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ON THE BEHAVIOR OF L^2 HARMONIC FORMS ON COMPLETE MANIFOLDS AT INFINITY AND ITS APPLICATIONS

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Abstract. We investigate the behavior of L^2 harmonic one forms on complete manifolds and as an application, we show the space of L^2harmonic one forms on a complete Riemannian manifold of nonnegative Ricci curvature outside a compact set with bounded $n/2$-norm of Ricci curvature satisfying the Sobolev inequality is finite dimensional.

1. Introduction

In this paper, all manifolds are complete, oriented and Riemannian unless explicitly stated otherwise. Let (M, g) be a complete Riemannian manifold and $H^pL_2(M)$ denote the space of L^2 harmonic p-forms on M, i.e., p-forms $\omega \in \Omega^p(M)$ such that

$$\Delta\omega = 0, \quad \int_M \omega \wedge \star\omega = \int_M |\omega|^2 \, dv_g < \infty. \tag{1.1}$$

It is clear that $H^pL_2(M)$ is naturally isomorphic to $H^{n-p}L_2(M)$ under the Hodge star operator $\star, n = dim(M)$.

The basic fact for the L^2 harmonic forms is the following

Lemma 1.1 ([Gro1], [Ya1]). *If ω is L^2 harmonic p-form on a complete Riemannian manifold M, then it is closed and co-closed, i.e., $d\omega = \delta\omega = 0$.*

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Obviously harmonic 0-forms are harmonic functions and there are several well-known results for harmonic functions on a complete noncompact Riemannian manifold. For instance, from Lemma 1.1 it is easy to see that every harmonic L^2-function on a complete Riemannian manifold M is constant. For harmonic 1-forms, if M has nonnegative Ricci curvature, then there is no L^2 harmonic one forms on M ([Ya3]). However, if M has nonnegative Ricci curvature outside a compact set, then there should be a nontrivial L^2 harmonic one form ([Don]). In fact, Donnelly has proved the following

THEOREM 1.2 ([Don]). *Assume M is a complete Riemannian manifold of nonnegative Ricci curvature outside a compact set. Then for fixed $p \geq 2$, the space of L^{4p} harmonic one forms on M is finite dimensional.*

Note that L^2 harmonic one forms are not necessarily in the class of L^{4p}, $p \geq 2$. In this article, we will investigate the behavior of L^2 harmonic forms and prove finiteness of dimension for the space of L^2 harmonic 1-forms on a complete noncompact Riemannian manifold as an application.

2. Behavior of harmonic one forms at infinity

Assume M is a complete noncompact Riemannian n-manifold ($n \geq 3$) of nonnegative Ricci curvature outside a compact set K satisfying the Sobolev inequality, i.e.,

$$\left(\int_M \phi^{2\mu}\right)^{\frac{1}{\mu}} \leq C_s \int_M |\nabla\phi|^2 \quad \text{for all} \quad \phi \in C_c^1(M), \tag{2.1}$$

with some positive constant C_s and $\mu = \frac{n}{n-2}$.

Suppose K is contained in a geodesic ball $B(p, r_o)$ of radius $r_o > 0$ for a point $p \in M$. The volume comparison theorem shows for $r \geq r_o$ sufficiently large

$$vol(B(p,r)) \leq V r^n, \tag{2.2}$$

where $V > 0$ is a positive constant. We denote $M \setminus B(p,r)$ by $D(r)$ for $r > r_o$.

In this section, we shall prove the following

THEOREM 2.1. *Let M be a complete Riemannian n-manifold ($n \geq 3$) of nonnegative Ricci curvature outside a compact set satisfying the Sobolev inequality. Suppose furthermore*

$$\int_M |Ric|^{n/2} < \infty.$$

If ω is an L^2 harmonic one form on M, then for $r > r_o$,

$$\sup_{D(r)} |\omega| \leq C \cdot r^{-n/2},$$

where $C > 0$ is a positive constant depending only on C_s, K, V, n and $k = \max |Ric|$.

EXAMPLE. Let (S^n, g_1), $n \geq 3$, be the standard sphere and $(\mathbf{R}^n, g_o)$ the Euclidean flat space. Let $M = S^n \# \mathbf{R}^n$ be a connected sum of S^n and $\mathbf{R}^n$. The Riemannian metric g on M is obtained from g_1 and g_o by smoothing on the gluing part. Then obviously one has $Ric(M, g) \geq -k$ for some constant $k > 0$ and $Ric(M, g) \geq 0$ outside a compact set. In fact, if letting K be a compact set containing the part S^n, then $Ric \equiv 0$ on $M - K$ and so

$$\int_M |Ric|^{n/2} = \int_K |Ric|^{n/2} < \infty.$$

It is also well-known (cf. [Aub]) that such a manifold M satisfies the Sobolev inequality (2.1). It follows from Theorem 3.1 that the space of L^2 harmonic one-forms is finite dimensional. Furthermore it is known ([Don]) that such a manifold admits a non-constant bounded harmonic function.

