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A UNIFORM ESTIMATE ON CONVOLUTION OPERATORS WITH THE ARCLENGTH MEASURE ON NONDEGENERATE SPACE CURVES

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Abstract. The L^p-L^q mapping properties of convolution operators with measures supported on curves in $\mathbb{R}^3$ have been studied by many authors. Oberlin provided examples of nondegenerate compact space curves whose arclength measures enjoy L^p-improving properties. This was later extended by Pan who showed that such properties hold for all nondegenerate compact space curves. In this paper, we will prove that the operator norm of the convolution operator with the arclength measure supported on a nondegenerate compact space curve depends only on certain quantities of the underlying curve.

1. Introduction

Let $\Gamma : I \to \mathbb{R}^3$ be a (sufficiently) smooth curve and consider the (euclidean) arclength measure μ_Γ on $\mathbb{R}^3$ associated with Γ. The mapping properties of the convolution operator T_{μ_Γ} defined by

$$(1) \qquad T_{\mu_\Gamma} f(x) = \int_{\mathbb{R}^3} f(x-y) \quad d\mu_\Gamma(y) = \int_I f\big(x-\Gamma(t)\big)\, dt$$

have been studied by many authors.

Oberlin [1] showed that T_{μ_Γ} maps $L^{\frac{3}{2}}(\mathbb{R}^3)$ boundedly into $L^2(\mathbb{R}^3)$ in case $\Gamma(t) = (t, t^2, t^3)$, $t \in [0,1]$. Later Pan [2] extended this result to nondegenerate compact curves in $\mathbb{R}^3$. Namely, he proved :

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Theorem 1.1 (Pan [2]). *Let* Γ *be a compact* C^3 *curve which has nonzero curvature and torsion at every point. Then, there exists a constant* C *such that*

$$\left\|T_{\mu_\Gamma}\ f\right\|_{L^2(\mathbb{R}^3)} \ \leq \ C\ \left\|f\right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}$$

for any $f \in L^{\frac{3}{2}}(\mathbb{R}^3)$.

The main purpose of this paper is to obtain a uniform estimate on the operator norm $\|T_{\mu_\Gamma}\|_{L^{\frac{3}{2}} \to L^2}$ in terms of certain quantities related to Γ. To be more precise, we will prove :

Theorem 1.2. *Let* $\Gamma : I \to \mathbb{R}^3$ *be a compact* C^4 *curve and assume that*

$$\sum_{j=1}^{4} \left|\Gamma^{(j)}(t)\right| \ \leq \ M \tag{2}$$

and

$$\left|\det \begin{bmatrix} \Gamma'(t) \\ \Gamma''(t) \\ \Gamma'''(t) \end{bmatrix}\right| \ \geq \ \delta \ > \ 0 \tag{3}$$

for $t \in I$. *Then, there exists a constant* C *depending only on* δ, M *and* $|I|$ *such that*

$$\left\|T_{\mu_\Gamma} * f\right\|_{L^2(\mathbb{R}^3)} \ \leq \ C \left\|f\right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}$$

for any $f \in L^{\frac{3}{2}}(\mathbb{R}^3)$.

The proof will be based on a T^*T argument by Oberlin [1], which was later adapted by Pan [2]. The author would like to take this opportunity to express his sincere gratitude to Stephen Wainger and Andreas Seeger for bringing this problem into his mind. Also, he would like to thank Jong-Guk Bak for stimulating conversations.

