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BANACH-SAKS PROPERTY ON THE DUAL OF SCHLUMPRECHT SPACE

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Abstract. In this paper, we show that Schlumprecht space is reflexive and the Dual of Schlumprecht space has the Banach-Saks property and study behavior of block basic sequence in Schlumprecht space.

1. Introduction

S. Banach and S. Saks [BS] showed that every bounded sequence in $L_p[0,1]$, $1 < p < \infty$, has a subsequence with arithmetic means converging in norm. J. Schrier [Sc] showed that $C[0,1]$ does not have this property. The above results lead us to consider the following question. What Banach space X has the *Banach-Saks property* i.e., every bounded sequence in X admits a subsequence whose arithmetic means converges in norm. S. Kakutani [Ka] showed that uniformly convex Banach spaces have the Banach-Saks property. T. Nishiura and D. Waterman [NW] showed that Banach spaces with the Banach-Saks property are reflexive. A. Baernstein [Ba] proved the converse by providing an example of a reflexive Banach space which does not have the Banach-Saks property. C. Seifert [Se1] showed that the dual of Baernstein space has the Banach-Saks property. In this paper, we introduce arbitrarily distortable Banach space - Schlumprecht space [Sh] and show that it is reflexive, not uniformly convex and its dual has the Banach-Saks property.

Schlumprecht [Sh] introduced a Banach space which is arbitrarily distortable. We introduce some basic definitions and construct Schlumprecht space S. The vector space of all real valued sequences

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(x_n) whose elements are eventually zero is denoted by c_{00} ; (e_i) denotes the usual unit vector basis of c_{00}, i.e., $e_i(j) = 1$ if $i = j$ and $e_i(j) = 0$ if $i \neq j$. For $x = \sum_{i=1}^{n} \alpha_i e_i \in c_{00}$, the set $\text{supp}(x) = \{i \in \mathbb{N} : \alpha_i \neq 0\}$ is called the support of x. If E and F are two finite subsets of $\mathbb{N}$ we write $E < F$ if $\max(E) < \min(F)$, and for $x, y \in c_{00}$ we write $x < y$ if $\text{supp}(x) < \text{supp}(y)$. For $E \subset \mathbb{N}$ and $x = \sum_{i=1}^{\infty} x_i e_i \in c_{00}$ we put $E(x) := \sum_{i \in E} x_i e_i$.

The following lemma is essential to define the Schlumprecht space and we refer to [Sh] for the its proof. From now on, we mean $f(x)$ as $\log_2(x+1)$.

LEMMA 1.1. *[Sh] Let* $f(x) = \log_2(x+1)$, *for* $x \geq 1$. *Then* f *has the following properties :*

(1) $f(1) = 1$ *and* $f(x) < x$ *for all* $x > 1$,
(2) f *is strictly increasing to* ∞,
(3) $\lim_{x \to \infty} (f(x)/x^q) = 0$ *for all* $q > 0$,
(4) *the function* $g(x) = x/f(x)$, $x \geq 1$ *is concave, and*
(5) $f(x) \cdot f(y) \geq f(x \cdot y)$ *for* $x, y \geq 1$.

On c_{00} we define a norm $|\cdot|_k$ by induction for each $k \in \mathbb{N}$. For $x = \sum x_n e_n \in c_{00}$ we let $|x|_0 = \max_{n \in \mathbb{N}} |x_n|$. Assuming that $|x|_k$ is defined for some $k \in \mathbb{N}$ we put

$$|x|_{k+1} = \max_{\substack{l \in \mathbb{N} \\ E_1 < E_2 < \cdots < E_l \\ E_i \subset \mathbb{N}}} \frac{1}{f(l)} \sum_{i=1}^{l} |E_i(x)|_k.$$

Since $f(1) = 1$,

$$|x|_{k+1} \geq |E(x)|_k = |x|_k,$$

where $E = \text{supp}(x)$. It follows that $(|x|_k)$ is increasing for any $x \in c_{00}$. Since $f(l) > 1$ for all $l \geq 2$ and

$$\frac{1}{f(l)} \sum_{k=1}^{l} |E_k(e_i)|_k \leq \frac{1}{f(l)},$$

it follows that

$$|e_i|_k = 1 \qquad \text{for any } i \in \mathbb{N} \text{ and } k \in \mathbb{N}_0.$$

We put

$$\|x\| = \max_{k\in\mathbb{N}} |x|_k, \qquad \text{for } x \in c_{00}.$$

Then $\|\cdot\|$ is a norm on c_{00} and we define the Schlumprecht space S as the completion of c_{00} with respect to $\|\cdot\|$.

The following proposition states some easy facts about S.

