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# REMARKS ON THE REIDEMEISTER NUMBER OF A $G$-MAP

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Abstract. For a $G$-map $\phi : X \to X$, we define and characterize the Reidemeister number $R_G(\phi)$ of $\phi$. Also, we prove that $R_G(\phi)$ is a $G$-homotopy invariance and we obtain a lower bound of $R_G(\phi)$.

## 1. Introduction

For a self map $\phi$ of a compact connected polyhedron $X$, let $\pi_1(X, x_0)$ and $L(\phi)$ denote the fundamental group of $X$ based at $x_0 \in X$ and the set of all liftings of $\phi$ on a universal covering space, respectively. For two $\tilde{f}_1, \tilde{f}_2 \in L(\phi)$, set $\tilde{f}_1 \sim \tilde{f}_2$ if there exists an $[\alpha] \in \pi_1(X, x_0)$ such that $\tilde{f}_2 = [\alpha] \circ \tilde{f}_1 \circ [\alpha]^{-1}$. This is an equivalence relation on $L(\phi)$. The cardinal number of the set of all equivalence classes is called the Reidemeister number of $\phi$ and denoted by $R(\phi)$.

The following theorem is well known.

Theorem. *(1) The number $R(\phi)$ is a homotopy invariance.*

*(2) There exists a homomorphism $\tilde{f}_\#$ of $\pi_1(X, x_0)$ induced by an element $\tilde{f}$ of $L(\phi)$. For any two $[\alpha], [\beta] \in \pi_1(X, x_0)$, set $[\alpha] \sim [\beta]$ if there exists a $[\gamma] \in \pi_1(X, x_0)$ such that $[\beta] = [\gamma] \circ [\alpha] \circ \tilde{f}_\#([\gamma]^{-1})$. Then this is an equivalence relation on $\pi_1(X, x_0)$ and the cardinal number of the set of equivalence classes is equal to $R(\phi)$.*

*(3) There exists a homomorphism $H_1(\phi)$ of the 1-st homology group $H_1(X)$ of $X$ induced by $\phi$ such that $R(\phi) \geq |Coker(i_{H_1(X)} \cdot H_1(\phi)')|$, where $i_{H_1(X)} \cdot H_1(\phi)'$ is a function of $H_1(X)$ defined by $(i_{H_1(X)} \cdot H_1(\phi)')([z]) = [z] \cdot H_1(\phi)([z]^{-1})$ for $[z] \in H_1(X)$.*

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The purpose of this note is to define the Reidemeister number $R_G(\phi)$ of a $G$-map $\phi$ and to generalize the above theorem to the case of a $G$-map.

We shall assume throughout this note that $X$ is a connected, locally path connected, and semi-locally 1-connected space, that $G$ is a topological group acting effectively on $X$, and that $\phi : X \to X$ is a $G$-map.

## 2. Main results

Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a based universal covering projection. For each $g \in G$, let $\theta_g : X \to X$ be the homeomorphism defined by $\theta_g(x) = gx$ for $x \in X$. Since $\tilde{X}$ is simply connected, the map $\theta_g \circ \phi$ of $X$ can be covered by maps of $\tilde{X}$. Let $L(\theta_g \circ \phi)$ be the set of all such liftings of $\theta_g \circ \phi$ and let $\mathcal{L}(\phi)$ be the disjoint union of the collection $\{L(\theta_g \circ \phi) | g \in G\}$. Clearly, $\mathcal{L}(i_X)$ is a subgroup of the group of all homeomorphisms of $\tilde{X}$.

Definition 1. Two liftings $\tilde{f}_1, \tilde{f}_2 \in \mathcal{L}(\phi)$ are said to be *conjugate* if there exists $\tilde{l} \in \mathcal{L}(i_X)$ such that $\tilde{f}_2 = \tilde{l} \circ \tilde{f}_1 \circ \tilde{l}^{-1}$. The equivalence classes by conjugacy are called *$G$-lifting classes* and the $G$-lifting classes of $\tilde{f}$ is denoted by $[\tilde{f}]_G$. If $\mathcal{L}'(\phi)$ is the set of all $G$-lifting classes, then $R_G(\phi) = |\mathcal{L}'(\phi)|$, the order of $\mathcal{L}'(\phi)$, is called the *Reidemeister number* of $\phi$.

