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CENTRAL LIMIT THEOREM ON HYPERGROUPS

Lee, Jae Won and Park, Sung Wook

Abstract. On the basis of Heyer and Zeuner's results we will treat the central limit theorem for probability measures on hypergroup.

0. Introduction

The purpose of this article is to verify a central limit theorem for random variables which take their values in a hypergroup. However the special case that G is the one-dimensional hypergroup will not be considered in any details because of the amount of material that involved on real line. Real line central limit theorem began with looking at limit distributions of the form as

$$\frac{X_1 + X_2 + \cdots + X_n - a_n}{b_n} \quad \text{as} \quad n \to \infty \tag{1}$$

for a sequence $\{X_n\}$ of i.i.d. random variables, where a_n and b_n are constants. The theory also broadened to consider the more general problem of finding the limiting distributions of the row sums $\sum_{i=1}^{k_n} X_{n,i}$ of triangular arrays $\{X_{n,i} : i = 1, \cdots, k_n; n \geq 1\}$ in which the random variables $\{X_{n,i} : i = 1, \cdots, k_n\}$ are independent for each n. For hypergroup-valued random variable, scaling, as represented by the division b_n in formula (1), is generally impossible and central limit theorem on hypergroups has therefore tended to concentrate on triangular arrays.

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Let G be a hypergroup and $M_+^b(G)$ and $M^1(G)$ denote the spaces of bounded positive measures and probability measures on G furnished with the weak topology T_w, respectively . Let $(\mu_i)_{i\geq 1}$ be a sequence of measures in $M^1(G)$. Then there is a probability space $(\Omega, \mathcal{F}, P)$ consisting of a nonempty Ω, a σ-field $\mathcal{F}$ of subsets of Ω and a probability measure P on $(\Omega, \mathcal{F})$ and there exists for each $i \geq 1$ a measurable function $X_i : \Omega \to G$ such that $X_i(P) = \mu_i$. Moreover the sequence $(X_i)_{i\geq 1}$ constructed as before is called to be *independent* in the sense that for all $n \geq 1$ and disjoint Borel sets $B_1, \cdots, B_n$ the formula

$$P\left[\bigcap_{i=1}^{n}\{X_i \in B_i\}\right] = \prod_{i=1}^{n} P[X_i \in B_i]$$

holds. Given any probability space $(\Omega, \mathcal{F}, P)$ and a hypergroup G we introduce a G-valued random variable as a measurable function $X : \Omega \to G$ and its distribution on the image measure $X(P) = \mu$ of P under X.

1. Preliminaries

Let G be a hypergroup and let $\mathbb{T}$ denote the unit circle group. Denote $G^\wedge$ by the dual group of G. Then $G^\wedge$ is the set of all continuous homeomorphisms of G into $\mathbb{T}$ and the dual group of $G^\wedge$ can be identified with the original hypergroup G. Denote $\chi(x)$ by the value of the homeomorphism $\chi \in G^\wedge$ at the point $x \in G$. A function $g \in \mathcal{C}(G \times G^\wedge)$ is called a *local inner product* for G if g has the following properties :

(1) $\sup_{x\in G}\sup_{\chi\in K}|g(x,\chi)| < \infty$ for all compact subset K of $G^\wedge$.
(2) $g(x, \chi_1\chi_2) = g(x,\chi_1) + g(x,\chi_2)$ and $g(x^{-1},\chi) = -g(x,\chi)$ for all $x \in G$, $\chi \in G^\wedge$.
(3) For every compact subset K of $G^\wedge$ there is a neighborhood U of the identity e such that $\chi(x) = \exp[ig(x,\chi)]$ holds for all $x \in U, \chi \in K$.
(4) For every compact subset K of $G^\wedge$, $\lim_{x\to e}\sup_{\chi\in K} g(x,\chi) = 0$.

