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SOME EQUATIONS ON THE SUBMANIFOLDS OF A MANIFOLD GSX_n

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Abstract. On a generalized Riemannian manifold X_n, we may impose a particular geometric structure by the basic tensor field $g_{\lambda\mu}$ by means of a particular connection $\Gamma_\lambda{}^\nu{}_\mu$. For example, Einstein's manifold X_n is based on the Einstein's connection defined by the Einstein's equations. Many *recurrent* connections have been studied by many geometers, such as Datta and Singel, M. Matsumoto, and E.M. Patterson. The purpose of the present paper is to study some relations between a *generalized semisymmetric g-recurrent manifold* GSX_n and its submanifold.

All considerations in this present paper deal with the general case $n \geq 2$ and all possible classes.

1. Introduction

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system y^ν, with coordinate transformation $y^\nu \to \bar{y}^\nu$, for which

$$Det\left(\frac{\partial y}{\partial \bar{y}}\right) \neq 0. \tag{1.1}$$

The manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$, which may be split into a symmetric part $h_{\lambda\mu}$ and a skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{1.2}$$

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where

$$\mathcal{G} = Det(g_{\lambda\mu}) \neq 0, \qquad \mathcal{H} = Det(h_{\lambda\mu}) \neq 0. \tag{1.3}$$

Hence, we may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^{\nu}_{\mu} \tag{1.4}$$

and X_n is assumed to be connected by a real nonsymmetric connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ with the following transformation rule:

$$\bar{\Gamma}_{\lambda}{}^{\nu}{}_{\mu} = \frac{\partial \bar{y}^{\nu}}{\partial y^{\alpha}}\left(\frac{\partial y^{\beta}}{\partial \bar{y}^{\lambda}}\frac{\partial y^{\gamma}}{\partial \bar{y}^{\mu}}\Gamma_{\beta}{}^{\alpha}{}_{\gamma} + \frac{\partial^2 y^{\alpha}}{\partial \bar{y}^{\lambda}\partial \bar{y}^{\mu}}\right). \tag{1.5}$$

This connection may also be decomposed into its symmetric part $\Lambda_{\lambda}{}^{\nu}{}_{\mu}$ and its skew-symmetric part $S_{\lambda\mu}{}^{\nu}$, called the torsion tensor of $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$:

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \Lambda_{\lambda}{}^{\nu}{}_{\mu} + S_{\lambda\mu}{}^{\nu} \tag{1.6}$$

where

$$\Lambda_{\lambda}{}^{\nu}{}_{\mu} = \Gamma_{(\lambda}{}^{\nu}{}_{\mu)}, \qquad S_{\lambda\mu}{}^{\nu} = \Gamma_{[\lambda}{}^{\nu}{}_{\mu]}. \tag{1.7}$$

Now, we will define a manifold GSX_n.

A connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ is said to be *semisymmetric* if its torsion tensor is of the form

$$S_{\lambda\mu}{}^{\nu} = 2\delta^{\nu}_{[\lambda}X_{\mu]} \tag{1.8}$$

for an arbitrary vector $X_{\mu} \neq 0$.

Hereafter we assume that X_{μ} is a non-null vector.

A particular differential geometric structure may be imposed on X_n by the tensor field $g_{\lambda\mu}$ by means of the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ defined by the following g-recurrent condition:

$$D_{\omega}g_{\lambda\mu} = -4X_{\omega}g_{\lambda\mu}. \tag{1.9}$$

Here, D_{ω} is the symbolic vector of the covariant derivative with respect to the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$.

Definition 1.1. The connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ which satisfies (1.8) is called g-recurrent connection.

Definition 1.2. A connection which is both semisymmetric and g-recurrent is called a GS-connection.

A generalized Riemannian manifold X_n on which the differential geometric structure is imposed by $g_{\lambda\mu}$ through a GS-connection is called an n-dimensional GS-manifold and will be denoted by GSX_n.

The following theorems have been proved ([3])[1].

