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A NOTE ON NORMAL SUBGROUPS OF M-GROUPS

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Abstract. For an M-group G, it is shown that a normal subgroup of G whose order is coprime to its index is an M-group.

A finite group is an *M-group* if every irreducible complex character is induced from a linear (i.e., degree 1) character of a subgroup. It is known that all M-groups are solvable (*cf.* [3]), but no non-character theoretic description of the class of M-groups has been found. Part of the difficulty of finding such a group theoretic characterization is undoubtedly related to the fact that subgroups of M-groups need not, themselves, be M-groups. We have an interesting question.

Question. *For an M-group G, is every normal subgroup of G an M-group?*

In this note, we give a partial answer for the question: if G is an M-group, then the normal subgroup N of G with $(|N|, |G : N|) = 1$, is an M-group. All groups in this note are assumed to be finite. Let $Irr(G)$ be the set of all irreducible complex characters of G.

Let N be a normal subgroup of G and let $\theta \in Irr(N)$ be invariant in G. Under these hypotheses we say that (G, N, θ) is a *character triple* (*cf.*[3]).

Let $Ch(G|\theta)$ denotes the set of characters χ of G such that χ_N is a multiple of θ. Let $Irr(G|\theta)$ be the set of irreducible constituents of θ^G. Note that if $N \subseteq H \subseteq G$, then (H, N, θ) is a character triple and $\chi_H \in Ch(H|\theta)$ whenever $\chi \in Ch(G|\theta)$. If $\tau : U \longrightarrow V$ is an isomorphism of groups and $\phi \in Irr(U)$, let $\phi^\tau \in Irr(V)$ denote the corresponding character, so that $\phi^\tau(u^\tau) = \phi(u)$, for all $u \in U$.

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Definition 1. Let (G, N, θ) and (Γ, M, ϕ) be character triples. A pair (τ, σ) is called an isomorphism of character triples (G, N, θ) and (Γ, M, ϕ) if $\tau : G/N \longrightarrow \Gamma/M$ is an isomorphism and σ is the union of the maps σ_H, where σ_H is defined as follows; for $N \subseteq H \subseteq G$, let H^τ denote the inverse image in Γ of $\tau(H/N)$, for every such H, $\sigma_H : Ch(H|\theta) \longrightarrow Ch(H^\tau|\phi)$ is the map which satisfies the following conditions for H, K with $N \subseteq K \subseteq H \subseteq G$ and $\chi, \phi \in Ch(H|\theta)$

(1) $\sigma_H(\chi + \phi) = \sigma_H(\chi) + \sigma_H(\phi)$,
(2) $[\chi, \phi] = [\sigma_H(\chi), \sigma_H(\phi)]$,
(3) $\sigma_K(\chi_K) = (\sigma_H(\chi))_{K^\tau}$,
(4) $\sigma_H(\chi\beta) = \sigma_H(\chi)\beta^\tau$ for $\beta \in Irr(H/N)$.

Lemma 2. ([3]). *Let (τ, σ) be an isomorphism of character triples (G, N, θ) and (Γ, M, ϕ). Then σ_H is a bijection of $Ch(H|\theta)$ onto $Ch(H^\tau|\phi)$ for all H with $N \subseteq H \subseteq G$. Furthermore, $\chi(1)/\theta(1) = \sigma_H(\chi)(1)/\phi(1)$ for all $\chi \in Ch(H|\theta)$.*

Proof. If $\sigma_H(\chi_1) = \sigma_H(\chi_2)$ for $\chi_i \in Ch(H|\theta)$, we have $[\chi_i, \phi] = [\sigma_H(\chi_i), \sigma_H(\phi)]$ is independent of i for all $\phi \in Irr(H|\theta)$. It follows that $\chi_1 = \chi_2$ and hence σ_H is one-to-one.

For $\chi \in Ch(H|\theta)$, write $e(\chi) = \chi(1)/\theta(1)$ and similarly set $e(\eta) = \eta(1)/\phi(1)$ for $\eta \in Ch(H^\tau|\phi)$. Note that $\sigma_\eta(\theta) \in Irr(M|\phi)$ and so $\sigma_N(\theta) = \phi$. We have $\chi_N = e(\chi)\theta$ and $\eta_M = e(\eta)\phi$ and thus

$$e(\sigma_N(\chi))\phi = (\sigma_H(\chi))_M = \sigma_N(\chi_N) = \sigma_N(e(\chi)\theta) = e(\chi)\phi$$