The sequence $\{\mu_n\}_{n\geq 1}$ of probability measures on G is said to *converge weakly* to the probability measure μ on G as $n \to \infty$ if, for every bounded continuous function $f : G \to \mathbb{R}$,

$$\int_G f(x)\mu_n(dx) \to \int_G f(x)\mu(dx) \quad \text{as} \quad n \to \infty\,.$$

We will define the characteristic function $\hat{\mu}$ of the probability measure μ by its Fourier transformation $\hat{\mu}$ defined on $G^{\wedge}$ as following : for all $\chi \in G^{\wedge}$,

$$\hat{\mu}(\chi) = \int_G \chi(x)\mu(dx)\,.$$

Definition 1. A triangular array $\{\mu_{n,j} \;:\; j = 1, 2, \cdots, k_n; n = 1, 2 \cdots\}$ of probability measures on G is called *uniformly infinitesimal* if

$$\lim_{n\to\infty} \sup_{1\le j\le k_n} \mu_{n,j}(U^c) = 0$$

for every neighborhood U of e in G, or equivalently, if

$$\lim_{n\to\infty} \sup_{1\le j\le k_n} |1 - \hat{\mu}_{n,j}(\chi)| = 0$$

for every $\chi \in G^{\wedge}$ where U^c is the complement of U and U is assumed to be measurable.

The triangular array

$$\{X_{n,j} \;:\; j = 1, 2, \cdots, k_n; n = 1, 2 \cdots\}$$

of G-valued random variables is said to be *uniformly infinitesimal* if the triangular array

$$\{\mu_{n,j} \;:\; j = 1, 2, \cdots, k_n; n = 1, 2 \cdots\}$$

of probability measures on G is uniformly infinitesimal, where $\mu_{n,j}$ is the distribution of $X_{n,j}$. Then for any $\mu \in M^1(G)$ and fixed local inner product g on $G \times G^{\wedge}$, the function

$$y \;\mapsto\; \exp\left[i\int_G g(x,\chi)\mu(dx)\right]$$

is a continuous homeomorphism of $G^{\wedge}$ into $\mathbb{T}$ and so is an element of the dual group of $G^{\wedge}$. Therefore, by the Pontryagin duality theorem, there is a fixed point x_0 of G such that for all $\chi \in G^{\wedge}$

$$\chi(x_0) = \exp\left[i\int_G g(x,\chi)\mu(dx)\right]\,.$$

Now let τ be the continuous homeomorphism $t \mapsto e^{it}$ from $\mathbb{R}$ into $\mathbb{T}$. Then $\mu \in M^1(\mathbb{T})$ is called a *normal distribution* if $\mu = \tau(\nu_{a,\sigma^2})$ for some $a \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and $\nu_{a,\sigma^2} \in M^1(G)$. In fact, measure $\mu(B)$ of a set B can be computed as

$$\mu(B) = \int_B \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}(u-a+2n\pi)^2\right] du$$

and notes that μ is normal if and only if $\hat{\mu}(k) = \exp[iak - (\sigma k)^2/2]$ for all $k \in \mathbb{Z}$.

DEFINITION 2. Let G be a hypergroup. A measure $\mu \in M^1(G)$ is called a *Gauss measure in the sense of Parthasarathy* if

(1) $\mu \in \mathcal{I}_0(G)$.
(2) For any factorization of μ of the form $\mu = \exp(\tau) * \lambda$ one has $\tau = a\varepsilon_e$ for some positive number a,

where τ is a positive bounded measure on G and $\lambda \in \mathcal{I}_0(G)$, $\mathcal{I}_0(G)$ denotes the collection of all weakly infinitely divisible measures in $M^1(G)$. And the class of Gauss measure in $M^1(G)$ will be abbreviated by $\mathcal{R}_P(G)$.

The class $\mathcal{I}_0(G)$ is a sequentially closed subsemigroup of $M^1(G)$. From this point we will use the notation $\exp(\nu) = \exp(\nu - \|\nu\|\varepsilon_e)$ for any measure $\nu \in M^1(G)$. Then one can have the following properties(see [1]).