Proposition 1.2. *[Sh]*

(1) *The sequence of unit vectors (e_i) is a 1-subsymmetric and 1-unconditional basis of the Schlumprecht space S ; i.e., for any $x = \sum_{i=1}^{\infty} x_i e_i \in S$, any strictly increasing sequence $(n_i) \subset \mathbb{N}$ and any $(\epsilon_i)_{i\in\mathbb{N}} \in \{-1,1\}^{\mathbb{N}}$ it follows that*

$$\|x\| = \left\|\sum_{i=1}^{\infty} x_i e_i\right\| = \left\|\sum_{i=1}^{\infty} \epsilon_i x_i e_{n_i}\right\|.$$

(2) *For $x \in S$,*

$$\|x\| = \max\left\{ |x|_0, \sup_{\substack{l\geq 2\\ E_1<E_2<\cdots<E_l\\ E_i\subset\mathbb{N}}} \frac{1}{f(l)} \sum_{i=1}^{l} \|E_i(x)\| \right\}$$

(3) *For $n \in \mathbb{N}$ we have that*

$$\left\|\sum_{i=1}^{n} e_i\right\| = \frac{n}{f(n)}.$$

The following is the main result of [Sh].

Theorem 1.3. *The Schlumprecht space S is arbitrarily distortable and does not contain an isomorphic copy of l_1.*

2. Banach-Saks Property on the Dual of Schlumprecht space

In this chapter, we show that S is reflexive, not uniformly convex and S^* has the Banach-Saks Property. Finally, we carefully examine the rather special behavior of block basic sequence in Schlumprecht space.

THEOREM 2.1. *S is a reflexive space.*

Proof. It suffices to show that S does not contain c_0, by Theorem 1.3 and Proposition 1.2.(1). Suppose c_0 is isomorphic to a subspace of S. Then there exists a sequence $\{y_n\}$ of S which is equivalent to the unit vectors $\{e_n\}$ of c_0, that is, there exists $m,\ M > 0$ such that

$$m\|\sum a_n e_n\|_{c_0} \le \|\sum a_n y_n\| \le M\|\sum a_n e_n\|_{c_0}.$$

Since the unit vector $\{e_n\}$ of c_0 is convergent weakly to zero and bounded away from zero in norm, so is $\{y_n\}$. By the Bessaga-Pelcynski selection principle, there exists a subsequence $\{y'_n\}$ of $\{y_n\}$ which is equivalent to a normalized block basis $\{u_j\}$ of unit vectors $\{e_n\}$ of S. Then we have

$$\left\|\sum_{k=1}^{n} y'_k\right\| \le M$$

and

$$\begin{aligned}\left\|\sum_{k=1}^{n} u_k\right\| &\ge \frac{1}{f(n)}\sum_{k=1}^{n}\|E_k(\sum_{k=1}^{n} u_k)\| \\ &= \frac{n}{f(n)},\end{aligned}$$

where E_k=supp (u_k).

Since $\frac{n}{f(n)} \to \infty$ as $n \to \infty$ by Lemma 1.1.(3), we get the contradiction to the fact that $\{u_j\}$ is equivalent to $\{y'_k\}$. □

We show that S is not uniformly convex. For this, we need following easy lemma.

LEMMA 2.2. *Let* $x = \left(\frac{f(2)}{2} \pm \epsilon\right) e_1 + \left(\frac{f(2)}{2} \mp \epsilon\right) e_2 \in S$ *and* $\frac{f(2)}{2} \leq \frac{f(2)}{2} + \epsilon \leq 1$. *Then* $\|x\| = 1$.

Proof. Since the number of supp(x) is 2, by Proposition 1.2.(2),

$$\begin{aligned}\|x\| &= \max\left\{|x|_0, \frac{1}{f(2)}\left(\|E_1(x)\| + \|E_2(x)\|\right)\right\}\\ &= \max\left\{\frac{f(2)}{2} + \epsilon, 1\right\} = 1.\end{aligned}$$

□

Using Lemma 2.2, we get the following proposition.

PROPOSITION 2.3. *The Schlumprecht space S is not uniformly convex.*

Proof. . Let $\epsilon > 0$, $\frac{f(2)}{2} + \epsilon \leq 1$ and

$$\begin{aligned}x &= \left(\frac{f(2)}{2} + \epsilon\right) e_1 + \left(\frac{f(2)}{2} - \epsilon\right) e_2\\ y &= \left(\frac{f(2)}{2} - \epsilon\right) e_1 + \left(\frac{f(2)}{2} + \epsilon\right) e_2.\end{aligned}$$

Then $\|x\| = \|y\| = 1$ and

$$\begin{aligned}\|x+y\| &= f(2)\|e_1 + e_2\|\\ &= f(2)\frac{2}{f(2)} \qquad \text{by Proposition 1.2.(3)}\\ &= 2.\end{aligned}$$

□

By Proposition 2.3 and Theorem 2.1, we can ask a natural question : does S or S^* has the Banach-Saks property ? The following Lemma is the criterion for testing for the Banach-Saks property.

LEMMA 2.4. *[Se2] Suppose X is a reflexive Banach space whose basis is $\{x_n\}$. Then X has the Banach-Saks property if and only if every bounded block basic sequence with respect to $\{x_n\}$ admits a subsequence whose arithmetic means converges to zero in norm.*

Now we are ready to get our main theorem which is focused throughout this paper.

