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SUBSERIES CONVERGENCE AND SEQUENCE-EVALUATION CONVERGENCE

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Abstract. We show a series of improved subseries convergence results, e.g., in a sequentially complete locally convex space X every weakly c_0-Cauchy series on X must be c_0-convergent. Thus, if X contains no copy of c_0, then every weakly c_0-Cauchy series on X must be subseries convergent.

Let X be a locally convex space. A series $\sum x_j$ on X is said to be weakly c-convergent if for every $\{t_j\} \in c$ the series $\sum_{j=1}^{\infty} t_j x_j$ converges in $(X, weak)$, i.e., for every $\{t_j\} \in c$ there is an $x_0 \in X$ such that

$$\sum_{j=1}^{\infty} t_j f(x_j) = \lim_{n\to\infty} f(\sum_{j=1}^{n} t_j x_j) = f(x_0)$$

for each $f \in X'$, the dual of X(= the family of continuous linear functionals on X). In this case, x_0 is the weak sum of the series $\sum t_j x_j$ and we write $x_0 = w - \sum_{j=1}^{\infty} t_j x_j$. Similarly a series $\sum x_j$ on X is said to be c-convergent if for every $\{t_j\} \in c$ the series $\sum_{j=1}^{\infty} t_j x_j$ converges in X.

Since $c_0 \subseteq c$, if $\sum x_j$ is weakly c-convergent then $\sum x_j$ is weakly c_0-convergent and, by the Orlicz-Pettis theorem, $\sum x_j$ is c_0-convergent. Therefore we have

Proposition 1. *If $\sum x_j$ is weakly c-convergent, then for all $f \in X'$*

$$(*) \qquad \sum_{j=1}^{\infty} |f(x_j)| < +\infty.$$

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Proof. See [1], Theorem 2. □

Of course, if $\sum x_j$ is (weakly) c_0-convergent, then $(*)$ holds and the converse is true if X is sequentially complete.

Note that with the norm $\|\{t_j\}\|_\infty = sup_j|t_j|$, c_0, c and l^∞ are Banach spaces. For a locally convex space X, let $\sigma(X, X')$, $\tau(X, X')$ and $\beta(X, X')$ denote the weak topology, the Mackey topology and the strong topology, respectively. $\tau(X, X')$ is just the topology of uniform convergence on weak* $(\sigma(X', X))$ compact balanced convex sets in X' and $\beta(X, X')$ is just the topology of uniform convergence on weak* bounded sets in X'. If $(X, \|\cdot\|)$ is a Banach space, then $\tau(X, X') = \beta(X, X') = \|\cdot\|$ by the Banach-Alaoglu theorem (see [2]).

For a locally convex space X (with the locally convex topology μ) and an operator $T : c \to X$ we say that T is continuous means T is $\|\cdot\| - \mu$ continuous. But $\mu \leq \tau(X, X') \leq \beta(X, X')$ so $\|\cdot\| - \beta(X, X')$ continuity is stronger than continuity $(= \|\cdot\|_\infty - \mu$ continuity). However, by the Hellinger-Toeplitz theorem, if $(Y, \|\cdot\|)$ is a Banach space and $T : Y \to X$ is continuous, i.e., $\|\cdot\| - \mu$ continuous, then T is $\|\cdot\| - \beta(X, X')$ continuous because $\beta(Y, Y') = \|\cdot\|$. Thus, for $T : c \to X$, the continuity of T is equivalent to the $\|\cdot\|_\infty - \beta(X, X')$ continuity.

It is well known that if $\sum x_j$ is a (weakly) c_0-convergent series on a locally convex space X, then letting $T\{t_j\} = \sum_{j=1}^\infty t_j x_j$ for each $\{t_j\} \in c_0$, T is $\|\cdot\|_\infty - \beta(X, X')$ continuous linear operator and, hence, T is $\|\cdot\|_\infty - \beta(X, X')$ continuous. Note that in this case the series $\sum_{j=1}^\infty t_j x_j$ converges with respect to the original topology on X and the more strong $\tau(X, X')$, the Mackey topology. But in the case of c-convergence, a weakly c-convergent series need not be c-convergent. The following result shows that weakly c-convergent series also gives $\|\cdot\|_\infty - \beta(X, X')$ continuous operators.

