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A REMARK ON FORMALITY

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Abstract. In this paper we prove two independent theorems concerning formality of a nilmanifold and a differential graded algebra using the well-known theorem of Deligne-Griffiths-Morgan-Sullivan. We first give a rational homotopy theoretic proof to the statement that a nilmanifold is formal if and only if it is a torus. And then we study some conditions with which formality of one dga implies formality of the other in an extension of dga's.

1. Minimal models and KS-extensions

We recall here the basic facts and notation we shall need from Sullivan's theory of minimal models, for which the basic references are [3, 4, 8]. We assume the reader to be familiar with the basics of *d*ifferential *g*raded *a*lgebras [2] over a field k of characteristic 0.

Definition. A dga (M, d) is called *minimal*, if :

i) $M = \Lambda V$ is freely generated for some graded k-vector space V;

ii) d is decomposable in the following sense : there exists an ordering in the set $\{x_a, a \in I\}$ of all free generators of M such that $x_\beta < x_a \Longrightarrow \deg(x_\beta) < \deg(x_a)$ and such that $dx_a \in \Lambda(V_{<a})$, $V_{<a}$ denoting the span of the $x_\beta < x_a$.

Notation. If $\{x_1, x_2, \dots\}$ is a basis for V, then we write $V = \langle x_1, x_2, \dots \rangle$ and $\Lambda V = \Lambda(x_1, x_2, \dots)$.

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Remark. When $A = \Lambda V$ is connected, that is $A^0 = k$, ii) is equivalent to $d : V \to \Lambda^{\geq 2} V$. $\Lambda^{\geq m} V$ denotes the differential ideal of ΛV having additive basis the monomials $x_{i_1} \cdots x_{i_k}$ with $k \geq m$.

Definition. i) A minimal model for a dga A is a minimal dga M_A and a dga map $\rho_A : M_A \to A$ such that the induced homomorphism on cohomology ρ_A^* is an isomorphism.

ii) A minimal model for a space X is a minimal model of the dga $A^*(X)$, the rational polynomial forms on X.

Example 1. i) $\Lambda(CP(n)) = \Lambda(x_2, y_{2n+1}),\ dy = x^{n+1}$.
ii) $\Lambda(T^n) = \Lambda(x_1^1, x_1^2, \ldots, x_1^n),\ d = 0$.

The aim of the second part of this section is to describe the *algebraic fibrations*, which serve as models for fibrations [9]. Only augmented algebras are considered, that is, (A, d_A) is always endowed with a homomorphism $\varepsilon : A \to k$ such that $\operatorname{Ker} \varepsilon = \oplus_{k>0} A^k$.

Definition. A KS-extension is a sequence of augmentation preserving dga morphisms

$$(A, d_A) \xrightarrow{\imath} (A \otimes \Lambda V, d) \xrightarrow{\rho} (\Lambda V, d)$$

with the following conditions:

i) $\imath(a) = a \otimes 1,\ \rho = \varepsilon_A \otimes id_{\Lambda V}$, where ε_A is the augmentation of A.

ii) there exists an ordered homogeneous basis $\{x_a : a \in I\}$ for V indexed by a well ordered set I such that $d(1 \otimes x_a) \in A \otimes \Lambda(V_{<a})$.

We will also call simply $(A, d_A) \xrightarrow{\imath} (A \otimes \Lambda V, d)$ a KS-extension.

2. Minimal model of a nilmanifold

Definition. A *nilmanifold* M is a compact homogeneous space of the form N/π where N is a simply connected Lie group and π is a lattice, that is, a discrete co-compact subgroup of N.

It is well known that N is diffeomorphic to some $\mathbb{R}^n$ and therefore, M is $K(\pi, 1)$. Furthermore, this entails the fact that π is a finitely generated torsion free nilpotent group.

The general theory of nilmanifolds is contained in [6]. We only need a minimal model of a nilmanifold. Following [7] we decompose $M = K(\pi, 1)$ into a tower S^1-bundles

$$S^1 \to M_{i-1} \xrightarrow{\tau_i} \mathbb{C}P(\infty), \quad i = 2, \ldots, n$$

which is, in fact, the Postnikov decomposition of M with k-invariants the τ_i. Note that $[M_{i-1}, \mathbb{C}P(\infty)] = [M_{i-1}, K(\mathbb{Z}, 2)] = H^2(M_{i-1}; \mathbb{Z})$.

Lemma 1. [7] *The minimal model of a nilmanifold M^n of dimension n has the form*

$$\Lambda(M^n) = (\Lambda(x_1, \ldots, x_n),\ d),\ \deg(x_i) = 1$$

with $dx_i = \tau_i$, where τ_i is a cocycle representing the class $\tau_i \in H^2(M_{i-1}; \mathbb{Z})$.

3. Formality of a dga and the theorem of Deligne-Griffiths-Morgan-Sullivan

The basic reference for this section is [1]. Let M be a minimal dga and $H^*(M)$ the cohomology of M viewed as a dga with the differential 0.

Definition. i) M is *formal* if there is a dga map $\Psi : M \to H^*(M)$ inducing the identity on cohomology.

ii) A dga (A, d_A) is a *formal consequence* of its cohomology algebra if its minimal model is formal.

iii) A smooth manifold M is *formal* if the de Rham algebra $\Omega^*(M)$ is a formal consequence of its cohomology algebra.

Example 2. Consider the 3-dimensional Heisenberg group $U_3(\mathbb{R})$ and mod out by $U_3(\mathbb{Z})$. The resulting manifold M is a 3-manifold obtained as a principle bundle,

$$S^1 \to M \to T^2.$$

The minimal model of M is given by

$$\Lambda(M) = \Lambda(x, y, z),\ \deg(x) = \deg(y) = \deg(z) = 1$$

with $dx = 0 = dy$ and $dz = xy$. Thus xz, for example, is closed but not exact. But since $x \cdot H^1(M) = 0$, there can be no map of $M \to H^*(M)$ inducing the identity in cohomology. Hence M is not formal.

