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SOME CHARACTERIZATIONS OF CR-SUBMANIFOLDS WITH $(n-1)$ CR-DIMENSION IN A COMPLEX PROJECTIVE SPACE

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Abstract. The purpose of this paper is to give some characterizations of n-dimensional CR-submanifolds with $(n-1)$ CR-dimension immersed in a complex projective space $CP^{\frac{n+p}{2}}$, in terms of the Riemannian curvature tensor R.

1. Introduction

Let M be a connected real n-dimensional submanifold of real codimension p of a complex manifold $\overline{M}$ with complex structure J. If the maximal J-invariant subspace $JT_xM \cap T_xM$ of T_xM has constant dimension for any x in M, then M is called a *CR-submanifold* and the constant is called the *CR-dimension* of M ([8,9]). Now let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension of $\overline{M}$. Then M admits an induced almost contact structure ([11,15,16]). A typical example of an n-dimensional CR-submanifold of $(n-1)$ CR-dimension is a real hypersurface. When the ambient manifold $\overline{M}$ is a complex projective space, real hypersurfaces are investigated by many authors ([2,4,5,6,7,10,12,13,14]) in connection with the shape operator and the induced almost contact structure.

Recently, from these results, the several authors ([8,11]) studied about an n-dimensional CR-submanifold of $(n-1)$ CR-dimension in a complex projective space $CP^{\frac{n+p}{2}}$. Especially, by using the Erbacher's

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reduction theorem ([3]), Okumura and Vanhecke [11] proved the following theorem, which is focused on the induced almost contact metric structure F on M and A_1 a special kind of shape operators.

THEOREM A. *Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension immersed in a complex projective space $CP^{\frac{n+p}{2}}$. If the normal vector field $\xi_1 := \xi$ appeared in* (2.1) *is parallel with respect to the normal connection and if F and A_1 commute, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$, where M_1 and M_2 belong to some odd-dimensional spheres (π is the Hopf-fibration $S^{n+p+1}(1) \to CP^{\frac{n+p}{2}}$).*

The purpose of this paper is to give some characterizations of CR-sub- manifolds of $(n-1)$ CR-dimension in $CP^{\frac{n+p}{2}}$, in terms of the Riemannian curvature tensor R. We first have a classification of CR-submanifold of $(n-1)$ CR-dimension in $CP^{\frac{n+p}{2}}$ satisfying $\mathcal{L}_{U_1}R = 0$, where $\mathcal{L}_{U_1}$ denotes the Lie derivative in the direction of the structure vector field U_1.

THEOREM 1. *Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension immersed in $CP^{\frac{n+p}{2}}$ and let there exist an orthonormal basis $\{\xi_\alpha\}_{\alpha=1,\dots,p}$ ($\xi_1 := \xi$) of normal vectors to M each of which is parallel with respect to the normal connection. If $\mathcal{L}_{U_1}R = 0$, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$, where M_1 and M_2 belong to some odd-dimensional spheres.*

Next, we also have a classification of CR-submanifold of $(n-1)$ CR-dimension in $CP^{\frac{n+p}{2}}$ satisfying $\nabla_{U_1}R = 0$, where $\nabla_{U_1}R$ denotes the covariant derivative in the direction of the structure vector field U_1. Namely, we prove the following theorem

THEOREM 2. *Let M be as in Theorem 1 with $n \geq 3$. If $\nabla_{U_1}R = 0$ and $g(A_1U_1, U_1) \neq 0$, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$, where M_1 and M_2 belong to some odd-dimensional spheres.*

2. Preliminaries

Let M be an n-dimensional Riemannian manifold isometrically immersed in a complex space form $\overline{M} = M^{\frac{n+p}{2}}(c)$ and denote by $(J, \overline{g})$ the Kähler structure on $\overline{M}$. For x of M we denote by T_xM and $T_xM^{\perp}$

the tangent space and normal space of M at x, respectively. From now on we assume that M is an n-dimensional CR-submanifold of $(n-1)$ CR-dimension, that is, $dim(JT_xM \cap T_xM) = n-1$. This implies that $dimM$ is odd ([11]).

Note that the definition of CR-submanifold of $(n-1)$ CR-dimension meets the definition of CR-submanifold in the sense of Bejancu [1].