2. Proof of Theorem 1.2

The inequalities (2) and (3) imply

$$\begin{aligned} M^2\left|\Gamma'(t)\right| \;&\geq\; \left|\Gamma'(t)\right|\left|\Gamma''(t)\right|\left|\Gamma'''(t)\right| \\ &\geq\; \left|\det\begin{bmatrix}\Gamma'(t)\\ \Gamma''(t)\\ \Gamma'''(t)\end{bmatrix}\right| \;\geq\; \delta, \end{aligned}$$

and we obtain

$$\left|\Gamma'(t)\right| \;\geq\; \frac{\delta}{M^2}$$

for $t \in I$. Writing $\Gamma(t) = \big(\gamma_1(t), \gamma_2(t), \gamma_3(t)\big)$, we have

$$\left|\gamma_1'(t)\right| + \left|\gamma_2'(t)\right| + \left|\gamma_3'(t)\right| \;\geq\; \left|\Gamma'(t)\right| \;\geq\; \frac{\delta}{M^2}. \tag{4}$$

Estimates (2) and (4) allow us to divide the interval I into at most $N = \left[\frac{6M^3|I|}{\delta}\right] + 1$ disjoint subintervals on each of which

$$\left|\gamma_k'(t)\right| \;\geq\; \frac{\delta}{6M^2}, \tag{5}$$

for some fixed $k = 1, 2, 3$. So, we may assume (5) holds throughout on the interval I. Furthermore, we can assume $k = 1$. We make a change of variable $s = \gamma_1(t)$ in the integral defining $T_{\mu_\Gamma} f(x)$ and get

$$\left|T_{\mu_\Gamma}\, f(x)\right| \;\leq\; \frac{6M^2}{\delta} \int_{\gamma_1(I)} |f| \left(x - \tilde{\Gamma}(s)\right)\, ds$$

where

$$\tilde{\Gamma}(s) \;=\; \left(s, \gamma_2\left(\gamma_1^{-1}(s)\right), \gamma_2\left(\gamma_1^{-1}(s)\right)\right) \;\equiv\; \big(s, \tilde{\gamma}_1(s), \tilde{\gamma}_2(s)\big) \;\equiv\; \big(s, \tilde{\gamma}(s)\big)$$

for $s \in \gamma_1(I)$. By chain rule, we get

$$\sum_{j=2}^{4} \left|\tilde{\gamma}^{(j)}(s)\right| \;\leq\; C_1(\delta, M).$$

Moreover, we have

$$\left|\det\begin{bmatrix}\tilde{\gamma}''(s)\\ \tilde{\gamma}'''(s)\end{bmatrix}\right| \;=\; \left|\frac{1}{\gamma_1'(s)^6}\det\begin{bmatrix}\Gamma'\left(\gamma_1^{-1}(s)\right)\\ \Gamma''\left(\gamma_1^{-1}(s)\right)\\ \Gamma'''\left(\gamma_1^{-1}(s)\right)\end{bmatrix}\right| \;\geq\; \left(\frac{\delta}{M^2}\right)^6 \delta.$$

Therefore it suffices to prove the following lemma :

LEMMA 2.1. *Suppose* $\gamma_1, \gamma_2 \in C^4(I)$ *and let* $\gamma(t) = \big(\gamma_1(t), \gamma_2(t)\big)$ *for* $t \in I$. *Suppose*

$$\sum_{j=2}^{4} \left|\gamma^{(j)}(t)\right| \;\leq\; M \tag{6}$$

$$\left|\det\begin{bmatrix} \gamma''(t) \\ \gamma'''(t) \end{bmatrix}\right| \;\geq\; \delta \;>\; 0 \tag{7}$$

for $t \in I$. *Define the measure* ν *on* $\mathbb{R}^3$ *by*

$$\int_{\mathbb{R}^3} f \; d\nu \;=\; \int_I f\big(t, \gamma(t)\big) \; dt$$

for $f \in C_0^\infty(\mathbb{R}^3)$. *Then, there exists a constant* C *depending only on* M, δ *and* $|I|$ *such that*

$$\big\|\nu * f\big\|_{L^2(\mathbb{R}^3)} \;\leq\; C\,\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}$$

whenever $f \in L^{\frac{3}{2}}(\mathbb{R}^3)$.

Proof of Lemma 2.1. We state a lemma by Pan [2].