THEOREM 2. *Let X be a locally convex space and $\sum x_j$ a weakly c-convergent series on X. Define $T : c \to X$ by $T\{t_j\} = w - \sum_{j=1}^\infty t_j x_j$, $\{t_j\} \in c$. Then T is a continuous linear operator and, hence, T is $\|\cdot\|_\infty - \beta(X, X')$ continuous.*

Proof. If $\{t_j\} \in c$, then

$$\sum_{j=1}^{\infty} t_j f(x_j) = \lim_n \sum_{j=1}^{n} t_j f(x_j) = \lim_n f(\sum_{j=1}^{n} t_j x_j) = f(w - \sum_{j=1}^{\infty} t_j x_j)$$

for all $f \in X'$. Suppose that $\lim_\alpha \{t_{\alpha j}\} = \{t_j\}$ in (c, weak). It is well known that $f \in c'$ if and only if there exists a $\gamma \in \mathbb{C}$ and a

$$\{\gamma_j\} \in l^1 = \{\{\delta_j\} : \sum_{j=1}^{\infty} |\delta_j| < +\infty\}$$

such that

$$f\{s_j\} = \gamma \lim_j s_j + \sum_{j=1}^{\infty} \gamma_j s_j$$

for $\{s_j\} \in c$. Therefore,

$$\lim_\alpha [\gamma \lim_j t_{\alpha j}] + \lim_\alpha \sum_{j=1}^{\infty} t_{\alpha j} \gamma_j = \gamma \lim_j t_j + \sum_{j=1}^{\infty} t_j \gamma_j$$

for every $\gamma \in \mathbb{C}$ and $\{\gamma_j\} \in c$. Letting $\gamma = 0$, we then have $\lim_\alpha \sum_{j=1}^{\infty} t_{\alpha j} \gamma_j = \sum_{j=1}^{\infty} t_j \gamma_j$ for all $\{\gamma_j\} \in l^1$.

Now let $f \in X'$ be arbitrary. By Proposition 1, $\{f(x_j)\} \in l^1$. Therefore,

$$\lim_\alpha f(T\{t_{\alpha j}\}) = \lim_\alpha f(w - \sum_{j=1}^{\infty} t_{\alpha j} x_j) = \lim_\alpha \sum_{j=1}^{\infty} t_{\alpha j} f(x_j) = \sum_{j=1}^{\infty} t_j f(x_j)$$

$$= f(w - \sum_{j=1}^{\infty} t_j x_j) = f(T\{t_j\}).$$

This shows that T is weak-weak continuous. By the Hellinger-Toeplitz theorem ([2], P. 169, Corollary. 6), T is $\beta(c, c') - \beta(X, X')$ continuous. But $\beta(c, c') = \| \cdot \|_\infty$ so T is $\| \cdot \|_\infty - \beta(X, X')$ continuous. □

A series $\sum x_j$ on a locally convex space X is said to be weakly c-Cauchy if for every $\{t_j\} \in c$, $\{\sum_{j=1}^n t_j x_j\}_{n=1}^\infty$ is a Cauchy sequence in (X, weak), i.e., for each $f \in X'$,

$$\{\sum_{j=1}^n t_j f(x_j)\}_{n=1}^\infty = \{f(\sum_{j=1}^n t_j x_j)\}_{n=1}^\infty$$

is a Cauchy sequence in $\mathbb{C}$. Clearly, $\sum x_j$ is weakly c-Cauchy if and only if for every $\{t_j\} \in c$ and $f \in X'$ the series $\sum_{j=1}^\infty t_j f(x_j)$ converges. The following result shows that a weakly c-Cauchy series on a sequentially complete locally convex space must be c_0-convergent. Note that Banach spaces are sequentially complete locally convex spaces.

