Kangweon-Kyungki Math. Jour. **6** (1998), No. 2, pp. 213–220

CONFORMAL CHANGE OF THE TENSOR $S_{\lambda\mu}{}^{\nu}$ IN 5-DIMENSIONAL g-UFT

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Abstract. We investigate change of the torsion tensor induced by the conformal change in 5-dimensional g-unified field theory. These topics will be studied for the second class in 5-dimensional case.

1. Introduction

The conformal change in a generalized 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by HLAVATÝ([8],1957). CHUNG([6],1968) also investigated the same topic in 4-dimensional $^*g$-unified field theory.

The Einstein's connection induced by the conformal change for all classes in 3-dimensional case, for the second and third classes in 5-dimensional case, and for the first class in 5-dimensional $^*g$-UFT, and for the second class in 6-dimensional g-UFT were investigated by CHO ([1],1992, [2],1994, [3],1996, [4],1995).

In the present paper, we investigate change of the torsion tensor $S_{\omega\mu}{}^{\nu}$ induced by the conformal change in 5-dimensional g-unified field theory. These topics will be studied for the second class in 5-dimensional case.

2. Preliminaries

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may

Received June 5, 1998.
1991 Mathematics Subject Classification: 53A45, 83E99.
Key words and phrases: Einstein connection, conformal change.
Supported by the Institute for Basic Science of Inha University, 1997.

be reffered to CHUNG([5],1988; [3],1988), CHO([1],1992; [2],1994; [3],1996; [4],1995).

2.1. n-dimensional g-unified field theory

The n-dimensional g-unified field theory (n-g-UFT hereafter) was originally suggested by HLAVATÝ([8],1957) and systematically introduced by CHUNG([7],1963).

Let X_n[1] be an n-dimensional generalized Riemannian manifold, reffered to a real coordinate system x^ν obeying coordinate transformations $x^\nu \to x^{\nu'}$, for which

$$\operatorname{Det}\left(\left(\frac{\partial x}{\partial x'}\right)\right) \neq 0. \tag{2.1}$$

In the usual Einstein's n-dimensional unified field theory, the manifold X_n is endowed with a general real nonsymmetric tensor g_{λ_μ} which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$[2] :

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$\operatorname{Det}((g_{\lambda\mu})) \neq 0 \qquad \operatorname{Det}((h_{\lambda\mu})) \neq 0. \tag{2.3}$$

Therefore we may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.4}$$

In our n-g-UFT, the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of the tensors in X_n in the usual manner.

The manifold X_n is connected by a general real connection $\Gamma^\nu_{\omega\mu}$ with the following transformation rule :

$$\Gamma^{\nu'}_{\omega'\mu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\omega'}} \cdot \frac{\partial x^\gamma}{\partial x^{\mu'}}\Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial x^{\omega'}\partial x^{\mu'}}\right) \tag{2.5}$$

[1]Throughout the present paper, we assumed that $n \geq 2$.

[2]Throughout this paper, Greek indices are used for holonomic components of tensors. In X_n all indices take the values $1, \cdots, n$ and follow the summation convention.

and satisfies the system of Einstein's equations

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha} \tag{2.6}$$

where D_ω denotes the covariant derivative with respect to $\Gamma^{\nu}_{\lambda\mu}$ and

$$S_{\lambda\mu}{}^{\nu} = \Gamma^{\nu}_{[\lambda\mu]} \tag{2.7}$$

is the *torsion tensor* of $\Gamma^{\nu}_{\lambda\mu}$. The connection $\Gamma^{\nu}_{\lambda\mu}$ satisfying (2.6) is called the *Einstein's connection.*

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \cdots$ are frequently used :

(2.8)a
$$\begin{aligned}&\mathfrak{g} = \mathrm{Det}((g_{\lambda\mu})) \neq 0, \quad \mathfrak{h} = \mathrm{Det}((h_{\lambda\mu})) \neq 0,\\ &\mathfrak{k} = \mathrm{Det}((k_{\lambda\mu})),\end{aligned}$$

(2.8)b
$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \qquad k = \frac{\mathfrak{k}}{\mathfrak{h}},$$

(2.8)c
$$K_p = k_{[\alpha_1}{}^{\alpha^1} \cdots k_{\alpha_p]}{}^{\alpha_p}, \quad (p = 0, 1, 2, \cdots)$$

(2.8)d
$$^{(0)}k_\lambda{}^\nu = \delta^\nu_\lambda, \qquad ^{(1)}k_\lambda{}^\nu = k_\lambda{}^\nu, \qquad ^{(p)}k_\lambda{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu,$$

(2.8)e
$$K_{\omega\mu\nu} = \nabla_\nu k_{\omega\mu} + \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu},$$

(2.8)f
$$\sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

where ∇_ω is the symbolic vector of the convariant derivative with respect to the Christoffel symbols $\left\{ {\nu \atop \lambda\mu} \right\}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

(2.9)a
$$K_0 = 1; K_n = k \text{ if } n \text{ is even}; \qquad K_p = 0 \text{ if } p \text{ is odd},$$