Theorem 1.3. *If the system (1.8) admits a solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, it must be of the form*

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \Lambda_{\lambda}{}^{\nu}{}_{\mu} + 2\delta^{\nu}_{[\lambda} X_{\mu]}. \tag{1.10}$$

Theorem 1.4. *If the system (1.9) admits a solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, it must be of the form*

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\} - V^{\nu}{}_{\lambda\mu} - 2S^{\nu}{}_{(\lambda\mu)} + S_{\lambda\mu}{}^{\nu} \tag{1.11}$$

where

$$V^{\nu}{}_{\lambda\mu} = 2X^{\nu} h_{\lambda\mu} - 4X_{(\lambda}\delta^{\nu}_{\mu)}. \tag{1.12}$$

Theorem 1.5. *If the system (1.9) admits a solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ with its semi-symmetric torsion tensor, it must be of the form*

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\} + 2\delta^{\nu}_{\lambda} X_{\mu}. \tag{1.13}$$

2. Preliminaries

[1]Numbers in brackets refer to the references at the end of the paper.

This section is a brief collection of basic concepts, results, and notations needed in the present paper[1].

Let X_m be a submanifold of X_n defined by a system of sufficiently differentiable equations

$$y^\nu = y^\nu(x^1, \ldots\ldots, x^m) \tag{2.1}$$

where the matrix of derivatives

$$B_i^\nu = \frac{\partial y^\nu}{\partial x^i}$$

is of rank m. Hence at each point of X_m, there exists *the first set* $\{B_i^\nu, \underset{x}{N}^\nu\}$ *of n linearly independent nonnull vectors.*

The m vectors B_i^ν are tangential to X_m and the $n-m$ vectors $\underset{x}{N}^\nu$ are normal to X_m and mutually orthogonal. That is

$$h_{\alpha\beta} B_i^\alpha \underset{x}{N}^\beta = 0, \qquad h_{\alpha\beta} \underset{x}{N}^\alpha \underset{y}{N}^\beta = 0 \qquad \text{for } x \neq y. \tag{2.2}$$

The process of determining the set $\{\underset{x}{N}^\nu\}$ is not unique unless $m = n-1$.

However, we may choose their magnitudes such that

$$h_{\alpha\beta} \underset{x}{N}^\alpha \underset{x}{N}^\beta = \varepsilon_x \tag{2.3}$$

where $\varepsilon_x = \pm 1$ according as the left-hand side of (2.3) is positive or negative.

[1]In our further considerations in the present paper, we use the following types of indices ($m < n$): (1) Lower Greek indices α, β, γ,..., running from 1 to n and used for the holonomic components of tensors in X_n. (2) Capital Latin indices A,B,C,..., running from 1 to n and used for the C-nonholonomic components of tensors in X_n at points of X_m. (3) Lower Latin indices i, j, k,..., with the exception of x, y, and z, running from 1 to m. (4) Lower Latin indices x, y, z, running from $m+1$ to n. The summation convention is operative with respect to each set of the above indices within their range, with exception of x, y, z.

3. The induced connection on X_m of GSX_n $(m < n)$

If $\Gamma_\lambda{}^\nu{}_\mu$ is a connection on X_n, the connection $\Gamma_i{}^k{}_j$ defined by

$$\Gamma_i{}^k{}_j = B^k_\gamma(B^\gamma_{ij} + \Gamma_\alpha{}^\gamma{}_\beta B^\alpha_i B^\beta_j), \qquad B^\gamma_{ij} = \frac{\partial B^\gamma_i}{\partial x^j} = \frac{\partial^2 y^\gamma}{\partial x^i \partial x^j} \tag{3.1}$$

is called the *induced connection* of $\Gamma_\lambda{}^\nu{}_\mu$ on X_m of X_n.

The following statements have been already proved([3]):

(a) The torsion tensor $S_{ij}{}^k$ of the induced connection $\Gamma_i{}^k{}_j$ is the induced tensor of the torsion tensor $S_{\lambda\mu}{}^\nu$ of the connection $\Gamma_\lambda{}^\nu{}_\mu$. That is

$$S_{ij}{}^k = S_{\alpha\beta}{}^\gamma B^\alpha_i B^\beta_j B^k_\gamma. \tag{3.2}$$

(b) The induced connection $\{{}_i{}^k{}_j\}$ of $\{{}_\lambda{}^\nu{}_\mu\}$ is the Christoffel symbol defined by h_{ij}. That is

$$\{{}_i{}^k{}_j\} = \frac{1}{2} h^{kp}(\partial_i h_{jp} + \partial_j h_{ip} - \partial_p h_{ij}). \tag{3.3}$$

(c) On an X_m of GSX_n, the induced connection $\Gamma_i{}^k{}_j$ is of the form

$$\Gamma_i{}^k{}_j = \{{}_i{}^k{}_j\} + 2\delta^k_i X_j. \tag{3.4}$$

Here $\{{}_i{}^k{}_j\}$ are the induced Christoffel symbols defined by (3.3) and X_j is the induced vector on X_m of a vector $X_\mu \neq 0$ determining $\Gamma_\lambda{}^\nu{}_\mu$.