First we shall prove some a priori estimates for a nonnegative function u which satisfies

$$\Delta u \geq -fu \quad \text{on} \quad M, \tag{2.3}$$

with nonnegative function $f \leq k$, k constant.

Lemma 2.2. *Suppose $f \in L^{n/2}(M)$, and $u \in L^2(M)$. Then one has*

$$\left(\int_{D(2r)} u^{2\mu}\right)^{\frac{1}{\mu}} \leq C(r^{-2}+1)\int_{D(r)} u^2, \tag{2.4}$$

where $C>0$ depends only on C_s, k and V.

Proof. For any constant $\alpha \geq 1$, multiplying (2.3) by $\phi^2 u^{2\alpha-1}$, we have

$$k\int \phi^2 u^{2\alpha} \geq -\int \phi^2 u^{2\alpha-1}\Delta u,$$

for any compactly supported Lipschitz function ϕ on M. Integration by parts gives

$$\begin{aligned} -\int \phi^2 u^{2\alpha-1}\Delta u &= 2\int \phi u^{2\alpha-1}\langle\nabla\phi,\nabla u\rangle + (2\alpha-1)\int \phi^2 u^{2\alpha-2}|\nabla u|^2 \\ &\geq 2\int \phi u^{2\alpha-1}\langle\nabla\phi,\nabla u\rangle + \alpha\int \phi^2 u^{2\alpha-2}|\nabla u|^2. \end{aligned}$$

By (2.1), and using the following identity

$$\begin{aligned} \int |\nabla(\phi u^\alpha)|^2 &= \int |\nabla\phi|^2 u^{2\alpha} \\ &\quad + 2\alpha\int \phi u^{2\alpha-1}\langle\nabla\phi,\nabla u\rangle + \alpha^2\int \phi^2 u^{2\alpha-2}|\nabla u|^2, \end{aligned}$$

we get

$$k\alpha\int \phi^2 u^{2\alpha} + \int |\nabla\phi|^2 u^{2\alpha} \geq C_s^{-1}\left(\int (\phi^2 u^{2\alpha})^\mu\right)^{\frac{1}{\mu}}. \tag{2.5}$$

Let us now choose $\phi(r)$ to be the cut-off function such that $0 \leq \phi \leq 1, \phi = 0$ in $B(p,r)\cup D(2r')$ and $\phi = 1$ in $D(2r)\setminus D(2r')$ with $|\nabla\phi| \leq C_1(r^{-1}+{r'}^{-1})$ for $2r < r'$. Substituting this ϕ into (2.5), one gets

$$\begin{aligned} C_s^{-1}\left(\int_{D(2r)\setminus D(r')} u^{2\alpha\mu}\right)^{\frac{1}{\mu}} &\leq \\ & C_1\left(\frac{1}{r}+\frac{1}{r'}\right)^2\int_{supp|\nabla\phi|} u^{2\alpha} + k\alpha\int_{D(r)\setminus D(2r')} u^{2\alpha}. \end{aligned}$$

Letting $\alpha = 1$ and $r' \to \infty$, one gets (2.4) □

Lemma 2.3. *Suppose $u \in L^2(M)$. Then*

$$\sup_{D(r)\setminus D(2r)} u \leq C r^{-n/2} \int_{D(r/2)} u^2.$$

Proof. Let $\beta \leq \alpha$. From (2.5) one has

$$\beta \int \phi^2 u^{2\beta} + \int |\nabla\phi|^2 u^{2\beta} \geq C_s^{-1} \left(\int (\phi^2 u^{2\beta})^{\mu} \right)^{\frac{1}{\mu}}. \tag{2.6}$$

For $r_1 < r_2 < r_3 < r_4$ with $r_1 - r_2 = r_3 - r_4$, we take ϕ so that $0 \leq \phi \leq 1, \phi \equiv 0$ in $B(p,r) \cup D(r_4), \phi = 1$ in $D(r_2) \setminus D(r_3)$ with $|\nabla\phi| \leq C_2(r_2 - r_1)^{-1}$. Then substituting this ϕ into (2.6) we get

(2.7)

$$C_s^{-1} \left(\int_{D(r_2)\setminus D(r_3)} u^{2\beta\mu} \right)^{\frac{1}{\mu}} \leq C_2 \left(\beta + (r_2 - r_1)^{-2} \right) \int_{D(r_1)\setminus D(r_4)} u^{2\beta}.$$