Theorem 2. *If two maps $\phi_1, \phi_2 : X \to X$ are $G$-homotopic, then $R_G(\phi_1) = R_G(\phi_2)$.*

*Proof.* Let $H : X \times I \to X$ be a $G$-homotopy from $\phi_1$ to $\phi_2$. If $\tilde{f} \in L(\theta_g \circ \phi_1)$, then $(p \circ \tilde{f})(\tilde{x}) = (\theta_g \circ \phi_1)(p(\tilde{x})) = (\theta_g \circ H)(p(\tilde{x}), 0)$ for all $\tilde{x} \in \tilde{X}$. Since $\tilde{X} \times I$ is connected, there exists a unique lifting $\tilde{F} : \tilde{X} \times I \to \tilde{X}$ of $\theta_g \circ H$ such that $\tilde{F}(\cdot, 0) = \tilde{f}$ and $p \circ \tilde{F} = \theta_g \circ H \circ (p \times i_I)$. Let $\tilde{F}(\cdot, 1) = f'$. Since $f' \in L(\theta_g \circ \phi_2)$, we have a function $\Phi_g : L(\theta_g \circ \phi_1) \to L(\theta_g \circ \phi_2)$ defined by $\Phi_g(\tilde{f}) = f'$.

Claim 1. $\Phi_g$ is one-to-one: Let $\tilde{f}_1$ and $\tilde{f}_2$ be distinct elements of $L(\theta_g \circ \phi_1)$ and let $\tilde{F}_1$ and $\tilde{F}_2$ be the liftings of $\theta_g \circ H$ such that $\tilde{F}(\cdot, 0) = \tilde{f}_1$ and $\tilde{F}_2(\cdot, 0) = \tilde{f}_2$. If $\tilde{F}_1(\tilde{x}_0, 1) = \tilde{F}_2(\tilde{x}_0, 1)$, then $\tilde{F}_1 = \tilde{F}_2$. This is a contradiction. Thus $\Phi_g(\tilde{f}_1) = \tilde{F}_1(\cdot, 1) \neq \tilde{F}_2(\cdot, 1) = \Phi_g(\tilde{f}_2)$.

Claim 2. $\Phi_g$ is onto: Let $f' \in L(\theta_g \circ \phi_2)$. If we define $H' : X \times I \to X$ by $H'(x,t) = H(x, 1-t)$ for $(x,t) \in X \times I$, then $\theta_g \circ H'$ is a homotopy from $\theta_g \circ \phi_2$ to $\theta_g \circ \phi_1$. Let $\tilde{F}' : \tilde{X} \times I \to \tilde{X}$ be the lifting of $\theta_g \circ H'$ with $\tilde{F}'(\cdot, 0) = f'$. Define $\tilde{F} : \tilde{X} \times I \to \tilde{X}$ by $\tilde{F}(\tilde{x}, t) = \tilde{F}'(\tilde{x}, 1-t)$ for $(\tilde{x}, t) \in \tilde{X} \times I$. Since $p \circ \tilde{F}(\tilde{x}, t) = p \circ \tilde{F}'(\tilde{x}, 1-t) = \theta_g \circ H'(p(\tilde{x}), 1-t) = \theta_g \circ H(p(\tilde{x}), t)$ for $(\tilde{x}, t) \in \tilde{X} \times I$ and $\tilde{F}(\tilde{x}, 0) \in L(\theta_g \circ \phi_1)$, we have $\tilde{F}$ is the lifting of $\theta_g \circ H$ such that $\tilde{F}(\tilde{x}, 1) = f' = \Phi_g(\tilde{F}(\tilde{x}, 0))$.

Let $\Phi : \mathcal{L}(\phi_1) \to \mathcal{L}(\phi_2)$ be the function whose restriction to $L(\theta_g \circ \phi_1)$ is equal to $\Phi_g$ for every $g \in G$. By Claim 1 and Claim 2, $\Phi$ is one-to-one and onto.