LEMMA 1. *For any $\mu \in \mathcal{I}_0(G)$ the following statements are equivalent:*

(1) *μ has a nontrivial idempotent factor,*
(2) *there is a $\chi_0 \in G^\wedge$ with $\hat{\mu}(\chi_0) = 0$.*

LEMMA 2. *Let $(\nu_n)_{n\geq 1}$ be a sequence in $M_+^b(G)$ and $(\mu_n)_{n\geq 1}$ the sequence of Poisson measures $\mu_n := \exp(\nu_n)$ for $n = 1, 2, \cdots$. We assume that*
(a) $(\mu_n)_{n\geq 1}$ is shift compact,
(b) any limit point μ of any sequence of shifts of $(\mu_n)_{n\geq 1}$ does not

admit an idempotent factor.
Then

(1) *For any neighborhood U of the identity e of G the restricted sequence $(\mathbf{1}_{G\setminus U}\nu_n)_{n\geq 1}$ is relatively compact*
(2) *For all $\chi \in G^\wedge$ we have $\sup_{n\geq 1}\int(1-Re\chi(x))\nu_n(dx) < \infty$.*

2. Two results

THEOREM 1. *For every $\mu \in \mathcal{R}_P(G)$ there are unique element $x_0 \in G$ and a positive quadratic form ϕ such that for all $\chi \in G^\wedge$, $\hat{\mu}(\chi) = \chi(x_0)\exp(-\phi(\chi))$ holds.*

Proof. For $\mu \in \mathcal{R}_P(G)$ and $n \geq 1$ there exist $\mu_n \in M^1(G)$ and $x_n \in G$ such that $\mu = \mu_n^n * \varepsilon_{x_n}$. Then we can assume, without loss of generality, that $\lim_{n\to\infty}\mu_n = \varepsilon_e$ holds. Hence the family $\{\mu_{n,j} : j = 1,\cdots,n; n \geq 1\}$ with $\mu_{n,j} := \mu_n$ for all $j = 1,2,\cdots,n$ is an infinitesimal triangular system in $M^1(G)$. For every $n \geq 1$ we define $y_n \in G$ by

$$\chi(y_n) := \exp\left[-i\int_G g(x,\chi)\mu_n(dx)\right]$$

for all $\chi \in G^\wedge$ where g is a local inner product for G, and $\alpha_n := \mu_n * \varepsilon_{y_n}$, $\beta_n := \exp(\alpha_n)$, $\lambda_n := \beta_n^n * \varepsilon_{y_n^{-n}}$. One notes that accumulation points of shifts of $\{\mu_n^n : n \geq 1\}$ differ only by shifts of μ. Therefore no accumulation point of any shift of $\{\mu_n^n : n \geq 1\}$ admits an idempotent factor. So, the following relation holds : for any compact subset K of $G^\wedge$,

$$\lim_{n\to\infty}\sup_{\chi\in K}|(\lambda_n * \varepsilon_{x_n})^\wedge(\chi) - \hat{\mu}(\chi)| = 0\,.$$

In particular, for every $\chi \in G^\wedge$

$$\begin{aligned}|\hat{\mu}(\chi)| &= \lim_{n\to\infty}|(\lambda_n * \varepsilon_{x_n})^\wedge(\chi)| \\ &= \lim_{n\to\infty}\exp\left[n\int(\mathrm{Re}\chi(x)-1)\alpha_n(dx)\right]\,.\end{aligned}$$

The real-valued function ϕ on $G^\wedge$ defined by

$$\phi(\chi) := \lim_{n\to\infty} n\int(1-\mathrm{Re}\chi(x))\alpha_n(dx) \quad \text{for all} \quad \chi \in G^\wedge$$

is a positive quadratic form on $G^\wedge$. In fact, for every $n \geq 1$ we define measures $\tau_n := n\alpha_n \in M^b_+(G)$ such that $\exp(\tau_n)$ is a shift of $\lambda_n * \varepsilon_{x_n}$ for $n \geq 1$. Then