We set

$$\Gamma(\beta, r, r') = \left(\int_{D(r)\setminus D(r')} u^{\beta} \right)^{1/\beta}.$$

We have

$$\Gamma(2\beta\mu, r_2, r_3) \leq \{C(\beta + (r_2 - r_1)^{-2})\}^{1/2\beta} \Gamma(2\beta, r_1, r_4). \tag{2.8}$$

Taking $r_{1,m} = (1 - 2^{-m})r, r_{2,m} = r_{1,m+1}, r_{4,m} = (2 + 2^{-m})r, r_{3,m} = r_{4,m+1}$ and $\beta_m = 2\beta\mu^m$, we obtain

$$\Gamma(\beta_{m+1}, r_{1,m+1}, r_{4,m+1}) \leq \{C(\beta + 4 \cdot 4^m r^{-2})\}^{1/2\beta\mu^m} \Gamma(\beta_m, r_{1,m}, r_{4,m}).$$

Inductively we have

$$\Gamma(\beta_m, r_{1,m}, r_{4,m}) \leq \left(\prod_{m=1}^{\infty} \left\{ C(\beta + 4 \cdot 4^m \cdot r^{-2}) \right\}^{1/2\beta\mu^m} \right) \Gamma(2, r/2, 5r/2)$$

Since $\sum m\mu^{-m} < \infty$ and $\sum \mu^{-m} = n/2$, one has

$$\prod_{m=1}^{\infty} \left\{C(\beta + 4 \cdot 4^m \cdot r^{-2})\right\}^{1/2\beta\mu^m} \le \prod_{m=1}^{\infty} \left\{C(\beta + 4r^{-2})4^m\right\}^{1/2\beta\mu^m}$$
$$\le C_3(\beta + 4r^{-2})^{n/4\beta}$$

Hence, by setting $\beta = 1$, we get

$$\sup_{D(r)\setminus D(2r)} u \le C \cdot r^{-n/2}\{\int_{D(r/2)} u^2\}.$$ □

PROPOSITION 2.4. *Under the same hypotheses as above, one has*

$$\sup_{D(2r)} u \le C \cdot r^{-n/2} \int_{D(r)} u^2.$$

Proof. It follows from Lemma 2.3 directly. In fact, one has

$$\sup_{D(2r)\setminus D(4r)} u \le C \cdot (2r)^{-n/2} \int_{D(r)} u^2.$$

and $\int_{D(r)} u^2$ is bounded by $\int_M u^2$ which is independent of r. So the radius is becoming larger, the supremum is getting smaller. □

Proof of Theorem 2.1. It follows from Proposition 2.4. □

3. Applications

As an application, we shall prove the following

THEOREM 3.1. *Let M be a complete Riemannian n-manifold ($n \ge 3$) of nonnegative Ricci curvature outside a compact set satisfying the Sobolev inequality. If*

$$\int_M |Ric|^{n/2} < \infty,$$

then the vector space of L^2 harmonic one forms on M is finite dimensional.

The proof follows from a priori estimates of the behavior of L^2 harmonic forms at infinity. Since Theorem 3.1 is trivial if M is compact, we may assume M is noncompact.

Proof of Theorem 3.1. Let ω be a differential one form on M. From the Bochner-Weitzenböck formula, we have

$$\frac{1}{2}\Delta|\omega|^2 = |D\omega|^2 + Ric(\omega^\#, \omega^\#). \tag{3.1}$$

On the other hand, since

$$\frac{1}{2}\Delta|\omega|^2 = |\omega|\Delta|\omega| + |\nabla|\omega||^2, \tag{3.2}$$

from Kato's inequlaity

$$|\nabla|\omega|| \leq |D\omega|,$$

one has

$$\Delta|\omega| \geq -|Ric||\omega|. \tag{3.3}$$

We may assume from the hypothesis that

$$Ric(M) \geq -k,$$

where $k > 0$ depends on the compact set K and assume $K \subset B(p, r_o)$. Let ω be a L^2 harmonic one form on M. Then one has from (3.3)

$$\Delta|\omega| \geq -k|\omega|.$$

Applying Theorem 2.1, one has for $r_o < r$

$$\begin{aligned}\int_M |\omega|^{4p} &= \int_{B(p,r)} |\omega|^{4p} + \int_{D(r)} |\omega|^{4p} \\ &\leq \int_{B(p,r)} |\omega|^{4p} + C \cdot r^{-2np} \left(\int_{D(r/2)} |\omega|^2 \right)^{4p}.\end{aligned}$$

This implies $H^1L_2(M) \subset H^1L_{4p}(M)$ and so the proof follows from Theorem 1.2. □

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