Furthermore, our hypothesis implies that *there exists a unit vector field* ξ_1 *normal to* M *such that* $JTM \subset TM \oplus Span\{\xi\}$. Hence, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha\}_{\alpha=1,\dots,p}$ $(\xi_1 := \xi)$ of normal vectors to M, we have the following decomposition in tangential and normal components :

$$JX = FX + u^1(X)\xi_1, \tag{2.1}$$

$$J\xi_\alpha = -U_\alpha + P\xi_\alpha, \quad \alpha = 1,\dots,p. \tag{2.2}$$

It is easily seen that F and P are skew-symmetric linear endomorphisms acting on T_xM and $T_xM^\perp$, respectively. Moreover, the Hermitian property of J implies

$$g(FU_\alpha, X) = -u^1(X)\overline{g}(\xi_1, P\xi_\alpha), \tag{2.3}$$

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - \overline{g}(P\xi_\alpha, P\xi_\beta). \tag{2.4}$$

From $\overline{g}(JX, \xi_\alpha) = -\overline{g}(X, J\xi_\alpha)$, we get $g(U_\alpha, X) = u^1(X)\delta_{1\alpha}$ and hence

$$g(U_1, X) = u^1(X), \quad U_\alpha = 0, \quad \alpha = 2,\dots,p. \tag{2.5}$$

Next, applying J to (2.1), using (2.2) and (2.5) we have

$$F^2X = -X + u^1(X)U_1, \quad u^1(X)P\xi_1 = -u^1(FX)\xi_1. \tag{2.6}$$

Since P is skew-symmetric, (2.3) and the second equation of (2.6) give

$$u^1(FX) = 0, \quad P\xi_1 = 0, \quad FU_1 = 0. \tag{2.7}$$

So, (2.2) may be written in the form

$$J\xi_1 = -U_1, \quad J\xi_\alpha = P\xi_\alpha, \quad \alpha = 2,\dots,p \tag{2.8}$$

and further, we may put

$$P\xi_\alpha = \sum\nolimits_{\beta=2}^{p} P_{\alpha\beta}\xi_\beta, \quad \alpha = 2, \dots, p, \tag{2.9}$$

where $(P_{\alpha\beta})$ is a skew-symmetric matrix which satisfies

$$\sum\nolimits_{\beta=2}^{p} P_{\alpha\beta}P_{\beta\gamma} = -\delta_{\alpha\gamma}, \quad \alpha, \gamma = 2, \dots, p. \tag{2.10}$$

These results imply that (F, U_1, u^1, g) defines an almost contact metric structure on (M, g) ([16]).

Now, let $\overline{\nabla}$ and ∇ denote the Levi-Civita connection on $\overline{M}$ and M, respectively and denote by D the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^\perp$ of M. Then the Gauss and Weingarten equations are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.11}$$

$$\overline{\nabla}_X \xi_\alpha = -A_\alpha X + D_X \xi_\alpha, \quad \alpha = 1, \dots, p \tag{2.12}$$

for any tangent vector fields X and Y to M. Here h denotes the second fundamental form and A_α is the shape operator corresponding to ξ_α. They are related by $h(X, Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y)\xi_\alpha$.

Furthermore, putting

$$D_X \xi_\alpha = \sum\nolimits_{\beta=1}^{p} s_{\alpha\beta}(X)\xi_\beta, \tag{2.13}$$

it follows that $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of D. Next, the Gauss, Codazzi and Ricci equations are ([11]) :

$$\begin{aligned} \overline{g}(\overline{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) \\ &+ \sum_{\alpha=1}^{p} \{g(A_\alpha X, Z)g(A_\alpha Y, W) - g(A_\alpha Y, Z)g(A_\alpha X, W)\}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \overline{g}(\overline{R}(X, Y)Z, \xi_\alpha) &= g((\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X, Z) \\ &+ \sum_{\beta=1}^{p} \{g(A_\beta Y, Z)s_{\beta\alpha}(X) - g(A_\beta X, Z)s_{\beta\alpha}(Y), \end{aligned} \tag{2.15}$$

$$\overline{g}(\overline{R}(X,Y)\xi_\alpha,\xi_\beta) = \overline{g}(R^\perp(X,Y)\xi_\alpha,\xi_\beta) + g([A_\beta,A_\alpha]X,Y) \tag{2.16}$$

for any tangent vector fields X, Y, Z and W to M. $\overline{R}$ denotes the Riemannian curvature tensor of $\overline{M}$ and R that of M. $R^\perp$ is the curvature tensor of the normal connection D.