$$\lim_{n\to\infty} \lambda_n * \varepsilon_{x_n} = \mu,$$

and so the sequence $(\exp(\tau_n))_{n\geq 1}$ is shift compact and the sequence $(\mathbf{1}_{U^c} \cdot \tau_n)_{n\geq 1}$ of restrictions of τ_n is relatively compact for every neighborhood U of e by Lemma 2. Hence for any accumulation point $\tau = \lim_{n\to\infty} \mathbf{1}_{U^c} \cdot \tau_n$ of the sequence $(\mathbf{1}_{U^c} \cdot \tau_n)_{n\geq 1}$ there is an $\alpha \in \mathcal{I}_0(G)$ such that $\mu = \exp(\tau) * \alpha$ holds. And by assumption we have $\tau = \|\tau\|\varepsilon_e$. Without loss generality we assume that $\lim_{n\to\infty} \mathbf{1}_{U^c} \cdot \tau_n = \|\tau\|\varepsilon_e$. Let now f be a bounded continuous function on G such that $f(U^c) = 1$ and $f(e) = 0$. Then

$$\begin{aligned} \|\tau\| &= \lim_{n\to\infty} (\mathbf{1}_{U^c} \cdot \tau_n)(G) \\ &= \lim_{n\to\infty} (\mathbf{1}_{U^c} \cdot \tau_n)(f) \\ &= \|\tau\| f(e) = 0\,. \end{aligned}$$

Hence for all $\chi \in G^\wedge$,

$$\phi(\chi) = \lim_{n\to\infty} \int_U (1 - \mathrm{Re}\chi(x))\tau_n(dx) \tag{2}$$

whenever U is a neighborhood of e and $\mathrm{Re}\chi(x)$ denotes the real part of $\chi(x)$. The power series expansion of the cosine function $\cos\theta$, $0 \leq \theta \leq 2\pi$, together with the continuity of characters of G implies that for every $\varepsilon > 0$ and all $\chi_1, \chi_2 \in G^\wedge$ there is a neighborhood U of e such that

$$\begin{aligned} &2(1-\varepsilon)[(1 - \mathrm{Re}\chi_1(x)) + (1 - \mathrm{Re}\chi_2(x))] \\ \leq &(1 - \mathrm{Re}(\chi_1\chi_2)(x)) + (1 - \mathrm{Re}(\chi_1\chi_2^{-1})(x)) \\ \leq &2(1+\varepsilon)[(1 - \mathrm{Re}\chi_1(x)) + (1 - \mathrm{Re}\chi_2(x))]\,. \end{aligned}$$

holds for all $x \in U$. Integration over U with respect to τ_n, for all $n \geq 1$, and passage to the limit as $n \to \infty$ yields

$$\begin{aligned} 2(1-\varepsilon)(\phi(\chi_1) + \phi(\chi_2)) &\leq \phi(\chi_1\chi_2) + \phi(\chi_1\chi_2^{-1}) \\ &\leq 2(1+\varepsilon)(\phi(\chi_1) + \phi(\chi_2))\,. \end{aligned}$$

Since ε is arbitrary,

$$\phi(\chi_1\chi_2) + \phi(\chi_1\chi_2^{-1}) = 2(\phi(\chi_1) + \phi(\chi_2(x))$$

for all $\chi_1, \chi_2 \in G^\wedge$. Hence that ϕ is a positive quadratic form on $G^\wedge$, so $-\phi = \log|\hat{\mu}|$ for ϕ. Since $\mu \in \mathcal{R}_P(G)$ does not admit an idempotent factor, $\psi := \hat{\mu}/|\hat{\mu}|$ is in $(G^\wedge)^\wedge$, and by Pontryagin's duality theorem there is an $x_0 \in G$ with $\psi(\chi) = \chi(x_0)$ for all $\chi \in G^\wedge$ and therefore the desired result. First of all we note that for any neighborhood U of e and $\chi_1, \chi_2 \in G^\wedge$ the following relation holds :

$$\frac{\psi(\chi_1\chi_2)}{\psi(\chi_1)\psi(\chi_2)} = \lim_{n\to\infty} \exp \int_U [\mathrm{Im}\chi_1\chi_2(x) - \mathrm{Im}\chi_1(x) - \mathrm{Im}\chi_2(x)]\tau_n(dx)$$

where $\mathrm{Im}\chi(x)$ denotes the imaginary part of $\chi(x)$. It remains to be proved that the right side of this equation is 0. In fact, we can obtain the following inequality

