ON FUZZY T_2-AXIOMS AND FUZZY COMPACTNESS

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ABSTRACT. In this paper, the fuzzy T_2-axioms due to Hutton and Reilly, Ganguly and Saha and Sinha are characterized by using the notion of fuzzy closure. As consequences, we study the relation between the fuzzy T_2-axioms and give some examples which show that the axiom of fuzzy compactness, due to Ganguly and Saha, is not compatible with the fuzzy T_2-axioms.

1. Introduction

Several fuzzy T_2-axioms have been defined in different ways and investigated by many authors, such as Hutton and Reilly [4], Ganguly and Saha [3] and Sinha [5]. In particular, Sinha characterized these axioms and compared them. In Section 2, we provide another equivalent condition for these axioms and study the relation between them.

In [2], Ganguly and Saha introduced a new notion of fuzzy compactness and proved that this notion satisfies almost all the properties of compactness of set topology. But, in spite of the remarkable results, the notion is not compatible with some fuzzy T_2-axioms. In Section 3, we explain this fact.

Throughout this paper, we write a fuzzy topological space in short as an fts. For a fuzzy set A in an fts X, the set $\{x \in X | A(x) > 0\}$, denoted by A_0, is called the *support* of A and the intersection of all fuzzy closed sets containing A is called the *fuzzy closure* of A and denoted by $Cl(A)$. 0_X and 1_X denote, respectively, the constant fuzzy sets taking the values 0 and 1 on X. For definitions and notations which are not explained in this paper, we refer to [2],[3] and [5].

2. Fuzzy T_2-axioms

DEFINITION 2.1. ([3]) An fts X is said to be *fuzzy GS-T_2* if for every pair of distinct fuzzy points x_α and y_β, the following conditions are satisfied:

(1) if $x \neq y$, then x_α and y_β have fuzzy open nbds which are not q-coincident.

(2) if $x = y$ and $\alpha < \beta$ (say), then y_β has a fuzzy open q-nbd V and x_α has a fuzzy open nbd U such that $V_q U$.

DEFINITION 2.2. ([4]) An fts X is said to be *fuzzy HR-T_2* if for every fuzzy set A in X, there exists a collection $\{U_{ij} | i \in I, j \in J_i\}$ of fuzzy open sets in X such that

$$A = \bigcup_{i \in I} \left(\bigcap_{j \in J_i} U_{ij} \right) = \bigcup_{i \in I} \left(\bigcap_{j \in J_i} Cl(U_{ij}) \right).$$

DEFINITION 2.3. ([5]) An fts X is said to be *fuzzy S-T_2* if for any fuzzy point x_α,

$$x_\alpha = \bigcap_{U \in \mathcal{U}} Cl(U)$$

where $\mathcal{U}$ is the collection of fuzzy open neighborhoods of x_α.

DEFINITION 2.4. ([6]) An fts X is said to be *fuzzy SS-T_2* if for any distinct fuzzy points x_α and y_β with $x \neq y$, there exist fuzzy open nbds U and V of x_α and y_β, respectively, such that $U \cap V = 0_X$.

THEOREM 2.5. *For an fts X, the following are equivalent:*

1. *X is fuzzy S-T_2*
2. *for any two distinct fuzzy points x_α and y_β: (1) if $x \neq y$, then there exist fuzzy open sets U_1, U_2, V_1, V_2 in X such that $x_\alpha \in U_1, y_{\beta q} V_1, U_{1 q} V_1$ and $y_\beta \in V_2, x_{\alpha q} U_2, U_{2 q} V_2$; (2) if $x = y$ and $\alpha < \beta$ (say), then there exist fuzzy open sets U and V in X such that $x_\alpha \in U, x_{\beta q} V$ and $U_q V$.*
3. *for any two distinct fuzzy points x_α and y_β: (1) if $x \neq y$, then there exist fuzzy open nbds U and V of x_α and y_β, respectively,*

such that $x_\alpha \notin Cl(V)$ and $y_\beta \notin Cl(U)$; (2) if $x = y$ and $\alpha < \beta$ (say), then there exists a fuzzy open nbd U of x_α such that $x_\beta \notin Cl(U)$.

Proof. (1$\Leftrightarrow$2) Theorem 2.4 of [5].