Moreover, if the ambient space $\overline{M}$ is of constant holomorphic sectional curvature c, since

$$\begin{aligned}\overline{R}(\overline{X},\overline{Y})\overline{Z} = \frac{c}{4}\{&\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y} + \overline{g}(J\overline{Y},\overline{Z})J\overline{X}\\ &- \overline{g}(J\overline{X},\overline{Z})J\overline{Y} - 2\overline{g}(J\overline{X},\overline{Y})J\overline{Z}\}\end{aligned}$$

for any tangent vector fields $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ to $\overline{M}$, the Riemannian curvature tensor R of M given by

$$\begin{aligned}R(X,Y)Z =&\frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(FY,Z)FX\\ &- g(FX,Z)FY - 2g(FX,Y)FZ\}\\ &+\sum\nolimits_{\alpha=1}^{p}\{g(A_\alpha Y,Z)A_\alpha X - g(A_\alpha X,Z)A_\alpha Y\}\end{aligned} \tag{2.17}$$

for any tangent vector fields X, $Y$and Z to M.

In the sequel, we consider the case of a complex space form $\overline{M} = M^{\frac{n+p}{2}}(c)$ and $\overline{\nabla}J = 0$. Then by differentiating (2.1) and (2.2) covariantly and by comparing the tangential and normal parts, we have

$$(\nabla_Y F)X = u^1(X)A_1Y - g(A_1X,Y)U_1, \tag{2.18}$$

$$(\nabla_Y u^1)X = g(FA_1Y,X), \tag{2.19}$$

$$\nabla_X U_1 = FA_1X, \tag{2.20}$$

$$g(A_\alpha U_1,X) = -\sum\nolimits_{\beta=2}^{p} s_{1\beta}(X)P_{\beta\alpha}, \quad \alpha = 2,\dots,p \tag{2.21}$$

for any tangent vector fields X and Y to M.

In the rest of this paper *we suppose that there exists an orthonormal basis* $\{\xi_\alpha\}_{\alpha=1,\dots,p}$ *of normal vectors to* M *each of which is parallel with respect to the normal connection* D. Then from (2.13) we have

$$s_{\alpha\beta} = 0. \tag{2.22}$$

Hence, from (2.21) and (2.22) we obtain

$$A_\alpha U_1 = 0, \quad \alpha = 2, ..., p. \tag{2.23}$$

Moreover, from (2.22) and (2.23), the Codazzi equation (2.15) becomes

$$\begin{aligned}(\nabla_X A_1)Y - (\nabla_Y A_1)X \\ = \frac{c}{4}\{g(X,U_1)FY - g(Y,U_1)FX - 2g(FX,Y)U_1\},\end{aligned} \tag{2.24}$$

$$(\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X = 0, \quad \alpha = 2, \ldots, p \tag{2.25}$$

for any tangent vector fields X and Y to M. Also, by differentiating (2.23) covariantly and using (2.25) we get

$$(\nabla_{U_1} A_\alpha)U_1 = 0, \quad \alpha = 2, \ldots, p. \tag{2.26}$$

Especially, recently Kwon and Pak [8] proved the following lemma.

Lemma 2.1. *Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension immersed in a complex space form $M^{\frac{n+p}{2}}(c), c \neq 0$ and let there exist an orthonormal basis $\{\xi_\alpha\}_{\alpha=1,\ldots,p}$ of normal vectors to M each of which is parallel with respect to the normal connection. If $A_1U_1 = \lambda U_1$ for some function λ, then λ is locally constant.*