$$\begin{aligned} &|\mathrm{Im}\chi_1\chi_2(x) - \mathrm{Im}\chi_1(x) - \mathrm{Im}\chi_2(x)| \\ \le&|\mathrm{Im}\chi_1(x)| \cdot |1 - \mathrm{Re}\chi_2(x)| + |\mathrm{Im}\chi_2(x)| \cdot |1 - \mathrm{Re}\chi_1(x)| \end{aligned}$$

for any $x \in G$ and $\chi_1, \chi_2 \in G^\wedge$. Furthermore, Lemma 2 implies that for all $\chi \in G^\wedge$ the condition

$$\sup_{n\ge1} \int (1 - \mathrm{Re}\chi(x))\tau_n(dx) < \infty$$

holds. Now choosing a neighborhood U of e such that $|\mathrm{Im}\chi_1(x)|$ and $|\mathrm{Im}\chi_2(x)|$ are smaller than $\varepsilon > 0$, for all $x \in U$ one obtains

$$\left| \int_U [\mathrm{Im}\chi_1\chi_2(x) - \mathrm{Im}\chi_1(x) - \mathrm{Im}\chi_2(x)]\tau_n(dx) \right| \le 2c\varepsilon$$

where $c = \sup_{n\ge1} \int (1 - \mathrm{Re}\chi(x))\tau_n(dx)$ depends on only χ and does not depend on n. Thus the right-hand side of (2) is equal to unity for ε is arbitrary. On the other hand, the uniqueness of $x_0 \in G$ and ϕ on $G^\wedge$ in the representation

$$\hat{\mu}(\chi) = \psi(\chi)|\hat{\mu}(\chi)| = \chi(x_0)\exp(-\phi(\chi))$$

valid for $\chi \in G^\wedge$. $\square$

THEOREM 2. *Let $x_0 \in G$ and ϕ be a positive quadratic form on $G^\wedge$. Then there exists a measure $\mu \in \mathcal{R}_P(G)$ such that for all $\chi \in G^\wedge$*

$$\hat{\mu}(\chi) = \chi(x_0)\exp(-\phi(\chi))$$

holds.

Proof. We first show that for $x_0 \in G$ and a positive quadratic form ϕ on $G^\wedge$ there is a measure $\mu \in \mathcal{I}_0(G)$ such that the desired representation holds. To show this it suffices to establish the negative definiteness of ϕ, because that if ϕ is negative definite then $\exp(-\phi)$ is positive definite. Thus by Bochner's theorem there exists a measure $\nu \in M^1(G)$ with the property $\hat{\nu} = \exp(-\phi)$. Since for $n \geq 1$ one also has a positive quadratic form $(1/n)\phi$, there is a measure $\nu_n \in M^1(G)$ with $\hat{\nu}_n = \exp(-(1/n)\phi)$ and therefore $\hat{\nu} = (\hat{\nu}_n)^n = (\nu_n^n)^\wedge$. Thus $\mu = \varepsilon_{x_0} * \nu \in \mathcal{I}_0(G)$ and ν is an infinitely divisible measure on G such that

$$\begin{aligned}\hat{\mu}(\chi) &= \hat{\varepsilon}_{x_0}(\chi)\hat{\nu}(\chi)\\ &= \chi(x_0)\exp(-\phi(\chi))\end{aligned}$$

for all $\chi \in G^\wedge$ holds. Concerning the negative definiteness of ϕ we must show that for every $n \geq 1$, $\chi_i \in G^\wedge, i = 1, \cdots, n$ and complex numbers c_i, $i = 1, \cdots, n$, we have

$$\sum_{i,j=1}^{n} c_i\bar{c}_j(\phi(\chi_i) + \phi(\chi_j) - \phi(\chi_i\chi_j^{-1})) \geq 0\,.$$

