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ON NONVANISHING SUM OF INTEGRAL SQUARES OF $\mathbb{Q}(\sqrt{5})$

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Abstract. In this paper, we will determine all the totally positive integers which cannot be represented by the sum of k nonvanishing integral squares when $k \geq 4$.

1. Introduction

In 1770, Lagrange proved every positive integer is represented by the sum of four squares. In 1911, Dubouis[1] determined all the positive integers which cannot be represented by the sum of k nonvanishing squares when $k \geq 4$. If $k = 4$, they are 1, 3, 5, 9, 11, 17, 29, 41, $2 \cdot 4^n$, $6 \cdot 4^n$, $10 \cdot 4^n$ and $14 \cdot 4^n$ where $n \in \mathbb{Z}^+ \cup \{0\}$. If $k = 5$, they are 1, 2, 3, 4, 6, 7, 9, 10, 12, 15, 18 and 33. If $k \geq 6$, they are $1, 2, \dots, k-1, k+1, k+2, k+4, k+5, k+7, k+10$ and $k+13$. In 1941, Maass[3] proved every totally positive integers of $\mathbb{Q}(\sqrt{5})$ is represented by the sum of three squares. Four years later, Siegel[4] proved $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$ are the only totally real number fields all of whose totally positive integers are represented by the sum of (arbitrary number of) squares. In this paper, we will determine all the totally positive integers which cannot be represented by the sum of nonvanishing k squares when $k \geq 4$.

2. Main result

Two algebraic integers a, b of number field K is said to be equivalent if $a = bu^2$ for some unit u of K.

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Lemma 1. *Every totally positive integer of* $\mathbb{Q}(\sqrt{5})$ *is equivalent to* $a+b\epsilon^2$ *for some* a, $b\in\mathbb{Z}^+\cup\{0\}$ *where* $\epsilon=\frac{1+\sqrt{5}}{2}$.

proof. Let $S=\{a_l\epsilon^{2l}+a_{l+1}\epsilon^{2l+2}+\ldots+a_k\epsilon^{2k}\mid l,k\in\mathbb{Z}, l\le k$ and a_l, $a_{l+1},\ldots,a_k\in\mathbb{Z}^+\cup\{0\}\}$. Then, trivially $1\in S$. Suppose $\alpha=a+b\sqrt{5}$ be a totally positive integer of $\mathbb{Q}(\sqrt{5})$ with minimal trace among the totally positive integers of $\mathbb{Q}(\sqrt{5})$ which do not belong to S. If $\alpha-1$ is totally positive, trivially $\alpha\in S$. If $\alpha-1$ is not totally positive, $a>1$ and $b>0$. So $b\ge\frac{1}{2}$ and $a<b\sqrt{5}+1\le b(\sqrt{5}+2)\le 5b$. So $\mathrm{tr}(\alpha\epsilon^{-2})=\mathrm{tr}(\frac{3a-5b+(a+3b)\sqrt{5}}{2})=3a-5b<2a=\mathrm{tr}(\alpha)$. So $\alpha\epsilon^{-2}\in S$ and thus $\alpha\in S$, which is a contradiction. So every totally positive integer belongs to S. Let $\alpha\in S$. We choose a representation $\alpha=a_l\epsilon^{2l}+a_{l+1}\epsilon^{2l+2}+\ldots+a_k\epsilon^{2k}$ such that $k-l$ is minimal. If $k-l\ge 2$ and $a_l\ge a_k$, then

$$\alpha=(a_l-a_k)\epsilon^{2l}+(a_{l+1}+2a_k)\epsilon^{2l+2}+(a_{l+2}+a_k)\epsilon^{2l+4}$$
$$+\ldots+(a_{k-2}+a_k)\epsilon^{2k-4}+(a_{k-1}+2a_k)\epsilon^{2k-2}$$

because

$$\epsilon^{2k}=2\epsilon^{2k-2}+\epsilon^{2k-4}+\epsilon^{2k-6}+\ldots+\epsilon^{2l+4}+2\epsilon^{2l+2}-\epsilon^{2l}.$$

This is a contradiction. A similar contradiction can be deduced if $k-l\ge 2$ and $a_l\le a_k$, contradiction holds again. So $k-l\le 1$. This proves the Lemma. □

Remark. The author[2] proved that if $D=n^2-1$ and D is square-free, every totally positive integer of $\mathbb{Q}(\sqrt{D})$ is equivalent to $a+b\epsilon$ for some a, $b\in\mathbb{Z}\cup\{0\}$ where $\epsilon=n+\sqrt{D}$. His method of proof is similar to the above.