Finally, we suppose that U_1 is principal with corresponding principal curvature λ. Then, by Lemma 2.1, λ is constant on M and it satisfies

$$(\nabla_{U_1} A_1)U_1 = 0, \tag{2.27}$$

$$A_1FA_1 = \frac{c}{4}F + \frac{1}{2}\lambda(A_1F + FA_1) \tag{2.28}$$

by virtue of (2.20) and (2.24). Hence from (2.24) and (2.28), we get

$$\nabla_{U_1} A_1 = -\frac{1}{2}\lambda(A_1F - FA_1). \tag{2.29}$$

3. Proof of Theorem 1

In this section, we are concerned with the proof of Theorem 1. Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension immersed in a complex space form $M^{\frac{n+p}{2}}(c)$. Then M admits an almost contact metric structure (F, U_1, u^1, g). The Lie derivative $\mathcal{L}_{U_1}R$ of R with respect to the structure vector field U_1 satisfies

$$(\mathcal{L}_{U_1}R)(X,Y,Z) = \mathcal{L}_{U_1}(R(X,Y)Z) - R(\mathcal{L}_{U_1}X,Y)Z - R(X,\mathcal{L}_{U_1}Y)Z - R(X,Y)\mathcal{L}_{U_1}Z \tag{3.1}$$

for any vector fields X, Y and Z on M. From now on we shall prove the following lemma.

Lemma 3.1. *Let M be as in Lemma 2.1. If $\mathcal{L}_{U_1}R = 0$, then $A_1F = FA_1$.*

Proof. Let T_0 be a distribution defined by the subspace $T_0(x) = \{u \in T_xM : g(u, U_1(x)) = 0\}$ of the tangent space T_xM of M at any point x, which is called the *holomorphic distribution*. Suppose that the structure vector field U_1 is not necessarily principal. Then we can put $A_1U_1 = \lambda U_1 + \mu V$, where V is a unit vector field in T_0, λ and μ are smooth functions on M. From (2.17), (2.18), (2.20), (3.1) and our assumption, we have

(3.2)
$$\begin{aligned}
0 = &\frac{c}{4}\mu[\{u^1(Y)g(Z,V) - u^1(Z)g(Y,V)\}FX \\
&- \{u^1(X)g(Z,V) - u^1(Z)g(X,V)\}FY \\
&- 2\{u^1(X)g(Y,V) - u^1(Y)g(X,V)\}FZ \\
&+ g(FY,Z)\{u^1(X)V - g(X,V)U_1\} \\
&- g(FX,Z)\{u^1(Y)V - g(Y,V)U_1\} \\
&- 2g(FX,Y)\{u^1(Z)V - g(Z,V)U_1\}] \\
&- \frac{c}{4}\{g(FY,Z)F(A_1F - FA_1)X - g(FX,Z)F(A_1F - FA_1)Y \\
&- 2g(FX,Y)F(A_1F - FA_1)Z + g((A_1F - FA_1)Y,Z)X \\
&- g((A_1F - FA_1)X,Z)Y + g((A_1F^2 - F^2A_1)Y,Z)FX
\end{aligned}$$

$$
\begin{aligned}
&- g((A_1F^2 - F^2A_1)X, Z)FY - 2g((A_1F^2 - F^2A_1)X, Y)FZ\} \\
&+ g((\nabla_{U_1}A_1)Y, Z)A_1X - g((\nabla_{U_1}A_1)X, Z)A_1Y \\
&+ g(A_1Y, Z)\{(\nabla_{U_1}A_1)X + (A_1F - FA_1)A_1X\} \\
&- g(A_1X, Z)\{(\nabla_{U_1}A_1)Y + (A_1F - FA_1)A_1Y\} \\
&- \sum_{\alpha=2}^{p}[\{g(A_1FA_\alpha Y, Z) - g((\nabla_{U_1}A_\alpha)Y, Z) - g(A_\alpha FA_1Y, Z)\}A_\alpha X \\
&- g(A_\alpha Y, Z)\{(\nabla_{U_1}A_\alpha)X - FA_1A_\alpha X + A_\alpha FA_1X\}] \\
&+ \sum_{\alpha=2}^{p}[\{g(A_1FA_\alpha X, Z) - g((\nabla_{U_1}A_\alpha)X, Z) - g(A_\alpha FA_1X, Z)\}A_\alpha Y \\
&- g(A_\alpha X, Z)\{(\nabla_{U_1}A_\alpha)Y - FA_1A_\alpha Y + A_\alpha FA_1Y\}]
\end{aligned}
$$

for any vector fields X, Y and Z on T_xM. Putting $Z = U_1$ and taking X and Y in the holomorphic distribution T_0 in (3.2) and using (2.23) and (2.26), we have

$$
\begin{aligned}
(3.3)\quad 0 =& \frac{c}{4}\mu\{g(Y, FV)X - g(X, FV)Y\} + g((\nabla_{U_1}A_1)U_1, Y)A_1X \\
&- g((\nabla_{U_1}A_1)U_1, X)A_1Y \\
&+ \mu[g(Y, V)\{(\nabla_{U_1}A_1)X + (A_1F - FA_1)A_1X\} \\
&- g(X, V)\{(\nabla_{U_1}A_1)Y + (A_1F - FA_1)A_1Y\}] \\
&+ \sum\nolimits_{\alpha=2}^{p}\mu\{g(A_\alpha Y, FV)A_\alpha X - g(A_\alpha X, FV)A_\alpha Y\}.
\end{aligned}
$$