LEMMA 2.2. *Let* $\psi : J \to \mathbb{R}^2$ *be a compact* C^2 *curve. Let* A_1, A_2 *be arcs on* $S^1 \subset \mathbb{R}^2$ *and* S_1, S_2 *be corresponding sectors in the plane. Assume that*

1. $l(A_1) \leq \frac{\pi}{4}$;
2. $l(A_2) \leq \frac{\pi}{4}$;
3. $d(A_1 \cup -A_1, A_2 \cup -A_2) = d_0 > 0$.

Suppose for every $t \in J$,

1. $\psi'(t) \;\in\; S_1$;
2. $\psi''(t) \;\in\; S_2$;
3. $|\psi'(t)| \;\geq\; \delta$; *and*
4. $\delta \;\leq\; |\psi''(t)| \;\leq M$

for some positive constants δ *and* M. *Also, we assume* $|J| \leq M$. *Then, there is a constant* $C(d_0, \delta, M)$ *such that*

$$\big\|\sigma * f\big\|_{L^3(\mathbb{R}^2)} \;\leq\; C(d_0, \delta, M)\,\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^2)},$$

where σ is the measure on $\mathbb{R}^2$ given by

$$\int_{\mathbb{R}^2} f\ d\sigma \;=\; \int_J f\big(\psi(t)\big)\ dt$$

for $f \in C_0^\infty\left(\mathbb{R}^2\right)$.

Proof of Lemma 2.2. We refer to [2]. □

We observe that

$$\begin{aligned}\delta \;&\le\; \left|\det\begin{bmatrix}\gamma''(t)\\ \gamma'''(t)\end{bmatrix}\right| \\ &=\; \big|\gamma''(t)\big|\,\big|\gamma'''(t)\big|\left|\det\begin{bmatrix}\frac{\gamma''(t)}{|\gamma''(t)|}\\ \frac{\gamma'''(t)}{|\gamma'''(t)|}\end{bmatrix}\right| \;\le\; M\big|\gamma''(t)\big|,\end{aligned}$$

which means

$$\big|\gamma''(t)\big| \;\ge\; \frac{\delta}{M}.$$

Similarly, we get

$$\big|\gamma'''(t)\big| \;\ge\; \frac{\delta}{M}.$$

Let

$$h_1 \;=\; \inf\left\{\left|\frac{\gamma''(t)}{|\gamma''(t)|} - \frac{\gamma'''(t)}{|\gamma'''(t)|}\right| : t \in I\right\}$$

and

$$h_2 \;=\; \inf\left\{\left|\frac{\gamma''(t)}{|\gamma''(t)|} + \frac{\gamma'''(t)}{|\gamma'''(t)|}\right| : t \in I\right\}.$$

From

$$\begin{aligned}\left|\frac{\gamma''(t)}{|\gamma''(t)|} \pm \frac{\gamma'''(t)}{|\gamma'''(t)|}\right|^2 \;&=\; 2 \;\pm\; 2\,\frac{\gamma''(t)}{|\gamma''(t)|}\cdot\frac{\gamma'''(t)}{|\gamma'''(t)|} \\ &\ge\; 2 \;-\; 2\left(1 - \left|\frac{\gamma''(t)}{|\gamma''(t)|}\times\frac{\gamma'''(t)}{|\gamma'''(t)|}\right|^2\right)^{\frac{1}{2}},\end{aligned}$$

we obtain

$$h_1 \;\ge\; \sqrt{2 - 2\sqrt{1 - \frac{\delta^2}{M^4}}} \;\ge\; \frac{\delta}{M^2} \tag{8}$$

and

$$h_2 \;\geq\; \sqrt{2 - 2\sqrt{1 - \frac{\delta^2}{M^4}}} \;\geq\; \frac{\delta}{M^2}. \tag{9}$$