Theorem 1. *Let* $k\ge 4$, *and* α *be a totally positive integer of* $\mathbb{Q}(\sqrt{5})$. *Then,* α *cannot be represented by the sum of nonvanishing* k *squares if and only if* α *is equivalent to* $a+b\epsilon^2$ *for some* a, $b\in\mathbb{Z}^+\cup\{0\}$ *with* $a+b<k$ *or* α *is equivalent to one of* $1+5\epsilon^2$ *and* $5+\epsilon^2$ *and* $k=4$.

Proof. Let S_k be the set of all elements which is represented by the sum of nonvanishing k squares. We will prove that if $\alpha = p + q\sqrt{5}(p, q \in \mathbb{Q})$ is totally positive, $\alpha \notin S_k$ for all $k \geq p - q + 1$. Let $\beta = r + s\sqrt{5}$ be a totally positive integer of $\mathbb{Q}(\sqrt{5})$ such that $r - s$ is minimal among the integers of $\mathbb{Q}(\sqrt{5})$ such that $\beta \in S_k$ for some $k \geq r - s + 1$. Let

$$\alpha = \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_k^2$$

where $\alpha_1\alpha_2 \ldots \alpha_k \neq 0$. If $\alpha_1^2 = t + u\sqrt{5}$, $t - u \geq 1$. So $\beta - \alpha_1^2 = (r-t) + (s-u)\sqrt{5}$ and $r - t - (s - u) = r - s - (t - u) \leq r - s - 1$. But $\beta - \alpha_1^2 \in S_{k-1}$ and $k - 1 \geq r - s - 1$, which is a contradiction. So we proved our assertion. Thus if $a + b < k$, as $a + b\epsilon^2 = (a + \frac{3b}{2}) + \frac{b}{2}\sqrt{5}$, $a + b\epsilon^2 \notin S_k$. We can easily see that $1 + 5\epsilon^2$ and $5 + \epsilon^2$ are not represented by the sum of nonvanishing k squares.

Conversely if $\alpha = a + b\epsilon^2 \in S_4$ and $a \geq 9$, by Maass' Theorem $\alpha - 9 = (a - 9) + b\epsilon^2$ is represented by the sum of three integral squares of $\mathbb{Q}(\sqrt{5})$. If $\alpha - 9 = 0$, $\alpha = 9 = \epsilon^2 + (\epsilon^{-1})^2 + (\sqrt{5})^2 + 1^2$. If $\alpha - 9 = \alpha_1^2$ for $\alpha_1 \neq 0$, $\alpha = \alpha_1^2 + 9 = \alpha_1^2 + 2^2 + 2^2 + 1^2$. If $\alpha - 9 = \alpha_1^2 + \alpha_2^2$ for $\alpha_1\alpha_2 \neq 0$, $\alpha = \alpha_1^2 + \alpha_2^2 + 9 = \alpha_1^2 + \alpha_2^2 + 2^2 + (\sqrt{5})^2$. If $\alpha - 9 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$ for $\alpha_1\alpha_2\alpha_3 \neq 0$, $\alpha = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 9 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 3^2$. So $\alpha \in S_4$. Similarly we can prove if $\alpha = a + b\epsilon^2 \in S_4$ and $b \geq 9$, $\alpha \in S_4$. And by brute force computation, we can show that if $\alpha = a + b\epsilon^2$, $a, b < 9$, $a + b \geq 4$, $\alpha \neq 1 + 5\epsilon^2$ and $\alpha \neq 5 + \epsilon^2$, $\alpha \in S_4$. So the Theorem is true for $k = 4$. Let $\alpha = p + q\epsilon^2 \notin S_l$ for some $4 < l \leq p + q$. Then, at least one of $\alpha - 1$ and $\alpha - \epsilon^2$ is totally positive and different from $1 + 5\epsilon^2$ and $5 + \epsilon^2$. So $\alpha - 1 \in S_{l-1}$ or $\alpha - \epsilon^2 \in S_{l-1}$, which is a contradiction. This proves the Theorem. □

References

[1] E. Dubouis, *Solution of a problem of J. Tannery*, Intermédiaire Math. **18** (1911), 55 – 56.

[2] Byeong Moon Kim, *Positive Universal Forms over Totally Real Fields, Thesis*, Seoul National University, 1997.

[3] H. Maass, *Über die Darstellung total positiver des Körpers $R(\sqrt{5})$ als Summe von drei Quadraten*, Abh. Math. Sem. Hamburg **14** (1941), 185 – 191.

[4] C. L. Siegel, *Sums of m-th Powers of Algebraic Integers*, Annals of Math. **46** (1945), 313 – 339.

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