Claim 3. For any $\tilde{l} \in \mathcal{L}(i_X)$ and any $\tilde{f} \in \mathcal{L}(\phi_1)$, $\Phi(\tilde{f} \circ \tilde{l}) = \Phi(\tilde{f}) \circ \tilde{l}$ and $\Phi(\tilde{l} \circ \tilde{f}) = \tilde{l} \circ \Phi(\tilde{f})$: Assume $\tilde{l} \in L(\theta_{g'} \circ i_X)$ and $\tilde{f} \in L(\theta_g \circ \phi_1)$. Clearly, $\tilde{f} \circ \tilde{l} \in L(\theta_{gg'} \circ \phi_1)$. Let $\tilde{F}$ and $\tilde{K}$ be the liftings of $\theta_g \circ H$ and $\theta_{gg'} \circ H$, respectively, such that $\tilde{F}(\cdot, 0) = \tilde{f}$ and $\tilde{K}(\cdot, 0) = \tilde{f} \circ \tilde{l}$. Define $\tilde{F}' : \tilde{X} \times I \to \tilde{X}$ by $\tilde{F}'(\tilde{x}, t) = \tilde{F}(\tilde{l}(\tilde{x}), t)$ for $(\tilde{x}, t) \in \tilde{X} \times I$. Because $H$ is a $G$-homotopy, $\tilde{F}'$ is the lifting of $\theta_{gg'} \circ H$ with $\tilde{F}'(\cdot, 0) = \tilde{f} \circ \tilde{l}$. By the uniqueness of lifting, $\tilde{F}' = \tilde{K}$. Therefore, $\Phi(\tilde{f} \circ \tilde{l}) = \tilde{K}(\cdot, 1) = \tilde{F}'(\cdot, 1) = \tilde{F}(\tilde{l}(\cdot), 1) = \Phi(\tilde{f}) \circ \tilde{l}$. Similarly, we can prove $\Phi(\tilde{l} \circ \tilde{f}) = \tilde{l} \circ \Phi(\tilde{f})$.

Now, assume that for $\tilde{f}_1, \tilde{f}_2 \in \mathcal{L}(\phi_1)$, $\tilde{f}_2 = \tilde{l} \circ \tilde{f}_1 \circ \tilde{l}^{-1}$. By Claim 3, $\Phi(\tilde{f}_2) = \Phi(\tilde{l} \circ \tilde{f}_1 \circ \tilde{l}^{-1}) = \tilde{l} \circ \Phi(\tilde{f}_1 \circ \tilde{l}^{-1}) = \tilde{l} \circ \Phi(\tilde{f}_1) \circ \tilde{l}^{-1}$. Thus $\Phi$ induces a one-to-one onto function from $\mathcal{L}'(\phi_1)$ to $\mathcal{L}'(\phi_2)$, and hence we have $R_G(\phi_1) = R_G(\phi_2)$. □

Lemma 3. *For any fixed* $\tilde{f}_0 \in L(\theta_{g_0} \circ \phi)$, *there exists a function* $\Psi : \mathcal{L}(\phi) \to \mathcal{L}(i_X)$ *induced by* $\tilde{f}_0$ *such that* $\Psi(\tilde{l} \circ \tilde{f}_0) = \tilde{l}$ *for every* $\tilde{l} \in \mathcal{L}(i_X)$.