Defining a real function ψ on $\mathbb{Z}^n$ by

$$\psi(m) := \phi\left(\sum_{i=1}^{n} m_i\chi_i\right)$$

for all $m := (m_1, \cdots, m_n) \in \mathbb{Z}^n$, one immediately verifies that

$$\psi(u+v) + \psi(u-v) = 2(\psi(u) + \psi(v))$$

for all $u, v \in \mathbb{Z}^n$. So, there is a unique continuous, positive symmetric bilinear function $\Psi : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{R}$ with $\Psi(u,u) = \psi(u)$ for all $u \in \mathbb{Z}^n$ which is uniquely determined by the matrix

$$A := (\Psi(e_i, e_j))_{i,j=1,\cdots,n}$$

with $e_i := (0, \cdots, 1, 0, \cdots, 0) \in \mathbb{Z}^n$ for all $i = 1, \cdots, n$ and 1 is the ith component of e_i and it can be extended to a bilinear form $\widetilde{\Psi}$ on $\mathbb{R}^n$ for which

$$\begin{aligned} \widetilde{\Psi}(\zeta u, \zeta u) &= \zeta^2 \widetilde{\Psi}(u,u) \\ &= \zeta^2 \Psi(u,u) \\ &= \zeta^2 \psi(u) \geq 0 \end{aligned}$$

holds whenever $\zeta \in \mathbb{R}$ and $u \in \mathbb{Z}^n$. Writing

$$D := \{x \in \mathbb{R}^n \,:\, x = \zeta u \quad \text{for } \zeta \in \mathbb{R} \quad \text{and } u \in \mathbb{Z}^n\}$$

then we have $\widetilde{\Psi}(z,z) \geq 0$ for all $z \in D$ and thus this inequality holds for all $z \in \mathbb{R}^n$ because of denseness of D in $\mathbb{R}^n$. This implies that $\widetilde{\Psi}$ is a positive bilinear form on $\mathbb{R}^n$. As ϕ is a positive definite form on $G^\wedge$, one obtains the following equalities : for all $i, j = 1, \cdots, n$

$$\begin{aligned} \phi(\chi_i) + \phi(\chi_j) - \phi(\chi_i \chi_j^{-1}) &= \psi(e_i) + \psi(e_j) - \psi(e_i e_j^{-1}) \\ &= \Psi(e_i, e_i) + \Psi(e_j, e_j) - \Psi(e_i e_j^{-1}, e_i e_j^{-1}) \\ &= 2\Psi(e_i, e_j) = 2\widetilde{\Psi}(e_i, e_j) \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{i,j=1}^n c_i \bar{c}_j (\phi(\chi_i) + \phi(\chi_j) - \phi(\chi_i \chi_j^{-1})) &= \sum_{i,j=1}^n c_i \bar{c}_j \widetilde{\Psi}(e_i, e_j) \\ &= 2\Psi\left(\sum_{i=1}^n c_i e_i, \sum_{i=1}^n c_i e_i\right) \geq 0\,. \end{aligned}$$

For every $\tau \in M_+^b(G)$ and $\chi_1, \chi_2 \in G^\wedge$ one has

$$\begin{aligned} &- \log|\exp(\tau)^\wedge(\chi_1 \chi_2)| - \log|\exp(\tau)^\wedge(\chi_1 \chi_2^{-1})| \\ &\leq 2(-\log|\exp(\tau)^\wedge(\chi_1)| - \log|\exp(\tau)^\wedge(\chi_2)|)\,. \end{aligned} \tag{3}$$

This relation follows from the inequality

$$1-\cos(\theta_1+\theta_2)\le(1-\cos\theta_1)+(1-\cos\theta_2),\quad 0\le\theta_1,\theta_2\le 2\pi\,.$$

In fact, we have

$$\log|\exp(\tau)^\wedge(\chi)|=\int(\mathrm{Re}\chi(x)-1)\tau(dx)$$

for all $\chi\in G^\wedge$ and consequently,

$$\begin{aligned}&-\log|\exp(\tau)^\wedge(\chi_1\chi_2)|-\log|\exp(\tau)^\wedge(\chi_1\chi_2^{-1})|\\ =&2\int(1-\mathrm{Re}\chi_1(x)\mathrm{Re}\chi_2(x))\tau(dx)\,.\end{aligned}$$