Again, putting $Y = Z = U_1$ and taking X in the holomorphic distribution T_0 in (3.2) and using (2.23), we have

$$
\begin{aligned}
(3.4)\quad \lambda(\nabla_{U_1}A_1)X =& \mu g(X, V)(\nabla_{U_1}A_1)U_1 - d\lambda(U_1)A_1X \\
&+ g((\nabla_{U_1}A_1)U_1, X)A_1U_1 \\
&+ \frac{c}{4}\mu g(X, FV)U_1 + \mu^2 g(X, V)(A_1F - FA_1)V \\
&- \lambda\mu^2 g(X, V)FV - \lambda(A_1F - FA_1)A_1X.
\end{aligned}
$$

Eliminating $(\nabla_{U_1}A_1)X$ and $(\nabla_{U_1}A_1)Y$ in (3.3) and (3.4) and using

(2.26), we obtain

(3.5)
$$
\begin{aligned}
0 =&\frac{c}{4}\mu[\lambda\{g(Y,FV)X - g(X,FV)Y\} + \mu\{g(X,FV)g(Y,V) \\
&- g(X,V)g(Y,FV)\}U_1] \\
&+ \lambda\{g((\nabla_{U_1}A_1)U_1,Y)A_1X - g((\nabla_{U_1}A_1)U_1,X)A_1Y\} \\
&+ \mu[g(Y,V)\{g((\nabla_{U_1}A_1)U_1,X)A_1U_1 - d\lambda(U_1)A_1X\} \\
&- g(X,V)\{g((\nabla_{U_1}A_1)U_1,Y)A_1U_1 - d\lambda(U_1)A_1Y\}] \\
&+ \sum\nolimits_{\alpha=2}^{p}\mu\{g(A_\alpha Y,FV)A_\alpha X - g(A_\alpha X,FV)A_\alpha Y\}
\end{aligned}
$$

for any vector fields X and Y in T_0. Now, putting $X = V$ and $Y = FV$ in (3.5) and using (2.26), we get

$$
\begin{aligned}
(3.6)\quad 0 =&\frac{c}{4}\mu(\lambda V - \mu U_1) + \lambda\{g((\nabla_{U_1}A_1)U_1,FV)A_1V \\
&- g((\nabla_{U_1}A_1)U_1,V)A_1FV\} \\
&- \mu\{g((\nabla_{U_1}A_1)U_1,FV)A_1U_1 - d\lambda(U_1)A_1FV\} \\
&+ \sum\nolimits_{\alpha=2}^{p}\mu\{g(A_\alpha FV,FV)A_\alpha V - g(A_\alpha V,FV)A_\alpha FV\}.
\end{aligned}
$$

Taking the inner product of (3.6) with U_1 and using (2.23), we obtain $\mu = 0$, that is, the structure vector field U_1 is principal. Hence, by Lemma 2.1, λ is constant. If $\lambda = 0$, then putting $X = U_1$ in (3.2) and using (2.23), (2.26) and $c \neq 0$, we get $A_1F - FA_1 = 0$. Next, suppose that $\lambda \neq 0$. Then from (3.4) and (2.27) we have

$$(\nabla_{U_1}A_1)X + A_1FA_1X - FA_1^2X = 0 \tag{3.7}$$

for any vector field X in T_0.