*Proof.* Assume $\tilde{f} \in L(\theta_g \circ \phi)$. Since $(\theta_{gg_0^{-1}} \circ p)(\tilde{f}_0(\tilde{x}_0)) = (p \circ \tilde{f})(\tilde{x}_0)$, there exists a unique map $\tilde{l} \in L(\theta_{gg_0^{-1}} \circ i_X)$ such that $\tilde{l}(\tilde{f}_0(\tilde{x}_0)) = \tilde{f}(\tilde{x}_0)$. Obviously, $\tilde{l} \circ \tilde{f}_0 \in L(\theta_g \circ \phi)$. By the uniqueness of lifting, $\tilde{l} \circ \tilde{f}_0 = \tilde{f}$. This induces a function $\Psi_g : L(\theta_g \circ \phi) \to L(\theta_{gg_0^{-1}} \circ i_X)$ defined by $\Psi_g(\tilde{f}) = \tilde{l}$, and hence we have a function $\Psi : \mathcal{L}(\phi) \to \mathcal{L}(i_X)$ whose restriction to $L(\theta_g \circ \phi)$ is equal to $\Psi_g$ for every $g \in G$. Now, let $\tilde{l} \in L(\theta_{g'} \circ i_X)$. Then $\Psi(\tilde{l} \circ \tilde{f}_0) \in L(\theta_{g' g_0 g_0^{-1}} \circ i_X) = L(\theta_{g'} \circ i_X)$ because $\tilde{l} \circ \tilde{f}_0 \in L(\theta_{g' g_0} \circ \phi)$. Since $\tilde{l} \circ \tilde{f}_0 = \Psi(\tilde{l} \circ \tilde{f}_0) \circ \tilde{f}_0$, we have $\Psi(\tilde{l} \circ \tilde{f}_0) = \tilde{l}$. □

LEMMA 4. *Let $\tilde{f} \in \mathcal{L}(\phi)$ be fixed. Then there exists a homomorphism $\tilde{f}_\# : \mathcal{L}(i_X) \to \mathcal{L}(i_X)$ induced by $\tilde{f}$.*

*Proof.* By the above lemma, there exists a function $\Psi : \mathcal{L}(\phi) \to \mathcal{L}(i_X)$ induced by $\tilde{f}$ such that $\Psi(\tilde{f} \circ \tilde{l}) \circ \tilde{f} = \tilde{f} \circ \tilde{l}$ for every $\tilde{l} \in \mathcal{L}(i_X)$. This induces a homomorphism $\tilde{f}_\# : \mathcal{L}(i_X) \to \mathcal{L}(i_X)$ defined by $\tilde{f}_\#(\tilde{l}) = \Psi(\tilde{f} \circ \tilde{l})$. In fact, if $\tilde{l}_1, \tilde{l}_2 \in \mathcal{L}(i_X)$, then

$$\begin{aligned}\Psi(\tilde{f} \circ (\tilde{l}_1 \circ \tilde{l}_2)) &= \Psi((\tilde{f} \circ \tilde{l}_1) \circ \tilde{l}_2)\\ &= \Psi(\Psi(\tilde{f} \circ \tilde{l}_1) \circ (\tilde{f} \circ \tilde{l}_2))\\ &= \Psi(\Psi(\tilde{f} \circ \tilde{l}_1) \circ \Psi(\tilde{f} \circ \tilde{l}_2) \circ \tilde{f})\\ &= (\Psi(\tilde{f} \circ \tilde{l}_1) \circ \Psi(\tilde{f} \circ \tilde{l}_2)),\end{aligned}$$

so $\tilde{f}_\#(\tilde{l}_1 \circ \tilde{l}_2) = \Psi(\tilde{f} \circ (\tilde{l}_1 \circ \tilde{l}_2)) = \Psi(\tilde{f} \circ \tilde{l}_1) \circ \Psi(\tilde{f} \circ \tilde{l}_2) = \tilde{f}_\#(\tilde{l}_1) \circ \tilde{f}_\#(\tilde{l}_2)$.□

THEOREM 5. *For any two $\tilde{l}_1, \tilde{l}_2 \in \mathcal{L}(i_X)$, set $\tilde{l}_1 \sim \tilde{l}_2$ if there exists an $\tilde{l} \in \mathcal{L}(i_X)$ such that $\tilde{l}_2 = \tilde{l} \circ \tilde{l}_1 \circ \tilde{f}_\#(\tilde{l}^{-1})$. This is an equivalence relation on $\mathcal{L}(i_X)$. Let $[\tilde{l}]_{\tilde{f}}$ be the equivalence class of $\tilde{l}$. If $\mathcal{L}'(i_X)$ is the set of all equivalence classes, then $R_G(\phi) = |\mathcal{L}'(i_X)|$.*

*Proof.* Define $\Psi' : \mathcal{L}(i_X) \to \mathcal{L}(\phi)$ by $\Psi'(\tilde{l}) = \tilde{l} \circ \tilde{f}$ for $\tilde{l} \in \mathcal{L}(i_X)$. Since $(\Psi \circ \Psi')(\tilde{l}) = \Psi(\tilde{l} \circ \tilde{f}) = \tilde{l}$ for $\tilde{l} \in \mathcal{L}(i_X)$ and $(\Psi' \circ \Psi)(f') = \Psi(f') \circ \tilde{f} = f'$ for $f' \in \mathcal{L}(\phi)$, the map $\Psi$ is one-to-one and onto.