(2.9)b $$g = 1 + K_2 + \cdots + K_{n-\sigma},$$

(2.9)c $$^{(p)}k_{\lambda\mu} = (-1)^{p(p)}k_{\mu\lambda}, \qquad ^{(p)}k^{\lambda\nu} = (-1)^{p(p)}k^{\nu\lambda}.$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by T :

(2.10)a $$\overset{pqr}{T} = \overset{pqr}{T}{}_{\omega\mu\nu} = T_{\alpha\beta\gamma}{}^{(p)}k_\omega{}^{\alpha(q)}k_\mu{}^{\beta(r)}k_\nu{}^\gamma,$$

(2.10)b $$T = T_{\omega\mu\nu} = \overset{000}{T},$$

(2.10)c $$2\overset{pqr}{T}{}_{\omega[\lambda\mu]} = \overset{pqr}{T}{}_{\omega\lambda\mu} - \overset{pqr}{T}{}_{\omega\mu\lambda},$$

(2.10)d $$2\overset{(pq)r}{T}{}_{\omega\lambda\mu} = \overset{pqr}{T}{}_{\omega\lambda\mu} + \overset{qpr}{T}{}_{\omega\lambda\mu}.$$

We then have

(2.11) $$\overset{pqr}{T}{}_{\omega\lambda\mu} = -\overset{qpr}{T}{}_{\lambda\omega\mu}.$$

If the system (2.6) admits $\Gamma^\nu_{\lambda\mu}$, using the above abbreviations it was shown that the connection is of the form

(2.12) $$\Gamma^\nu_{\omega\mu} = \left\{ {\nu \atop \omega\mu} \right\} + S_{\omega\mu}{}^\nu + U^\nu{}_{\omega\mu}$$

where

(2.13) $$U_{\nu\omega\mu} = 2\overset{001}{S}{}_{\nu(\omega\mu)}.$$

The above two relations show that *our problem of determining* $\Gamma^\nu_{\omega\mu}$ *in terms of* $g_{\lambda\mu}$ *is reduced to that of studying the tensor* $S_{\omega\mu}{}^\nu$. On the other hand, it has also been shown that the tensor $S_{\omega\mu}{}^\nu$ satisfies

(2.14) $$S = B - 3\overset{(110)}{S}$$

where

(2.15) $$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}k_{\omega]}{}^\alpha k_\nu{}^\beta.$$

2.2. Some results for the second class in 5-g-UFT

In this section, we introduce some results of 5-g-UFT without proof, which are needed in our subsequent considerations.

They may be referred to CHO([1],1992).

DEFINITION 2.1. In 5-g-UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be the second class , if $K_2 \neq 0$, $K_4 = 0$.

THEOREM 2.2 (MAIN RECURRENCE RELATIONS). *For the second class in 5-UFT, the following recurrence relation hold*

$$
{}^{(p+3)}k_{\lambda}{}^{\nu} = -K_2 {}^{(p+1)}k_{\lambda}{}^{\nu}, \qquad (p = 0, 1, 2, \cdots). \tag{2.16}
$$

THEOREM 2.3 (FOR THE SECOND CLASS IN 5-g-UFT). *A necessary and sufficient condition for the existence and uniqueness of the solution of (2.5) is*

$$
1 - (K_2)^2 \neq 0. \tag{2.17}
$$

If the condition (2.17) is satisfied, the unique solution of (2.14) is given by

$$
(1 - K_2{}^2)(S - B) = -2 \overset{(10)1}{B} + (K_2 - 1) \overset{110}{B} + 2 \overset{(20)2}{B} + 2 \overset{112}{B}. \tag{2.18}
$$

3. Conformal change of the 5-dimensional torsion tensor for the second class

In this final chapter we investigate the change $S_{\lambda\mu}{}^{\nu} \to \overline{S}_{\lambda\mu}{}^{\nu}$ of the torsion tensor induced by the conformal change of the tensor $g_{\lambda\mu}$, using the recurrence relations and theorems introduced in the preceding chapter.

We say that X_n and $\overline{X}_n$ are conformal if and only if

$$
\overline{g}_{\lambda\mu}(x) = e^{\Omega} g_{\lambda\mu}(x) \tag{3.1}
$$

where $\Omega = \Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the torsion tensor $S_{\lambda\mu}{}^{\nu}$. An explicit representation of the change of 5-dimensional torsion tensor $S_{\lambda\mu}{}^{\nu}$ for the second class will be exhibited in this chapter.

AGREEMENT 3.1. Throughout this section, we agree that, if T is a function of $g_{\lambda\mu}$, then we denote $\overline{T}$ the same function of $\overline{g}_{\lambda\mu}$. In particular, if T is a tensor, so is $\overline{T}$. Furthermore, the indices of T ($\overline{T}$) will be raised and/or lowered by means of $h^{\lambda\nu}(\overline{h}^{\lambda\nu})$ and/or $h_{\lambda\mu}(\overline{h}_{\lambda\mu})$.

The results in the following theorems are needed in our further considerations. They may be referred to CHO([1],1992, [2],1994, [3],1996).