Furthermore, putting $Y = Z = U_1$ in (3.2) and using (2.23) and (2.27), we see that (3.7) holds for any vector field X. This implies that

$$F(A_1^2 - \lambda A_1 - \frac{c}{4}I)X = 0 \tag{3.8}$$

for any vector field X, where I denotes the identity transformation and we have used (2.28) and (2.29). (3.8) is equivalent to

$$A_1^2 - \lambda A_1 - \frac{c}{4}(I - u^1 \otimes U_1) = 0,$$

from which it follows that A_1 satisfies $(A_1F - FA_1)^2 = 0$, where we have used that (2.28) and $A_1F^2 = F^2A_1 = -A_1 + \lambda u^1 \otimes U_1$. Hence, we have $A_1F - FA_1 = 0$. □

Proof of Theorem 1. Combining Lemma 3.1 and Theorem A, we have Theorem 1. □

4. Proof of Theorem 2

In this section, we are concerned with the proof of Theorem 2. Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension immersed in a complex space form $M^{\frac{n+p}{2}}(c)$. Then M admits an almost contact metric structure (F, U_1, u^1, g). The covariant derivative $\nabla_{U_1}R$ of R with respect to the structure vector field U_1 satisfies

$$(\nabla_{U_1}R)(X,Y,Z) = \nabla_{U_1}(R(X,Y)Z) - R(\nabla_{U_1}X,Y)Z - R(X,\nabla_{U_1}Y)Z - R(X,Y)\nabla_{U_1}Z \tag{4.1}$$

for any vector fields X, Y and Z on M. From now on we shall prove the following lemma.

Lemma 4.1. *Let M be as in Lemma 2.1 with $n \geq 3$. If $\nabla_{U_1}R = 0$, then $\nabla_{U_1}A_1 = 0$.*

Proof. From (2.17), (2.18), (4.1) and our assumption, we have

$$\begin{aligned} 0 = \frac{c}{4}[&\{u^1(Y)g(A_1U_1,Z) - u^1(Z)g(A_1U_1,Y)\}FX \\ &- \{u^1(X)g(A_1U_1,Z) - u^1(Z)g(A_1U_1,X)\}FY \\ &- 2\{u^1(X)g(A_1U_1,Y) - u^1(Y)g(A_1U_1,X)\}FZ \\ &+ g(FY,Z)\{u^1(X)A_1U_1 - g(A_1U_1,X)U_1\} \\ &- g(FX,Z)\{u^1(Y)A_1U_1 - g(A_1U_1,Y)U_1\} \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
&- 2g(FX,Y)\{u^1(Z)A_1U_1 - g(A_1U_1,Z)U_1\}] \\
&+ g((\nabla_{U_1}A_1)Y,Z)A_1X - g((\nabla_{U_1}A_1)X,Z)A_1Y \\
&+ g(A_1Y,Z)(\nabla_{U_1}A_1)X - g(A_1X,Z)(\nabla_{U_1}A_1)Y \\
&+ \sum\nolimits_{\alpha=2}^{p}\{g((\nabla_{U_1}A_\alpha)Y,Z)A_\alpha X - g((\nabla_{U_1}A_\alpha)X,Z)A_\alpha Y \\
&+ g(A_\alpha Y,Z)(\nabla_{U_1}A_\alpha)X - g(A_\alpha X,Z)(\nabla_{U_1}A_\alpha)Y\}
\end{aligned}
$$

for any vector fields X, Y and Z on T_xM. Suppose that the structure vector field U_1 is not necessarily principal. Then we can put $A_1U_1 = \lambda U_1 + \mu V$, where V is a unit vector field in T_0, and λ and μ are smooth functions on M. Let M_0 be the non-empty open subset of M consisting of points x at which $\mu(x) \neq 0$. Hereafter unless otherwise stated, we shall discuss on the subset M_0 of M. By the form $A_1U_1 = \lambda U_1 + \mu V$, (4.2) is reformed as

$$
\begin{aligned}
(4.3)\quad 0 =& \frac{c}{4}\mu[\{u^1(Y)g(Z,V) - u^1(Z)g(Y,V)\}FX \\
&- \{u^1(X)g(Z,V) - u^1(Z)g(X,V)\}FY \\
&- 2\{u^1(X)g(Y,V) - u^1(Y)g(X,V)\}FZ \\
&+ g(FY,Z)\{u^1(X)V - g(X,V)U_1\} \\
&- g(FX,Z)\{u^1(Y)V - g(Y,V)U_1\} \\
&- 2g(FX,Y)\{u^1(Z)V - g(Z,V)U_1\}] \\
&+ g((\nabla_{U_1}A_1)Y,Z)A_1X - g((\nabla_{U_1}A_1)X,Z)A_1Y \\
&+ g(A_1Y,Z)(\nabla_{U_1}A_1)X - g(A_1X,Z)(\nabla_{U_1}A_1)Y \\
&+ \sum\nolimits_{\alpha=2}^{p}\{g((\nabla_{U_1}A_\alpha)Y,Z)A_\alpha X - g((\nabla_{U_1}A_\alpha)X,Z)A_\alpha Y \\
&+ g(A_\alpha Y,Z)(\nabla_{U_1}A_\alpha)X - g(A_\alpha X,Z)(\nabla_{U_1}A_\alpha)Y\}
\end{aligned}
$$