Let $\tilde{l}_1, \tilde{l}_2 \in \mathcal{L}(i_X)$. Then $[\tilde{l}_1 \circ \tilde{f}]_G = [\tilde{l}_2 \circ \tilde{f}]_G$ if and only if there exists an $\tilde{l} \in \mathcal{L}(i_X)$ such that

$$\begin{aligned}\tilde{l}_2 \circ \tilde{f} &= \tilde{l} \circ (\tilde{l}_1 \circ \tilde{f}) \circ \tilde{l}^{-1}\\ &= (\tilde{l} \circ \tilde{l}_1) \circ (\tilde{f} \circ \tilde{l}^{-1})\\ &= (\tilde{l} \circ \tilde{l}_1) \circ (\tilde{f}_\#(\tilde{l}^{-1}) \circ \tilde{f})\end{aligned}$$

if and only if $\tilde{l}_2 = \tilde{l} \circ \tilde{l}_1 \circ \tilde{f}_\#(\tilde{l}^{-1})$ if and only if $[\tilde{l}_2]_{\tilde{f}} = [\tilde{l}_1]_{\tilde{f}}$. Thus $\Psi$ induces a one-to-one correspondence between $\mathcal{L}'(\phi)$ and $\mathcal{L}'(i_X)$. □

In [4], F. Rhodes defined the fundamental group $\sigma(X, x_0, G)$ of a $G$-space $X$, which is a generalization of the concept of the fundamental group of a topological space.

Lemma 6. *$\mathcal{L}(i_X)$ is isomorphic to $\sigma(X, x_0, G)$*

*Proof.* For any $\tilde{l} \in L(\theta_g \circ i_X) \subset \mathcal{L}(i_X)$, choose a path $\tilde{\gamma}$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{l}(\tilde{x}_0)$. Define $\iota : \mathcal{L}(i_X) \to \sigma(X, x_0, G)$ by $\iota(\tilde{l}) = [p\tilde{\gamma}; g]$. Since $\tilde{X}$ is simply connected, $\iota$ is well defined. To show that $\iota$ is a homomorphism, let $\tilde{l}_1 \in L(\theta_{g_1} \circ i_X)$ and $\tilde{l}_2 \in L(\theta_{g_2} \circ i_X)$. If $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are paths from $\tilde{x}_0$ to $\tilde{l}_1(\tilde{x}_0)$ and $\tilde{l}_2(\tilde{x}_0)$, respectively, then $\tilde{l}_2 \circ \tilde{\gamma}_1$ is a path in $\tilde{X}$ from $\tilde{l}_2(\tilde{x}_0)$ to $(\tilde{l}_2 \circ \tilde{l}_1)(\tilde{x}_0)$ and $p(\tilde{l}_2 \circ \tilde{\gamma}_1) = g_2(p \circ \tilde{\gamma}_1)$. Thus, if $\tilde{\gamma}_3$ is a path from $\tilde{x}_0$ to $(\tilde{l}_2 \circ \tilde{l}_1)(\tilde{x}_0)$, we have $\iota(\tilde{l}_2 \circ \tilde{l}_1) = [p \circ \tilde{\gamma}_3; g_2 g_1] = [p \circ \tilde{\gamma}_2 * g_2(p \circ \tilde{\gamma}_1); g_2 g_1] = [p \circ \tilde{\gamma}_2; g_2] * [p \circ \tilde{\gamma}_1; g_1]$. This shows that $\iota$ is a homomorphism.