THEOREM 2.5. *S^* has the Banach-Saks property, where S^* is the dual space of the Schlumprecht space.*

Proof. We note that $\{e_n\}$ is shrinking and the biothogonal functionals $\{e^*\}$ form a Schauder basis of S^*, since S is reflexive. Let $\{x_n^*\}$ be a bounded block basic sequence with respect to $\{e_n^*\}$, where $x_n^* = \sum_{j \in F_n} \alpha_j e_j^*, \quad F_1 < F_2 < \cdots F_n < \cdots$. Let $x = \sum_{j=1}^{\infty} x_j e_j \in S, \quad \|x\| = 1$. Then

$$\begin{aligned}
\left| \left\langle \sum_{m=1}^{n} x_m^*, x \right\rangle \right| &= \left| \sum_{m=1}^{n} \left\langle \sum_{j \in F_m} \alpha_j e_j^*, \sum_{j=1}^{\infty} x_j e_j \right\rangle \right| \\
&\leq \sum_{m=1}^{n} \left| \sum_{j \in F_m} \alpha_j x_j \right| \\
&= \sum_{m=1}^{n} |\langle x_m^*, x_{F_m} \rangle| \\
&\leq \sum_{m=1}^{n} M \|x_{F_m}\|, \qquad \text{where } M = \sup_m \|x_m^*\| \\
&\leq M f(n) \|x\|, \qquad \text{by Proposition 1.2 (2)} \\
&= M f(n).
\end{aligned}$$

Hence

$$\frac{1}{n} \left\| \sum_{m=1}^{n} x_m^* \right\| \leq \frac{M f(n)}{n} \to 0 \qquad \text{as } n \to \infty, \text{ by Lemma 1.1 (3)}$$

This completes the proof. □

Finally, we carefully examine the rather special behavior of block basic sequence in Schlumprecht space.

THEOREM 2.6. *Let* $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$, $(n = 1, 2, \cdots)$ *be a normalized block basic sequence of scalars* $\{e_n\}_{n=1}^{\infty}$. *Then for every sequence of scalars* $\{b_n\}_{n=1}^{\infty}$,

$$\left\| \sum_n b_n e_n \right\| \leq \left\| \sum_n b_n y_n \right\|.$$

Proof. We show for every choice of (b_n),

$$\left| \sum_n b_n e_n \right|_m \leq \left\| \sum_n b_n y_n \right\| \qquad \text{for every } m.$$

For $m = 0$,

$$\begin{aligned} \left| \sum_n b_n e_n \right|_0 &= \sup_n |b_n| \\ &= \sup_n \|b_n y_n\| \\ &\leq \sup_n \left\| \sum_n b_n y_n \right\|, \end{aligned}$$

since (e_n) is 1-unconditional. Suppose our result up to some positive integer m. Let $x = \sum b_n e_n,\ y = \sum b_n y_n$. Then for $E_1 < \cdots < E_l$,

$$\begin{aligned} \frac{1}{f(l)} \sum_{j=1}^{l} |E_j(x)|_m &\leq \frac{1}{f(l)} \sum_{j=1}^{l} \left\| \sum_{n \in E_j} b_n y_n \right\| \\ &\qquad \text{by the induction hypothesis} \\ &= \frac{1}{f(l)} \sum_{j=1}^{l} \|F_j(y)\| \\ &\qquad \text{where } F_j = \cup_{n \in E_j} \text{supp} y_n \\ &\leq \|y\|. \end{aligned}$$

Thus, $|x|_{m+1} \leq \|y\|$. This completes the proof. □

References

[Ba] Baernstein, A., II, *On reflexivity and summability*, Studia Math. **42** (1972), 91-94.

[BS] Banach, S. and Sak, S., *Sur la convergence forte dans les champs L^p*, Studia Math. **2** (1930), 51-57.

[Ka] Kakutani, S., *Weak convergence in uniformly convex spaces*, Tohoku Math. **45** (1938), 188-193.

[NW] Nishiura, T. and Waterman, *Reflexivity and summability*, Studia Math. **23** (1963), 53-57.

[Pe] Pettis, B. J., *A proof that every uniformly convex space is reflexive*, Duke Math. J. **5** (1939), 249-253.

[Sc] Schreier, J., *Ein Gegenbeispiel zur Theorie der schwachen Konvergenz*, Studia Math. **2** (1930), 58-62.

[Se1] Seifert, C. J., *The dual of Baernstein's space and the Banach-Saks property*, Bull. Acad. Polon. Sci. **26** (1978), 237-239.

[Se2] Seifert, C. J., *Averaging in Banach space, (dissertation)*, Kent State University (1977).

[Sh] Schlumprecht, Th., *An arbitrarily distortable Banach space*, Isr. J. Math. **76** (1991), 81-95.

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