(d) On an X_m of GSX_n, a necessary and sufficient condition for the induced connection $\Gamma_i{}^k{}_j$ to be g-recurrent is

$$\sum_x k_{x[i} \overset{x}{\Lambda}_{j]k} = 0, \qquad \text{where} \qquad \overset{x}{\Lambda}_{ij} = (\nabla_\beta \overset{x}{N}_\alpha) B^\alpha_i B^\beta_j. \tag{3.5}$$

Let $\overset{o}{D}_j$ be the symbolic vector of the generalized covariant derivative with respect to the $x's$. That is

$$\overset{o}{D}_j B^\alpha_i = B^\alpha_{ij} + \Gamma_\beta{}^\alpha{}_\gamma B^\beta_i B^\gamma_j - \Gamma_i{}^k{}_j B^\alpha_k. \tag{3.6}$$

Then the vector $\overset{o}{D}_j B_i^\alpha$ in X_n is normal to X_m and is given by

(3.7) $$\overset{o}{D}_j B_i^\alpha = -\sum_x \overset{x}{\Omega}_{ij} \underset{x}{N}^\alpha$$

where

(3.8) $$\overset{x}{\Omega}_{ij} = -(\overset{o}{D}_j B_i^\alpha)\overset{x}{N}_\alpha.$$

And we know that the tensors $\overset{x}{\Omega}_{ij}$ are the induced tensors on X_m of the tensor $D_\beta \overset{x}{N}_\alpha$ in X_n. That is

(3.9) $$\overset{x}{\Omega}_{ij} = (D_\beta \overset{x}{N}_\alpha) B_i^\alpha B_j^\beta.$$

The tensor $\overset{x}{\Omega}_{ij}$ will be called the *generalized coefficients of the second fundamental form* of X_m.

4. The generalized fundamental equations for X_m of GSX_n

On an X_m of GSX_n, the following identities hold ([2]):

(4.1) $$\overset{o}{D}_j B_i^\alpha = -\sum_x \overset{x}{\Lambda}_{ij} \underset{x}{N}^\alpha \qquad \text{where} \qquad \overset{x}{\Lambda}_{ij} = (\nabla_\beta \overset{x}{N}_\alpha) B_i^\alpha B_j^\beta$$

(Generalized Gauss formulas for an X_m of GSX_n)

(4.2) $$\overset{o}{D}_j \underset{x}{N}^\alpha = (\varepsilon_x h^{im} \overset{x}{\Lambda}_{mj}) B_i^\alpha + \sum_y (\varepsilon_y \underset{x}{\overset{y}{H}}_\gamma B_j^\gamma + 2\delta_x^y X_j) \underset{y}{N}^\alpha.$$

(Generalized Weingarten equations on an X_m of GSX_n)

In order to derive the generalized *Gauss-Codazzi equations,*we need the following curvature tensors of GSX_n and X_m:

(4.3) $$R_{\omega\mu\lambda}{}^\nu = 2(\partial_{[\mu} \Gamma_{|\lambda|}{}^\nu{}_{\omega]} + \Gamma_\lambda{}^\alpha{}_{[\omega} \Gamma_{|\alpha|}{}^\nu{}_{\mu]})$$

(4.4) $$R_{ijk}{}^h = 2(\partial_{[j} \Gamma_{|k|}{}^h{}_{i]} + \Gamma_k{}^p{}_{[i} \Gamma_{|p|}{}^h{}_{j]})$$

The following notation will be used in further considerations:

(4.5) $$\underset{x}{\overset{y}{H}}_\gamma = \varepsilon_y (\nabla_\gamma \underset{x}{N}^\alpha) \overset{y}{N}_\alpha$$

THEOREM 4.1. *On an X_m of GSX_n, the curvature tensors defined by (4.3) and (4.4) satisfy the following identities:*