Now, consider the following diagram

$$\begin{array}{ccccccccc} 0 & \to & \pi_1(X, x_0) & \to & \mathcal{L}(i_X) & \to & G & \to & 0 \\ & & \downarrow & & \iota \downarrow & & \downarrow & & \\ 0 & \to & \pi_1(X, x_0) & \to & \sigma(X, x_0, G) & \to & G & \to & 0, \end{array}$$

where each map is defined naturally. It is easy to show that each square is commutative and each low is exact. By the five lemma, $\iota$ is an isomorphism. □

Let $\phi_\# = \iota \circ \tilde{f}_\# \circ \iota^{-1}$. Then $\phi_\#$ is a homomorphism of $\sigma(X, x_0, G)$. By Lemma 6, It is possible to restate Theorem 5 as follows:

Theorem 7. *For any two $[\alpha_1; g_1], [\alpha_2; g_2] \in \sigma(X, x_0, G)$, set $[\alpha_1; g_1] \sim [\alpha_2; g_2]$ if there exists a $[\beta; h] \in \sigma(X, x_0, G)$ such that $[\alpha_2; g_2] = [\beta; h] * [\alpha_1; g_1] * \phi_\#([\beta; h]^{-1})$. This is an equivalence relation on $\sigma(X, x_0, G)$ and the order of the set of equivalence classes is equal to $R_G(\phi)$.*

Theorem 8. *If $\sigma(X, x_0, G)$ is abelian, then the subset $N = \{[\beta; h] * \phi_\#([\beta; h]^{-1}) | [\beta; h] \in \sigma(X, x_0, G)\}$ is a normal subgroup of $\sigma(X, x_0, G)$ and $R_G(\phi) = |\sigma(X, x_0, G)/N|$.*

*Proof.* It is clear that $N$ is a subgroup of $\sigma(X, x_0, G)$, and hence normal. Let $[\alpha; g] \in \sigma(X, x_0, G)$. From the fact that

$$\begin{aligned} &\{[\beta; h] * [\alpha; g] * \phi_\#([\beta; h]^{-1}) | [\beta; h] \in \sigma(X, x_0, G)\} \\ =&\{[\alpha; h] * ([\beta; g] * \phi_\#([\beta; h]^{-1})) | [\beta; h] \in \sigma(X, x_0, G)\} \\ =&[\alpha; g] * N, \end{aligned}$$

we have

$$R_G(\phi) = |\{[\alpha; g] * N | [\alpha; g] \in \sigma(X, x_0, G)\}| = |\sigma(X, x_0, G)/N|. \quad \square$$

Definition 9. ([3]) Let $f : G \to G$ be a group homomorphism. If there exists a positive integer $n$ such that $f^n(G)$ is abelian, then $f$ is said to be *eventually commutative.*

Clearly, every homomorphism of an abelian group is eventually commutative.

Definition 10. ([3]) For any group $(G, \cdot)$ and any homomorphism $f : G \to G$, the *Reidemeister operator* of $f$ on $G$ is the left action of $G$ on itself given by

$$(g_1, g_2) \to g_1 \cdot g_2 \cdot f(g_1^{-1}).$$

We write the set of orbits of the operation as $Coker(i_G \cdot f')$ with elements $< g >$ for $g \in G$.

Lemma 11. *([3]) Let $q : A \to B$ be a surjective homomorphism. If the diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ q\downarrow & & q\downarrow \\ B & \xrightarrow{g} & B \end{array}$$

*of groups and homomorphisms is commutative, then $|Coker(i_A \circ f)| \geq |Coker(i_B \cdot g)|$.*

Lemma 12. *Let $i_\sigma$ be the identity on $\sigma(X, x_0, G)$. Then $R_G(\phi) = |Coker(i_\sigma \cdot \phi'_\#)|$.*

*Proof.* If $j : \sigma(X, x_0, G) \to Coker(i_\sigma \cdot \phi'_\#)$ is the function defined by $j([\alpha; g]) =< [\alpha; g] >$ for $[\alpha; g] \in \sigma(X, x_0, G)$, then $j([\alpha_1; g_1]) = j([\alpha_2; g_2])$ if and only if there exists a $[\beta; h] \in \sigma(X, x_0, G)$ such that $[\alpha_2; g_2] = [\beta; h] * [\alpha_1; g_1] * \phi_\#([\beta; h]^{-1})$. By Theorem 7, the desired result follows. $\square$