THEOREM 3. *Let X be a sequentially complete locally convex space. If a series $\sum x_j$ on X is weakly c-Cauchy, then $\sum x_j$ is c_0-convergent, i.e., for each $\{t_j\} \in c_0$ the series $\sum_{j=1}^n t_j x_j$ converges.*

Proof. Suppose $\sum_{j=1}^\infty |f(x_j)| = +\infty$ for some $f \in X'$. There is an integer $n_1 > 1$ such that $\sum_{j=1}^{n_1} |f(x_j)| > 1$. There is an integer $n_2 > n_1$ such that $\sum_{j=1}^{n_2} |f(x_j)| > \sum_{j=1}^{n_1} |f(x_j)| + 2$. There is an $n_3 > n_2$ such that $\sum_{j=1}^{n_3} |f(x_j)| > \sum_{j=1}^{n_2} |f(x_j)| + 3$. Continuing this construction we have an integer sequence $1 = n_0 < n_1 < n_2 < n_3 < \cdots$ such that

$$\sum_{j=n_k+1}^{n_{k+1}} |f(x_j)| > k+1, \quad k = 0, 1, 2, 3, \cdots .$$

Let $t_1 = 0$, $t_j = \frac{1}{k+1} sgn\, f(x_j)$, $n_k < j \leq n_{k+1}$, $k = 0, 1, 2, 3, \cdots$. Then $t_j \to 0$ so $\{t_j\} \in c_0 \subseteq c$. But

$$\sum_{j=1}^N t_j f(x_j) = \sum_{j=2}^\infty t_j f(x_j) = \sum_{k=0}^N \sum_{j=n_k+1}^{n_{k+1}} \frac{1}{k+1} (sgn\, f(x_j)) f(x_j)$$

$$= \sum_{k=0}^N \frac{1}{k+1} \sum_{j=n_k+1}^{n_{k+1}} |f(x_j)| > \sum_{k=0}^N 1 = N+1,$$

for all $N \in \mathbb{N}$, i.e., $\sum_{j=1}^{\infty} t_j f(x_j)$ diverges. This contradicts that $\sum x_j$ is weakly c-Cauchy. So $\sum_{j=1}^{\infty} |f(x_j)| < +\infty$, foa all $f \in X'$. Let

$$A = \left\{ \sum_{j=1}^{n} \alpha_j x_j : n \in \mathbb{N}, |\alpha| \leq 1, 1 \leq j \leq n \right\}.$$

For every $f \in X'$,

$$\left| f(\sum_{j=1}^{n} \alpha_j x_j) \right| = \left| \sum_{j=1}^{n} \alpha_j f(x_j) \right| \leq \sum_{j=1}^{n} |\alpha_j||f(x_j)|$$

$$\leq \sum_{j=1}^{n} |f(x_j)| \leq \sum_{j=1}^{\infty} |f(x_j)| < +\infty,$$

for all $\sum_{j=1}^{n} \alpha_j x_j \in A$. This shows that A is weakly bounded and, hence, bounded by the Mackey theorem ([2], p.114, Theorem 1).

Now suppose that $\{t_j\} \in c_0$, i.e., $t_j \to 0$. Without loss of generality, we assume that for all j_0 there exists $j > j_0$ such that $t_j \neq 0$. Let U be a neighborhood of $0 \in X$. Letting $\alpha_k = sup_{j \geq k}|t_j|$, $\alpha_k \to 0$. Since A is bounded, there is a $\delta > 0$ such that $\alpha A \subseteq U$ for all $|\alpha| \leq \delta$. Since $\alpha_k \to 0$, there is a $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, then $|\alpha_k| \leq \delta$. Therefore, if $m > k \geq k_0$, then

$$\begin{aligned} \sum_{j=k}^{m} t_j x_j &= \alpha_k \sum_{j=k}^{m} \frac{t_j}{\alpha_k} x_j \\ &= \alpha_k \left(0x_1 + 0x_2 + \cdots + 0x_{k-1} + \sum_{j=k}^{m} \frac{t_j}{\alpha_k} x_j \right) \\ &\in \alpha_k A \subseteq U. \end{aligned}$$