Thus, from the above inequality, we obtain

$$\begin{aligned}&2\int_G(1-\mathrm{Re}\chi_1(x)\mathrm{Re}\chi_2(x))\tau(dx)\\ \le&2(-\log|\exp(\tau)^\wedge(\chi_1)|-\log|\exp(\tau)^\wedge(\chi_2)|)\end{aligned}$$

for all $\chi_1,\chi_2\in G^\wedge$. Moreover, given any measure $\tau\in M_+^b(G)$ we have the inequality

$$|\hat\tau(\mathbf{1})-\hat\tau(\chi)|^2\le 2\hat\tau(\mathbf{1})[\hat\tau(\mathbf{1})-\mathrm{Re}\hat\tau(\chi)] \tag{4}$$

valid for all $\chi\in G^\wedge$. From this inequality (4) it follows that for all $\chi\in G^\wedge$

$$\int(1-\mathrm{Re}\chi(x))\tau(dx)=0$$

implies $\hat\tau(\chi)=\hat\tau(\mathbf{1})=\|\tau\|$ and hence $\tau=\|\tau\|\varepsilon_e$. It remains to be shown that $\mu=\exp(\tau)*\lambda$ with $\tau\in M_+^b(G)$ and $\lambda\in\mathcal{I}_0(G)$ implies that $\tau=\|\tau\|\varepsilon_e$. Let $\tau_0\in M_+^b(G)$ and $\lambda\in\mathcal{I}_0(G)$ with $\mu=\exp(\tau_0)*\lambda$. Since $\hat\mu$ admits no zero by Lemma 1, $\hat\lambda$ does not either, so that λ admits no idempotent factor. Hence λ is the limit of a sequence of shifts of measures $\exp(\tau)$ with $\tau\in M_+^b(G)$. It follows from the inequality (3) that for all $\chi_1,\chi_2\in G^\wedge$ one has

$$\begin{aligned}&-\log|\hat\lambda(\chi_1\chi_2)|-\log|\hat\lambda(\chi_1\chi_2^{-1})|\\ \le&2(-\log|\hat\lambda(\chi_1)|-\log|\hat\lambda(\chi_2)|)\,.\end{aligned}$$

Addition of this inequality to the inequality (3) yields

$$\phi(\chi_1\chi_2) + \phi(\chi_1\chi_2^{-1}) \le 2(\phi(\chi_1) + \phi(\chi_2))$$

for all $\chi_1, \chi_2 \in G^\wedge$. Since ϕ is a positive quadratic form, we obtain equality in (3) if τ is replaced by τ_0. Since

$$|\exp(\tau_0)^\wedge(\chi)| = \int (1 - \mathrm{Re}\chi(x))\tau_0(dx)$$

for all $\chi \in G^\wedge$, we have

$$\begin{aligned} &\int (1 - \mathrm{Re}\chi_1\chi_2(x) + 1 - \mathrm{Re}\chi_1\chi_2^{-1}(x))\tau_0(dx) \\ =&2\int_G (1 - \mathrm{Re}\chi_1(x) + 1 - \mathrm{Re}\chi_2(x))\tau_0(dx)\,, \end{aligned}$$

for each $\chi_1, \chi_2 \in G^\wedge$ and therefore

$$\int (1 - \mathrm{Re}\chi_1(x))(1 - \mathrm{Re}\chi_2(x))\tau_0(dx) = 0$$

for all $\chi_1, \chi_2 \in G^\wedge$. If one choose $\chi_2 := \chi_1$, inequality (4) implies that $\tau_0 = \|\tau_0\|\varepsilon_e$, so that τ_0 is degenerate at the identity of G which is the desired result. □

Our Result. *From Theorem 1 and Theorem 2, we can obtain the following result : For any complex valued function f on $G^\wedge$ the following statements are equivalent*

(1) *There is a $\mu \in \mathcal{R}_P(G)$ such that $f = \hat{\mu}$.*
(2) *There is an $x_0 \in G$ and a positive quadratic form ϕ on $G^\wedge$ such that $f(\chi) = \chi(x_0)\exp(-\phi(\chi))$ for all $\chi \in G^\wedge$.*

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Department of Mathematics
Kumoh National University of Technology
Kumi, 730 - 701, Korea
E-mail: ljaewon@knut.kumoh.ac.kr