THEOREM 13. *Let* $H_1(\phi_\#) : H_1(\sigma(X, x_0, G)) \to H_1(\sigma(X, x_0, G))$ *be the homomorphism induced by* $\phi_\# : \sigma(X, x_0, G) \to \sigma(X, x_0, G)$, *where* $H_1$ *is the first integral homology functor from groups to abelian groups. Then*

$$R_G(\phi) \geq |Coker(i_{H_1(\sigma)} \cdot H_1(\phi_\#)')|,$$

*where* $i_{H_1(\sigma)}$ *is the identity on* $H_1(\sigma(X, x_0, G))$. *Furthermore, if* $\phi_\#$ *is eventually commutative, then*

$$R_G(\phi) = |coker(i_{H_1(\sigma)} \cdot H_1(\phi_\#)')|.$$

*Proof.* Let $q : \sigma(X, x_0, G) \to H_1(\sigma(X, x_0, G))$ be the natural projection. Since the following diagram commutes

$$\begin{array}{ccc} \sigma(X, x_0, G) & \xrightarrow{\phi_\#} & \sigma(X, x_0, G) \\ q \downarrow & & q \downarrow \\ H_1(\sigma(X, x_0, G)) & \xrightarrow{H_1(\phi_\#)} & H_1(\sigma(X, x_0, G)), \end{array}$$

we have, by Lemma 11 and Lemma 12,

$$R_G(\phi) = |Coker(i_\sigma \cdot \phi'_\#)| \geq |Coker(i_{H_1(\sigma)} \cdot H_1(\phi_\#)')|.$$

In the case $\phi_\#$ is eventually commutative, the result follows from Corollary 1.15 of [3]. □

REMARK. It is well-known that for a group $G$, $H_1(G)$ is isomorphic to $G/G'$ where $G'$ is the commutator subgroup of $G$. Thus, in the above theorem, $H_1(\sigma(X, x_0, G))$ and $Coker(i_{H_1(\sigma)} \cdot H_1(\phi_\#)')$ can be replaced by $\sigma(X, x_0, G)/\sigma'(X, x_0, G)$ and $Coker(i_{\sigma/\sigma'} \cdot (\bar{\phi}_\#)')$, respectively, where $i_{\sigma/\sigma'}$ is the identity homomorphism on $\sigma(X, x_0, G)/\sigma'(X, x_0, G)$ and $\bar{\phi}_\#$ is the homomorphism of $\sigma(X, x_0, G)/\sigma'(X, x_0, G)$ induced by $\phi_\#$.

THEOREM 14. *If* $N = \{[\alpha; g] * \phi_\#([\alpha; g]^{-1}) | [\alpha; g] \in \sigma(X, x_0, G)\}$, *then*

$$R_G(\phi) \geq |Coker(i_{H_1(\sigma)} \cdot H_1(\phi_\#)')| = |\sigma(X, x_0, G)/\sigma'(X, x_0, G)N|.$$

*Proof.* By the same method as in the proof of Lemma 5 in [2], we can prove that $\sigma'(X, x_0, G)N$ is a normal subgroup of $\sigma(X, x_0, G)$. Consider the commutative diagram in the proof of Theorem 13 and the composition $\eta \circ q$

$$\sigma(X, x_0, G) \xrightarrow{q} H_1(\sigma(X, x_0, G)) \xrightarrow{\eta} Coker(i_{H_1(\sigma)} \cdot H_1(\phi_{\#})'),$$

where $\eta$ is the natural projection (Note that $Coker(i_{H_1(\sigma)} \cdot H_1(\phi_{\#})')$ is a group because $H_1(\sigma(X, x_0, G))$ is abelian). Since $\eta \circ q$ is an epimorphism, $Coker(i_{H_1(\sigma)} \cdot H_1(\phi_{\#})')$ is isomorphic to $\sigma(X, x_0, G)/Ker(\eta \circ q)$. Now, applying the method used in the proof of Theorem 3 in [2], we have $Ker(\eta \circ q) = \sigma'(X, x_0, G)N$. Therefore, we obtain the desired result. □

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