which implies

$$e(\sigma_H(\chi)) = e(\chi)$$

By Frobenius reciprocity, we have

$$\theta^H = \sum_{\chi \in Irr(H|\theta)} e(\chi)\chi$$

and comparing degrees yields $\sum e(\chi)^2\theta(1) = |H : N|\theta(1)$ so that $\sum e(\chi)^2 = |H : N|$ where χ runs over $Irr(H|\theta)$. Similarly, $\sum e(\eta)^2 = |H^\tau : M| = |H : N|$ for $\eta \in Irr(H^\tau|\phi)$. Since σ_H maps $Irr(H|\theta)$ one-to-one into $Irr(H|\theta)$, we have

$$|H : N| = \sum e(\chi)^2 = \sum e(\sigma_H(\chi))^2 \leq \sum e(\eta)^2 = |H : N|.$$

It follows that every $\eta \in Irr(H^\tau|\phi)$ is of the form $\sigma_H(\chi)$ for some $\chi \in Irr(H|\theta)$ which proves the Lemma. □

Proposition 3. ([3]). *Let* $N \lhd G$ *and* $\chi \in Irr(G)$. *If* $\theta \in Irr(N)$ *is a constituent of* χ_N *then* $\chi(1)/\theta(1)$ *divides* $|G:N|$.

Proof. Let $T = I_G(\theta)$, the inertia group ($cf.$[3]) and let $\phi \in Irr(T)$ such that $\phi^G = \chi$ and $\phi_N = e\theta$ by the Clifford's theorem. Since $\chi(1) = |G:T|\phi(1)$, it suffices to show that $\phi(1)/\theta(1)$ divides $|T:N|$. Let (Γ, A, λ) be a character triple isomorphic to (T, N, θ) with λ linear. Let $\zeta \in Irr(\Gamma|\lambda)$ correspond to $\phi \in Irr(T|\theta)$. Then $\phi(1)/\theta(1) = \zeta(1)/\lambda(1) = \zeta(1)$ by Lemma 2. Since $A \subseteq Z(\zeta)$, we have $\zeta(1)$ divides $|\Gamma : A| = |T:N|$. □

Corollary 4. *Let* $N \lhd G$ *and* $\chi \in Irr(G)$. *If* $(\chi(1), |G:N|) = 1$, *then* χ_N *is irreducible.*

Proof. Let θ be an irreducible constituent of χ_N. Then by Proposition 3, $\chi(1)/\theta(1)$ divides $|G:N|$. Hence we have $\chi(1)/\theta(1) = 1$ since $(\chi(1), |G:N|) = 1$. So, $\chi(1) = \theta(1)$. Thus $\chi_N = \theta$ is irreducible. □

Theorem 5. *Let* G *be an* M*-group and suppose* $N \lhd G$ *with* $(N, |G:N|) = 1$. *Then* N *is an* M*-group.*

Proof. Let $\theta \in Irr(N)$ and let χ be an irreducible constituent of θ^G. Since G is an M-group, χ is a monomial. So $\chi = \lambda^G$ where $\lambda \in Irr(H)$ is linear for some $H \subseteq G$.

Let $\phi = \lambda^{NH}$. Then we have

$$\phi^G = (\lambda^{NH})^G = \lambda^G = \chi \in Irr(G).$$

Thus $\phi \in Irr(NH)$. Hence we get

$$\phi(1) = \lambda^{NH}(1) = |NH : H|\lambda(1) = |NH : H| = |N : N \cap H|.$$

This divides $|N|$. Since $|N|$ is coprime to $|G:N|$, we have $(\phi(1), |G:N|) = 1$. But $|NH:N|$ divides $|G:N|$. Thus we get $(\phi(1), |NH:N|) = 1$.

By Corollary 4, we obtain that ϕ_N is irreducible. But

$$\phi_N = (\lambda^{NH})_N = (\lambda_{N\cap H})^N.$$

So, ϕ_N is monomial. Since $\phi^G = \chi$, by Frobenius Reciprocity, ϕ is a constituent of χ_{NH}. Thus ϕ_N is an irreducible constituent of $(\chi_{NH})_N = \chi_N$. Since θ is an irreducible constituent of χ_N, by Clifford theorem $\theta = (\phi_N)^g = (\lambda^g_{N\cap H})^N$; i.e., induced from linear character for some $g \in G$. Hence θ is an monomial and the proof is completed. $\square$

Corollary 6. *Let G be an M-group. Then the Sylow p-subgroup of G is an M-group, where p is prime.*

Proof. Let S be a Sylow p-subgroup of G. Then $S \lhd G$ and $(|S|, |G : S|) = 1$. Thus by Theorem 5, S is an M-group. $\square$

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