of $\widehat{h}$ is the required fibrewise homotopy.

Proposition 2.8

Let (X, A) be a closed fibrewise pair over B. If (X, A) is fibrewise cofibred then the precomposition function

$u^* : C_B(X, E) \to C_B(A, E)$

is a fibrewise fibration for all fibrewise spaces E.

Proof. Let Y be a fibrewise space, f : Y $\to$ $C_B(X, E)$ be a fibrewise map, and let g : I $\times$ Y $\to$ $C_B(A, E)$ be a fibrewise homotopy such that $g \sigma o = u^* f$. Since u is a fibrewise cofibration, so is

$u \times_B 1Y : A \times_B Y \to X \times_B Y$.

Hence there exists a fibrewise homotopy $\widehat{h}$: $X \times_B Y \to C_B(I \times B, E)$ such that $\rho o \widehat{h} = \widehat{f}$ and $\widehat{h}(u \times_B 1Y) = \widehat{g}$, where $\widehat{f}$, $\widehat{g}$ are adjoints of f, g respectively. Then the adjoint h : $I \times Y \to C_B(X, E)$ is the required fibrewise homotopy.

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