Set $h = \min\left(h_1, h_2, \frac{\pi}{4}\right)$. Since

$$\begin{aligned}
\left|\frac{d}{dt}\;\frac{\gamma''(t)}{|\gamma''(t)|}\right| &\leq \frac{2M}{\delta^2}\;; \\
\left|\frac{d}{dt}\;\frac{\gamma'''(t)}{|\gamma'''(t)|}\right| &\leq \frac{2M}{\delta^2}\;; \\
\left|\gamma^{(3)}(t)\right| &\leq M\;; \\
\left|\gamma^{(4)}(t)\right| &\leq M,
\end{aligned}$$

we can decompose the interval I into at most $N(\delta, M)$ disjoint subintervals, $I = \bigcup_{k=1}^{n}$, $n \leq N$ such that for $1 \leq k \leq n$ and $t, t' \in I_k$, the following hold :

$$\left|\frac{\gamma''(t)}{|\gamma''(t)|} - \frac{\gamma''(t')}{|\gamma''(t')|}\right| \;\leq\; \frac{h}{2}\;; \tag{10}$$

$$\left|\frac{\gamma'''(t)}{|\gamma'''(t)|} - \frac{\gamma'''(t')}{|\gamma'''(t')|}\right| \;\leq\; \frac{h}{2}\;; \tag{11}$$

$$\left|\gamma''(t) - \gamma''(t')\right| \;\leq\; \frac{\delta}{4M}\;; \tag{12}$$

$$\left|\gamma'''(t) - \gamma'''(t')\right| \;\leq\; \frac{\delta}{4M}\;. \tag{13}$$

For $J \subset I$, let

$$\begin{aligned}
A_1(J) &= \left\{\frac{\gamma''(t)}{|\gamma''(t)|} : t \in J\right\}, \\
A_2(J) &= \left\{\frac{\gamma'''(t)}{|\gamma'''(t)|} : t \in J\right\},
\end{aligned}$$

and $S_1(J)$, $S_2(J)$ be the corresponding sectors. Let ν_k be the measures defined by

$$\int_{\mathbb{R}^3} f \; d\nu_k \;=\; \int_{I_k} f\big(t, \gamma(t)\big)\; dt.$$

We have $\nu = \sum_{k=1}^{n} \nu_k$. Thus, it suffices to prove

$$\left\|\nu_k * f\right\|_{L^2(\mathbb{R}^3)} \;\leq\; C\,\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}$$

for $k = 1, \cdots, n$, with the constant C depending only on δ, M. Let J be any of $I_k,\ k = 1, \cdots, n$. Then, we have

$$\begin{aligned}\left|\frac{\gamma''(t)}{|\gamma''(t)|} - \frac{\gamma'''(t')}{|\gamma'''(t')|}\right| &\geq \left|\frac{\gamma''(t')}{|\gamma''(t')|} - \frac{\gamma'''(t')}{|\gamma'''(t')|}\right| \\ &\quad - \left|\frac{\gamma''(t)}{|\gamma''(t)|} - \frac{\gamma''(t)}{|\gamma''(t)|}\right| \geq \frac{h}{2}, \\ \left|\frac{\gamma''(t)}{|\gamma''(t)|} + \frac{\gamma'''(t')}{|\gamma'''(t')|}\right| &\geq \left|\frac{\gamma''(t')}{|\gamma''(t')|} + \frac{\gamma'''(t')}{|\gamma'''(t')|}\right| \\ &\quad - \left|\frac{\gamma''(t)}{|\gamma''(t)|} - \frac{\gamma''(t)}{|\gamma''(t)|}\right| \geq \frac{h}{2},\end{aligned}$$

for $t, t' \in J$. In other words, we have :

$$d\big(A_1(J) \cup -A_1(J), A_2(J) \cup -A_2(J)\big) \geq \frac{h}{2}. \tag{14}$$