for any vector fields X, Y and Z on T_xM. Putting $Z = U_1$ and taking X and Y in the holomorphic distribution T_0 in (4.3) and using (2.23) and (2.26), we have

$$
\begin{aligned}
(4.4)\qquad 0 =& \frac{c}{4}\mu\{-g(Y,V)FX + g(X,V)FY - 2g(FX,Y)V\} \\
&+ g((\nabla_{U_1}A_1)U_1,Y)A_1X - g((\nabla_{U_1}A_1)U_1,X)A_1Y \\
&+ \mu\{g(Y,V)(\nabla_{U_1}A_1)X - g(X,V)(\nabla_{U_1}A_1)Y\}.
\end{aligned}
$$

Next, putting $Y = Z = U_1$ and taking X in the holomorphic distribution T_0 in (4.3) and using (2.20), (2.23) and (2.26) we have

$$(4.5)\qquad \lambda(\nabla_{U_1}A_1)X = g(\nabla_{U_1}A_1)U_1, X)A_1U_1 + \mu g(X,V)(\nabla_{U_1}A_1)U_1 - d\lambda(U_1)A_1X.$$

Combining (4.4) and (4.5) and using (2.26), we obtain

$$\begin{aligned}(4.6)\qquad 0 =& \frac{c}{4}\lambda\mu\{-g(Y,V)FX + g(X,V)FY - 2g(FX,Y)V\}\\ &+ \mu\{g(Y,V)g((\nabla_{U_1}A_1)U_1, X)\\ &- g(X,V)g((\nabla_{U_1}A_1)U_1, Y)\}A_1U_1\\ &+ \{\lambda g((\nabla_{U_1}A_1)U_1, Y) - \mu d\lambda(U_1)g(Y,V)\}A_1X\\ &- \{\lambda g((\nabla_{U_1}A_1)U_1, X) - \mu d\lambda(U_1)g(X,V)\}A_1Y\end{aligned}$$

for any vector fields X and Y in T_0. Let $L(U_1, V, FV)$ be a distribution defined by the subspace $L_x(U_1, V, FV)$ in the tangent space T_xM spanned by the vectors $U_1(x)$, $V(x)$ and $FV(x)$ at any point x in M, and let T_1 be the orthogonal complement in the tangent bundle TM of the distribution $L(U_1, V, FV)$. Then T_1 is not empty because of $n \geq 3$. For any unit vector field X in T_1, putting $Y = FX$ in (4.6) and using (2.26), we have

$$(4.7)\qquad \frac{c}{2}\mu V = g((\nabla_{U_1}A_1)U_1, FX)A_1X - g((\nabla_{U_1}A_1)U_1, X)A_1FX,$$

provided $\lambda \neq 0$. Suppose that there is a unit vector field X_0 in T_1 at which $g((\nabla_{U_1}A_1)U_1, X_0) = 0$. Then from (4.7) we obtain

$$(4.8)\qquad \frac{c}{2}\mu V = g((\nabla_{U_1}A_1)U_1, FX_0)A_1X_0 \neq 0.$$

Accordingly we can put $A_1X_0 = \omega(X_0)V$, where ω is a 1-form on M_0. Putting $X = X_0$, $Y = V$ in (4.6), we have

$$(4.9)\qquad \frac{c}{4}\lambda\mu FX_0 - \omega(X_0)\{\lambda g((\nabla_{U_1}A_1)U_1, V) - \mu d\lambda(U_1)\}V = 0.$$