This shows that $\{\sum_{j=1}^{n} t_j x_j\}_{n=1}^{\infty}$ is Cauchy and, hence, the series $\sum_{j=1}^{\infty} t_j x_j$ converges because X is sequentially complete. $\square$

THEOREM 4. *Let X be a sequentially complete locally convex space. For a series $\sum x_j$ on X, the following conditions are equivalent.*

(1) *$\sum x_j$ is a weakly unconditional Cauchy series, i.e., for all $f \in X'$, $\sum_{j=1}^{\infty} |f(x_j)| < +\infty$.*
(2) *For every $\{t_j\} \in l^{\infty}$, $\{\sum_{j \in \Delta} t_j x_j : \Delta \subseteq \mathbb{N} \quad \text{finite}\}$ is bounded.*
(3) *$\sum x_j$ is c_0-convergent, i.e., for every $\{t_j\} \in c_0$, the series $\sum_{j=1}^{\infty} t_j x_j$ converges.*
(4) *$\sum x_j$ is weakly c_0-Cauchy, i.e., the series $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in c_0$ and $f \in X'$.*
(5) *$\sum x_j$ is weakly c-Cauchy, i.e., the series $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in c$ and $f \in X'$.*
(6) *$\{\sum_{j=1}^{n} t_j x_j : n \in \mathbb{N}, |t_j| \leq 1, \quad 1 \leq j \leq n\}$ is bounded.*

Proof. By Theorem 2 of [1], (1)=(2)=(3) since X is sequentially complete. Since $c_0 \subseteq c$, (5)⇒(4). As in the proof of Theorem 3, (4) ⇒ (1) ⇒ (6) ⇒ (3) ⇒ (4). So (1)=(2)=(3)=(4)=(6) and (5)⇒(4). Suppose (4) holds. Then (1) holds because (1)=(4), i.e., $\sum_{j=1}^{\infty} |f(x_j)| < +\infty$, for all $f \in X'$. Since $\{t_j\} \in c \Rightarrow \{t_j\}$ is bounded,

$$\sum_{j=1}^{\infty} |t_j f(x_j)| = \sum_{j=1}^{\infty} |t_j||f(x_j)| \leq sup_j |t_j| \sum_{j=1}^{\infty} |f(x_j)| < +\infty.$$

This shows that $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for all $\{t_j\} \in c$. □

COROLLARY 5. *If X is a sequentially complete locally convex space, then (1)=(2)=(3)=(4)=(5)=(6)=(7)=(8)=(9)=(10).*
(7) $\sum_{j=1}^{\infty} |t_j f(x_j)| < +\infty$, for all $\{t_j\} \in c_0$, $f \in X'$.
(8) $\sum_{j=1}^{\infty} |t_j f(x_j)| < +\infty$, for all $\{t_j\} \in c$, $f \in X'$.
(9) $\sum_{j=1}^{\infty} |t_j f(x_j)| < +\infty$, for all $\{t_j\} \in l^{\infty}$, $f \in X'$.
(10) $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in l^{\infty}$, and $f \in X'$.

Proof. $\{t_j\} \in l^{\infty} \Rightarrow \{t_j \, sgn \, f(x_j)\} \in l^{\infty}$, so (9)=(10).
(1)⇒(9)⇒(8)⇒(7)⇒(4)⇒(1). □

Now we give the main result of this paper.

THEOREM 6. *Let X be a sequentially complete locally convex space. The following conditions are equivalent.*

(a) *X contains no copy of c_0.*
(b) *Each weakly c_0-Cauchy series on X is c-convergent, i.e., if $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in c_0$ and $f \in X'$, then $\sum_{j=1}^{\infty} t_j x_j$ converges for each $\{t_j\} \in c$.*
(c) *Each weakly c-Cauchy series on X is c-convergent, i.e., if $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in c$ and $f \in X'$, then $\sum_{j=1}^{\infty} t_j x_j$ converges for each $\{t_j\} \in c$.*