We will use the following criterion for formality.

LEMMA 2. (Deligne-Griffiths-Morgan-Sullivan) [1] *A minimal dga $(\Lambda V, d)$ is formal if and only if V decomposes as a direct sum $V = C \oplus N$ with $d(C) = 0$ and d injective on N such that every closed element in the ideal generated by N is exact.*

Nonexact cocycles in the ideal (N) are called *Massey products*.

4. Main theorems

We now present our main theorems.

THEOREM 1. *A nilmanifold M^n is formal if and only if it is a torus.*

Proof. Since the minimal model of a torus is given by the dga $(\Lambda(x_1, \ldots, x_n),\ 0)$ where each x_i has degree one and the differential is 0 (See Example 1), it is clearly formal. Conversely, let M have a minimal model of the form $(\Lambda(x_1, \ldots, x_n), d)$ where $\deg(x_i) = 1$, $i = 1, \ldots, n$ and $d \neq 0$. Then there exists k such that $dx_1, = \cdots = dx_{k-1} = 0$, $dx_k \neq 0$. By the minimality condition dx_k can be written as $dx_k = \sum_{i<j<k} x_i x_j$. Let $\{x_{i_1}, x_{i_2}, \ldots, x_{i_l}\}$, $i_1 < i_2 < \cdots < i_l$, be the set of different $x_i' s$ appearing in the summation of dx_k. We may assume that $l > 2$ (See Example 2). Consider the element $a = \Sigma x_{i_1}, \cdots \hat{x}_{i_s} \cdots x_{i_l}$ ranging all the permutations of $\{i_1, \ldots, i_l\}$. Suppose that ΛV is formal. Then V has a decomposition $V = C \oplus N$ as in Lemma 2. Then clearly $ax_k \in (N)$. Note that $d(ax_k) = (da)x_k \pm adx_k = \pm(\Sigma x_{i_1} \cdots \hat{x}_{i_s} \cdots x_{i_l})(\Sigma x_i x_j) = 0$ since each term reduces to 0. But it is not hard to see that ax_k is not a coboundary, which is a contradiction. Hence $d = 0$, and M has the rational homotopy type of a torus. We now follow the argument in [7] P.204 to conclude that M has the homotopy type of a torus. □

THEOREM 2. *Let $i : (\Lambda V, d) \to (\Lambda V \otimes \Lambda W, D)$ be a KS-extension. Then we have the followings:*

i) *if* $(\Lambda V, d)$ *is formal and* $\imath^*$ *is an epimorphism, then* $(\Lambda V \otimes \Lambda W, D)$ *is formal and,*

ii) *if* $(\Lambda V \otimes \Lambda W, D)$ *is formal and* $\imath^*$ *is a monomorphism, then* $(\Lambda V, d)$ *is formal.*

Proof. 1) Since $(\Lambda V, d)$ is formal, there exists a dga map $\Phi : \Lambda V \to H^*(\Lambda V)$ such that $\Phi^* = id$. We proceed by induction on the number n of generators of W. When $n = 1$, that is $W = \langle y \rangle$, define a map $\Psi : \Lambda V \otimes \Lambda(y) \to H^*(\Lambda V \otimes \Lambda(y))$ by $\Psi|_{\Lambda V} = i^*\Phi$ and $\Phi(y) = 0$. Ψ is indeed a dga map since $dy \in Z(\Lambda V)$, the cocycles in ΛV. Since $\imath^*$ is onto, each element in $H^*(\Lambda V \otimes \Lambda(y))$ has a preimage which maps identically into itself by $\Phi^* = id$. Hence, $\Phi = id$. Now assume that the statement is true when $n = k-1$ and $i_1^* : H^*(\Lambda V) \to H^*(\Lambda V \otimes \Lambda(y_1, \ldots, y_{k-1}, y_k))$ is an epimorphism. Note that $i_2^* : H^*(\Lambda V \otimes \Lambda(y_1, \ldots, y_{k-1})) \to H^*(\Lambda V \otimes \Lambda(y_1, \ldots, y_k))$ is also an epimorphism and $dy_k \in \Lambda V \otimes \Lambda(y_1, \ldots, y_{k-1})$. Repeating the above argument we conclude that $\Lambda V \otimes \Lambda(y_1, \ldots, y_k)$ is formal.

2) Suppose that $(\Lambda V \otimes \Lambda W, D) = (\Lambda(V \oplus W), D)$ is formal. By Lemma 2 there exists a decomposition $V \oplus W = C \oplus N$ with $D(C) = 0$ and D is injective on N such that every closed element in (N), the ideal generated by N in $\Lambda V \otimes \Lambda W$, is exact. By taking $C' = C \cap V$ and $N' = N \cap V$ we have $d(C) = 0$ and d is injective on N' since $D|_V = d$. Let $a \in (N')$, the ideal generated by N' in ΛV, and $da = 0$. Since $a \in (N') \subset (N)$, $a = Db$ for some $b \in \Lambda V \otimes \Lambda W$. Since $\imath^*([a]) = [Db] = 0$ and $\imath^*$ is a monomorphism we have $[a] = 0$. Hence $a = da'$ for some $a' \in \Lambda V$, which completes the proof. $\square$

Remark. For any non-formal dga $(\Lambda V, d)$ we may continuously add generators to kill the Massey products producing an extension $(\Lambda V \otimes \Lambda W, D)$ which is formal. Clearly $\imath^* : H^*(\Lambda V, d) \to H^*(\Lambda V \otimes \Lambda W, D)$ is not a monomorphism.

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