Thus, since FX_0 and V are orthonormal vector fields and $\lambda \neq 0$, (4.9) implies $\mu = 0$, a contradiction. Accordingly we get

$$(4.10)\qquad g((\nabla_{U_1}A_1)U_1, X) \neq 0$$

for any non-zero vector field X in T_1. On the other hand, putting $Y = FV$ in (4.6) and using (2.26), we have

$$g((\nabla_{U_1}A_1)U_1, FV)A_1X - g((\nabla_{U_1}A_1)U_1, X)A_1FV = 0 \tag{4.11}$$

for any X in T_1 under the assumption $\lambda \neq 0$. If $g((\nabla_{U_1}A_1)U_1, FV) = 0$, then from (4.10) and (4.11) we obtain $A_1FV = 0$. Now, we suppose that $g((\nabla_{U_1}A_1)U_1, FV) \neq 0$. From (4.10) and (4.11) we have

$$A_1X = \theta(X)A_1FV, \quad \theta(X) \neq 0 \tag{4.12}$$

for any non-zero vector field X in T_1, where θ is a 1-form on M_0. Hence, for any non-zero vector fields X and Y in T_1, we obtain

$$A_1\{\theta(Y)X - \theta(X)Y\} = 0, \quad \theta(X) \neq 0, \quad \theta(Y) \neq 0. \tag{4.13}$$

If we put $Z_1 = \theta(Y_1)X_1 - \theta(X_1)Y_1$ for given linearly independent vector fields X_1 and Y_1 in T_1, then from (4.13) $A_1Z_1 = 0$ and hence $A_1FV = 0$ by virtue of (4.12).

Next, putting $X = V$ and $Y = FV$ in (4.6), and using $A_1FV = 0$, we have

$$\frac{3c}{4}\lambda\mu V - g((\nabla_{U_1}A_1)U_1, FV)(\lambda A_1V - \mu A_1U_1) = 0. \tag{4.14}$$

Consequently, we get $g((\nabla_{U_1}A_1)U_1, FV) \neq 0$ which together with (4.11) we have $A_1X = 0$ for any vector field X in T_1. Hence, putting $X = U_1$ and taking $Y, Z \in T_1$ in (4.3), and using (2.23) and (2.26), we obtain

$$\lambda g((\nabla_{U_1}A_1)Y, Z)U_1 + \mu\{\frac{c}{4}g(FY, Z) + g((\nabla_{U_1}A_1)Y, Z)\}V = 0. \tag{4.15}$$

Accordingly it turns out to be $\mu = 0$ on M_0 provided $\lambda \neq 0$, a contradiction. This means that U_1 is principal on M', where M' denotes the open subset of M consisting of points x at which $\lambda(x) \neq 0$. Thus, putting $Y = Z = U_1$ in (4.3) and using (2.23), (2.26) and (2.27), we have $\nabla_{U_1}A_1 = 0$.

Now, let us denote by $Int(M - M')$ the interior of the subset $(M - M')$. Then $\lambda = 0$ on $Int(M - M')$. Suppose that U_1 is not principal on $Int(M - M')$. Then the subset M_1 of $Int(M - M')$ consisting of points x at which $\mu(x) \neq 0$ is non-empty open set. Hence, from (4.5) we have

$$g((\nabla_{U_1} A_1)U_1, X)V + g(X, V)(\nabla_{U_1} A_1)U_1 = 0 \tag{4.16}$$

on M_1 for any vector field X in T_0. Accordingly $g((\nabla_{U_1} A_1)U_1, Y) = 0$ for any vector field Y in T_0 orthogonal to V. Taking the inner product of (4.16) with X in T_0, we have $g((\nabla_{U_1} A_1)U_1, X)g(X, V) = 0$. Putting $X = V$ in this equation, we get $g((\nabla_{U_1} A_1)U_1, V) = 0$. Hence, putting $X = V$ in (4.16), we obtain $(\nabla_{U_1} A_1)U_1 = 0$ on M_1. Taking X and Y in T_0 orthogonal to V in (4.4) and using $(\nabla_{U_1} A_1)U_1 = 0$, we obtain $g(FX, Y) = 0$ on M_1, a contradiction. This means that U_1 is principal with corresponding principal curvature $\lambda = 0$. Accordingly we have $\nabla_{U_1} A_1 = 0$ on $Int(M - M')$ by virtue of (2.29). This completes the proof by the continuity of $\nabla_{U_1} A_1 = 0$. □

Proof of Theorem 2. Combining (2.29), Lemma 4.1 and Theorem A, we have Theorem 2. □

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