(4.6)
$$R_{ijk}{}^{h} = R_{\beta\gamma\epsilon}{}^{\alpha} B_i^{\beta} B_j^{\gamma} B_k^{\epsilon} B_{\alpha}^{h} + 2\sum_x \overset{x}{\Lambda}_{k[i}(\overset{x}{\Lambda}_{j]m}\varepsilon_x h^{hm} - \delta_{j]}^{h} X_x + k_{j]}{}^{h} X_x + k_{j]x} X^h)$$

(The generalized Gauss equations for an X_m of GSX_n)

(4.7)
$$2\overset{o}{D}_{[k}\overset{x}{\Lambda}_{j]i} = R_{\beta\gamma\epsilon}{}^{\alpha} B_k^{\beta} B_j^{\gamma} B_i^{\epsilon} \overset{x}{N}_{\alpha} + 6\overset{x}{\Lambda}_{i[k} X_{j]} + 2\sum_y \overset{y}{\Lambda}_{i[k}(B_{j]}^{\gamma}\varepsilon_x \underset{y}{\overset{x}{H}}{}_{\gamma} + X_{j]} k_y{}^{x} + k_{j]}{}^{x} X_y)$$

(The generalized Codazzi equations for an X_m of GSX_n)

Proof. In virtue of (3.1),(3.6),(4.3) and (4.4), we have

(4.8)
$$\begin{aligned} 2\overset{o}{D}_{[k}\overset{o}{D}_{j]} B_i^{\alpha} &= 2[\partial_{[k}(\overset{o}{D}_{j]} B_i^{\alpha}) - \Gamma_{[j}{}^{m}{}_{k]}(\overset{o}{D}_m B_i^{\alpha}) \\ &\quad - \Gamma_i^{m}{}_{[k}(\overset{o}{D}_{j]} B_m^{\alpha}) + \Gamma_{\beta}{}^{\alpha}{}_{\gamma}(\overset{o}{D}_{[j} B_{|i|}^{\beta}) B_{k]}^{\gamma}] \\ &= -R_{\epsilon\gamma\beta}{}^{\alpha} B_i^{\beta} B_j^{\gamma} B_k^{\epsilon} + R_{kji}{}^{m} B_m^{\alpha} + 4\sum_x \overset{x}{\Lambda}_{i[j} X_{k]} \underset{x}{N}^{\alpha} \end{aligned}$$

On the other hand, the equations (4.1) and (4.2) give

(4.9)
$$\begin{aligned} 2\overset{o}{D}_{[k}\overset{o}{D}_{j]} B_i^{\alpha} &= -2\sum_x \overset{o}{D}_{[k}(\overset{x}{\Lambda}_{j]i} \underset{x}{N}^{\alpha}) \\ &= 2\sum_x (\overset{o}{D}_{[j}\overset{x}{\Lambda}_{k]i}) \underset{x}{N}^{\alpha} + 2\sum_x \overset{x}{\Lambda}_{i[k}\overset{o}{D}_{j]}) \underset{x}{N}^{\alpha} \\ &= 2\sum_x (\overset{o}{D}_{[j}\overset{x}{\Lambda}_{k]i} + \overset{x}{\Lambda}_{i[k} X_{j]}) \underset{x}{N}^{\alpha} \\ &\quad + 2\sum_{x,y} \overset{x}{\Lambda}_{i[k}(B_{j]}^{\gamma}\varepsilon_x \underset{x}{\overset{y}{H}}{}_{\gamma} + X_{j]} k_x{}^{y} + k_{j]}{}^{y} X_x) \underset{y}{N}^{\alpha} \\ &\quad + 2\sum_x \overset{x}{\Lambda}_{i[k}(\overset{x}{\Lambda}_{j]m}\varepsilon_x h^{pm} - \delta_{j]}^{p} X_x + k_{j]x} X_x + k_{j]}{}^{p} X_x) B_p^{\alpha} \end{aligned}$$