THEOREM 3.2. *In n-g-UFT, the conformal change (3.1) induces the following changes :*

$$\begin{aligned} {}^{(p)}\overline{k}_{\lambda\mu} &= e^{\Omega(p)} k_{\lambda\mu}, \qquad {}^{(p)}\overline{k}_{\lambda}{}^{\nu} = {}^{(p)}k_{\lambda}{}^{\nu}, \\ {}^{(p)}\overline{k}^{\lambda\nu} &= e^{-\Omega(p)} k^{\lambda\nu} \end{aligned} \tag{3.2a}$$

$$\overline{g} = g, \qquad \overline{K_p} = K_p, \qquad (p = 1, 2, \cdots). \tag{3.2b}$$

THEOREM 3.3. *(For all classes in 5-g-UFT). The change of the tensor $B_{\omega\mu\nu}$ induced by the conformal change (3.1) may be given by*

$$\begin{aligned} \overline{B}_{\omega\mu\nu} =& e^{\Omega}(B_{\omega\mu\nu} + k_{\nu[\omega}\Omega_{\mu]} - k_{\omega\mu}\Omega_{\nu} \\ &- h_{\nu[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} + 2{}^{(2)}k_{\nu[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} + k_{\omega\mu}{}^{(2)}k_{\nu}{}^{\delta}\Omega_{\delta}). \end{aligned} \tag{3.3}$$

Now, we are ready to derive representations of the changes $S_{\omega\mu}{}^{\nu} \to \overline{S}_{\omega\mu}{}^{\nu}$ in 5-g-UFT for the second class induced by the conformal change (3.1).

THEOREM 3.4. *The conformal change (3.1) induces the following change :*

$$\begin{aligned} 2\overline{\overset{(10)1}{B}}_{\omega\mu\nu} =& e^{\Omega}[2\overset{(10)1}{B}_{\omega\mu\nu} + (-2{}^{(4)}k_{\nu[\omega}k_{\mu]}{}^{\delta} \\ &+ 2{}^{(2)}k_{\nu[\omega}k_{\mu]}{}^{\delta} - k_{\nu[\omega}{}^{(2)}k_{\mu]}{}^{\delta})\Omega_{\delta} - {}^{(3)}k_{\nu[\omega}\Omega_{\mu]}]. \end{aligned} \tag{3.4}$$

THEOREM 3.5. *The conformal change (3.1) induces the following change :*

$$\begin{aligned}\overline{\overset{ppq}{B}}{}_{\omega\mu\nu} =& e^{\Omega}[\overset{ppq}{B}{}_{\omega\mu\nu} + (-1)^{p}\{2^{(p+q+2)}k_{\nu[\omega}{}^{(p+1)}k_{\mu]}{}^{\delta} \\ &+ {}^{(2p+1)}k_{\omega\mu}{}^{(2+q)}k_{\nu}{}^{\delta} - {}^{(2p+1)}k_{\omega\mu}{}^{(q)}k_{\nu}{}^{\delta} \\ &+ {}^{(p+q+1)}k_{\nu[\omega}{}^{(p)}k_{\mu]}{}^{\delta} - {}^{(p+q)}k_{\nu[\omega}{}^{(p+1)}k_{\mu]}{}^{\delta}\}\Omega_{\delta}]. \\ &\qquad \begin{pmatrix} p = 0,1,2,3,4,\cdots \\ q = 0,1,2,3,4,\cdots \end{pmatrix}\end{aligned} \tag{3.5}$$

THEOREM 3.6. *The change $S_{\omega\mu}{}^{\nu} \rightarrow \overline{S}_{\omega\mu}{}^{\nu}$ induced by conformal change (3.1) may be represented by*

$$\begin{aligned}\overline{S}_{\omega\mu}{}^{\nu} =& S_{\omega\mu}{}^{\nu} + \frac{1}{C}[(3 - K_2 + K_2{}^2)^{(2)}k^{\nu}{}_{[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} \\ &+ 2K_2{}^2k^{\nu}{}_{[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} + (4K_2 + 2K_2{}^2)k^{\nu}{}_{[\omega}{}^{(2)}k_{\mu]}{}^{\delta}\Omega_{\delta} \\ &+ (1 - K_2 + 4K_2{}^2)k_{\omega\mu}{}^{(2)}k^{\nu\delta}\Omega_{\delta} + (-1 - K_2)k_{\omega\mu}\Omega^{\nu} \\ &+ (1 + K_2)k^{\nu}{}_{[\omega}\Omega_{\mu]} + (-1 - K_2{}^2)h^{\nu}{}_{[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta}]\end{aligned} \tag{3.6}$$

where $C = K_2{}^2 - 1$.

Proof. In virtue of (2.18) and Agreement (3.1), we have

$$(1 - \overline{K}_2{}^2)(\overline{S} - \overline{B}) = -2\,\overline{\overset{(10)1}{B}} + (\overline{K}_2 - 1)\,\overline{\overset{110}{B}} + 2\,\overline{\overset{(20)2}{B}} + 2\,\overline{\overset{112}{B}}. \tag{3.7}$$

The relation (3.6) follows by substituting (3.2), (3.3), (3.4), (3.5), (2.16), Definition (2.1), into (3.7). □

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