Proof. (a)⇒(b). Suppose $\sum_{j=1}^{\infty} \alpha_j f(x_j)$ converges for every $\{\alpha_j\} \in c_0$ and $f \in X'$. Let $\{t_j\} \in c$. Then $\alpha_j t_j \to 0$ for each $\{\alpha_j\} \in c_0$ so $\sum_{j=1}^{\infty} \alpha_j f(t_j x_j)$ converges for every $\{\alpha_j\} \in c_0$ and $f \in X'$. By theorem 4 ((3)=(4)), $\sum_{j=1}^{\infty} \alpha_j t_j x_j$ converges for each $\{\alpha_j\} \in c_0$, i.e., $\{t_j x_j\} \in CMC(X)$ (see [3]). Since X contains no copy of c_0, by Theorem 4 of [3], $\sum_{j=1}^{\infty} t_j x_j$ converges, i.e., (b) holds.
(b)⇒(c) : $c_0 \subseteq c$.
(c)⇒(a). Suppose X contains a copy of c_0. Say that $c_0 \subseteq X$. Let e_j denotes the sequence that has 1 at the j-th spot and 0 elsewhere, i.e., $e_j = (0, \cdots, 0, 1, 0, 0, \cdots)$. For every $\{t_j\} \in c$ and $f = \{\alpha_j\} \in l^1 = c_0'$,

$$\begin{aligned}
\sum_{j=1}^{n} |t_j f(e_j)| = \left|\sum_{j=1}^{n} f(t_j e_j)\right| &= \left|f(\sum_{j=1}^{n} t_j e_j)\right| \\
&= |f(t_1, t_2, \cdots, t_n, 0, 0, \cdots)| = \left|\sum_{j=1}^{n} t_j \alpha_j\right| \\
&\le \sum_{j=1}^{n} |t_j||\alpha_j| \le sup_j |t_j| \sum_{j=1}^{n} |\alpha_j| \\
&\le sup_j |t_j| \sum_{j=1}^{n} |\alpha_j| < +\infty,
\end{aligned}$$

for all $n \in \mathbb{N}$, i.e., for every $\{t_j\} \in c$ and $f \in c_0'$, $\sum_{j=1}^{\infty} t_j f(e_j)$ converges. However, letting $t_j = 1$ for all j, $\{t_j\} = \{1\} \in c$ but the series $\sum_{j=1}^{\infty} e_j$

diverges in c_0 :

$$\|\sum_{j=m}^{n} e_j\|_\infty = \|(0,\cdots,0,1,1,,\cdots,1,0,0,\cdots)\|_\infty = 1$$

for all $1 \leq m < n < +\infty$. If $\lim_n \sum_{j=1}^{n} e_j = x \in X\backslash c_0$, then

$$\lim_{m,n\to\infty} \|\sum_{j=m}^{n} e_j\|_\infty = 0.$$

So $\sum_{j=1}^{\infty} e_j$ diverges in X. This contradicts (c). □

COROLLARY 7. *If a sequentially complete locally convex space X contains no copy of c_0, then every weakly c-convergent series on X is c-convergent.*

By Theorem 4 of [3], we have

THEOREM 8. *Let X be a sequentially complete locally convex space. The followings are equivalent.*

(1°) *X contains no copy of c_0.*
(2°) *Each weakly c_0-Cauchy series on X is bounded multiplier convergent, i.e., if $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in c_0$ and $f \in X'$, then $\sum_{j=1}^{\infty} t_j x_j$ converges for each $\{t_j\} \in l^\infty$, the family of bounded number sequences.*

Proof. (1°)⇒(b). So if $\sum_{j=1}^{\infty} t_j f(x_j)$ converges for every $\{t_j\} \in c_0$ and $f \in X'$, then $\{x_k\} \in CMC(X)$ but (1°)⇒ $CMC(X) = BMC(X)$ by Theorem 4 of [3]. □

References

1. Li Ronglu and Min-Hyung Cho, *Weakly Unconditional Cauchy Series on Locally Convex Spaces*, Northeast Math. J., **11(2)** (1995), 187-190.
2. A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, New York (1978).

3. Li and Bu, *Locally Convex Spaces Containing No Copy of* c_0, J. Math. Anal. Appl., **172(1)** (1993), 205-211.

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