By means of (4.8) and (4.9), we have

$$R_{kji}{}^{m}B^{\alpha}_{m}=R_{\epsilon\gamma\beta}{}^{\alpha}B^{\epsilon}_{k}B^{\beta}_{i}B^{\gamma}_{j}+2\sum_{x}(\overset{o}{D}_{[j}\overset{x}{\Lambda}_{k]i}+3\overset{x}{\Lambda}_{i[k}X_{j]})\underset{x}{N}^{\alpha}$$
$$+2\sum_{x,y}\overset{x}{\Lambda}_{i[k}(B^{\gamma}_{j]}\varepsilon_{x}\underset{x}{\overset{y}{H}}_{\gamma}+X_{j]}k_{x}{}^{y}+k_{j]}{}^{y}X_{x})\underset{y}{N}^{\alpha}$$
$$+2\sum_{x}\overset{x}{\Lambda}_{i[k}(\overset{x}{\Lambda}_{j]m}\varepsilon_{x}h^{pm}-\delta^{p}_{j]}X_{x}+k_{j]x}X_{x}+k_{j]}{}^{p}X_{x})B^{\alpha}_{p} \tag{4.10}$$

Multiplying both sides of (4.10) by B^{h}_{α}, we have(4.6). Similarly, the identity (4.7) follows by multiplying $\overset{z}{N}_{\alpha}$ into both sides of (4.10). □

5. Parallelism. Paths

In this section we investigate parallelism and paths in X_n and GSX_n. Let C be any curve in X_n, given by

$$y^{\nu}=y^{\nu}(t). \tag{5.1}$$

Definition 5.1. A vector field V^{ν} is said to be parallel along C with respect to a connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ if it satisfies the following condition:

$$\frac{dy^{\alpha}}{dt}V^{[\lambda}D_{\alpha}V^{\nu]}=0,\qquad V^{\nu}\neq\rho\frac{dy^{\alpha}}{dt}D_{\alpha}V^{\nu},\qquad \rho\neq 0 \tag{5.2a}$$

or equivalently,

(5.2b)
$$V^{[\lambda}(\frac{dV^{\nu]}}{dt}+\Gamma_{\beta}{}^{\nu]}{}_{\alpha}V^{\beta}\frac{dy^{\alpha}}{dt})=0,\qquad V^{\nu}\neq\rho\frac{dy^{\alpha}}{dt}D_{\alpha}V^{\nu},\qquad \rho\neq 0.$$

In particular, the curves whose tangents are parallel along themselves are called the *paths* in X_n with respect to $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$. A path with respect to $\{\lambda{}^{\nu}{}_{\mu}\}$ is called a *geodesic* of X_n.

Therefore, a curve C in X_n, given by (5.1), is a path if it satisfies (5.3).

$$\frac{dy^{[\lambda}}{dt}(\frac{d^{2}y^{\nu]}}{dt^{2}}+\Gamma_{\alpha}{}^{\nu]}{}_{\beta}\frac{dy^{\alpha}}{dt}\frac{dy^{\beta}}{dt})=0 \tag{5.3}$$

As a consequence of (5.3), we have the following result:

THEOREM 5.2. *Every path C in GSX_n is a geodesic.*

THEOREM 5.3. *A necessary and sufficient condition that parallelism be the same along every curve in X_n with respect to two connections one of which is a GS connection is that other connection $\bar{\Gamma}_\lambda{}^\nu{}_\mu$ be given by*

$$\bar{\Gamma}_\lambda{}^\nu{}_\mu = \{{}_\lambda{}^\nu{}_\mu\} + 2\delta_\lambda^\nu A_\mu \qquad for\ an\ arbitrary\ vector\ A_\mu. \tag{5.4}$$

Proof. Suppose that parallelism is the same along every curve with respect to two connections $\Gamma_\lambda{}^\nu{}_\mu$ and $\bar{\Gamma}_\lambda{}^\nu{}_\mu$. Then $\bar{\Gamma}_\lambda{}^\nu{}_\mu$ is given by ([3])

$$\bar{\Gamma}_\lambda{}^\nu{}_\mu = \Gamma_\lambda{}^\nu{}_\mu + 2\delta_\lambda^\nu P_\mu \quad for\ an\ arbitrary\ vector\ P_\mu. \tag{5.5}$$

By means of (1.12) and (5.5) , we have (5.4). □

REMARK 5.4. As an immediate consequence of Theorem 5.3, we know that if parallelism is preserved along every curve in X_n with respect to a GS connection $\Gamma_\lambda{}^\nu{}_\mu$, then the other connection $\bar{\Gamma}_\lambda{}^\nu{}_\mu$ is also a GS connection.

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