On the other hand we have :

$$l\big(A_1(J)\big) \leq \frac{\pi}{4}, \tag{15}$$

$$l\big(A_2(J)\big) \leq \frac{\pi}{4}. \tag{16}$$

Write $J = [a, b]$. For $0 \leq u \leq b - a$, let $J(u) = [a, b - u]$; for $a - b \leq u < 0$, let $J(u) = [a - u, b]$. Let

$$\begin{aligned}\tilde{\gamma}_{1,u}(t) &= \frac{1}{u}\big(\gamma_1(t+u) - \gamma_1(t)\big), \\ \tilde{\gamma}_{2,u}(t) &= \frac{1}{u}\left(\gamma_2(t+u) - \gamma_2(t)\right),\end{aligned}$$

and

$$\tilde{\gamma}_u(t) = \big(\tilde{\gamma}_{1,u}(t), \tilde{\gamma}_{2,u}(t)\big),$$

for $|u| \leq b - a$ and $t \in J(u)$. We will show that there exists a constant C which is independent of u such that

$$\|\sigma_u * g\|_{L^3(\mathbb{R}^2)} \leq C\, \|g\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}, \tag{17}$$

for each $g \in L^{\frac{3}{2}}(\mathbb{R}^2)$ and $|u| \leq b - a$, where σ_u is the measure on $\mathbb{R}^2$ defined

$$\int_{\mathbb{R}^2} g\, d\sigma_u = \int_{J(u)} g\big(\tilde{\gamma}_u(t)\big)\, dt.$$

To prove (17), we use Lemma 2.2. Fix u, $|u| \le b-a$, and let $t \in J(u)$. Then, there exist $\tau_1, \tau_2 \in I_k$ such that $\tilde{\gamma}'_u(t) = \big(\gamma''_1(\tau_1), \gamma''_2(\tau_2)\big)$. Thus,

$$\left|\tilde{\gamma}'_u(t)\right| \;\ge\; \left|\gamma''(\tau_1)\right| - \left|\gamma''_2(\tau_2) - \gamma''_2(\tau_1)\right| \;\ge\; \frac{\delta}{2}.$$

So, we obtain

$$\left|\tilde{\gamma}'_u(t)\right| \;\ge\; \frac{\delta}{2}. \tag{18}$$

Similarly, we obtain

$$\left|\tilde{\gamma}''_u(t)\right| \;\ge\; \frac{\delta}{2}. \tag{19}$$

By mean value theorem, there is $\tau_3 \in I_k$ such that

$$\tilde{\gamma}'_{1,u}(t)\;\gamma''_2(\tau_3) \;=\; \tilde{\gamma}'_{2,u}(t)\;\gamma''_1(\tau_3)\,. \tag{20}$$

From (20), we get $\tilde{\gamma}'_u(t) \in A_1(I_k)$. Similar argument shows $\tilde{\gamma}_u{}''(t) \in A_2(I_k)$. Thus, by Lemma 2.2 we see (17) holds uniformly in u.

A well-known T^*T argument [1] finishes the proof of Lemma 2.1. □

Remark 2.3. As is well-known, the type set for T, $Tf = \mu * f$, is the trapezoid $\mathcal{T}$ with vertices at $(0,0), (1,1), \left(\frac{2}{3}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \frac{1}{3}\right)$. According to duality and Riesz-Thorin convexity [4], for any p, q with $\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{T}$, there exists a constant C depending only on $\delta, M, |I|, p$ and q.

References

[1] D. Oberlin. Convolution estimates for some measures on curves. *Proc. Amer. Math. Soc.* **99** (1987), no. 1, 56–60.

[2] Y. Pan. A remark on convolution with measures supported on curves. *Canad. Math. Bull.* **36** (1993), no. 2, 245–250.

[3] F. Ricci and E. M. Stein. Harmonic analysis on nilpotent groups and singular integrals. III. Fractional integration along manifolds. *J. Funct. Anal.* **86** (1989), no. 2, 360–389.

[4] E. M. Stein and G. Weiss. *Introduction to Fourier analysis in Euclidean spaces.* Princeton Univ. Press, Princeton, NJ, 1971.

[5] A. Zygmund. *Trigonometric series.* Cambridge Univ. Press, 1959.

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