(2$\Rightarrow$3) Assume $x \neq y$. By (1), there exist fuzzy open sets U, V, W_1, W_2 in X such that $x_\alpha \in U, y_{\beta q} W_1, U_q W_1$ and $y_\beta \in V, x_{\alpha q} W_2, V_q W_2$. This implies that $x_\alpha \in U, \beta > (1_X - W_1)(y), U \leq 1_X - W_1$ and $y_\beta \in V, \alpha > (1_X - W_2)(x), V \leq 1_X - W_2$. Therefore, U and V are nbds of x_α and y_β, respectively, such that $Cl(U)(y) \leq [Cl(1_X - W_1)](y) = (1_X - W_1)(y) < \beta$ and $Cl(V)(x) \leq [Cl(1_X - W_2)](x) = (1_X - W_2)(x) < \alpha$.

Now, assume that $x = y$. By (2), there exist fuzzy open sets U and V in X such that $x_\alpha \in U, x_{\beta q} V$ and $U_q V$. Then $x_\alpha \in U, \beta > (1_X - V)(x)$ and $U \leq 1_X - V$. Consequently, U is a fuzzy open nbd of x_α such that $Cl(U)(x) \leq [Cl(1_X - V)](x) = (1_X - V)(x) < \beta$.

(3$\Rightarrow$2) Assume $x \neq y$. By (1), there exist fuzzy open nbds U_1 and V_2 of x_α and y_β, respectively, such that $x_\alpha \notin Cl(V_2)$ and $y_\beta \notin Cl(U_1)$. Let $V_1 = 1_X - Cl(U_1)$ and $U_2 = 1_X - Cl(V_2)$. Then V_1 and U_2 are fuzzy open sets in X such that $\beta + V_1(y) = \beta + [1 - Cl(U_1)(y)] > \beta + (1 - \beta) = 1, \alpha + U_2(y) = \alpha + [1 - Cl(V_2)(x)] > \alpha + (1 - \alpha) = 1, V_1 = 1_X - Cl(U_1) \leq 1_X - U_1, U_2 = 1_X - Cl(V_2) \leq 1_X - V_2$. That is, $x_\alpha \in U_1, y_{\beta q} V_1, U_{1 q} V_1$ and $y_\beta \in V_2, x_{\alpha q} U_2, U_{2 q} V_2$.

Now, assume $x = y$. By (2), there exists a fuzzy open nbd U of x_α such that $x_\beta \notin Cl(U)$. Let $V = 1_X - Cl(U)$. Then V is a fuzzy open set in X such that $V \leq 1_X - U$ and $\beta + V(x) = \beta + [1 - Cl(U)(x)] > \beta + (1 - \beta) = 1$. That is, $x_\alpha \in U, x_{\beta q} V$ and $U_q V$. $\square$

As an application of Theorem 2.5, we obtain the following theorem.

THEOREM 2.6. *Let X be a finite fts. Then X is fuzzy S-T_2 if and only if X is fuzzy discrete space.*

Proof. ($\Leftarrow$) Clear.

($\Rightarrow$) Obviously, every fuzzy point in X is fuzzy closed. Now, let x_α be a fuzzy point in X. Assume $\gamma < \alpha$. By Theorem 2.5, there exists a fuzzy open nbd U_γ of x_γ such that $x_\alpha \notin Cl(U_\gamma)$. Note that $\gamma \leq U_\gamma(x) < \alpha$. Let $U = \cup_{0 < \gamma < \alpha} U_\gamma$. Then U is a fuzzy open set and $U(x) = \alpha$. Let $V = (1_X - \cup_{y \neq x} y_1) \cap U$. Since X is finite, V is a fuzzy open set in X. Furthermore, $V = x_\alpha$. $\square$

But there is a finite fuzzy HR-T_2 space which is not a discrete space as is seen from the following example.

EXAMPLE 2.7. Let $X = \{x, y\}$ and let $A_{\lambda\mu}$ be the fuzzy set in X defined by

$$A_{\lambda\mu}(z) = \begin{cases} \lambda & \text{for } z = x \\ \mu & \text{for } z = y. \end{cases}$$

Since the collection $\tau = \{A_{\lambda\mu} | 0 < \lambda \leq 1, 0 < \mu \leq 1\} \cup \{A_{00}\}$ is a fuzzy topology on X, $x_1 = A_{10}$ and $y_1 = A_{01}$ are not fuzzy open in X. Hence X is not a fuzzy discrete space under the fuzzy topology τ. Now, since $A_{\lambda\mu}$ is both fuzzy open and fuzzy closed for each $\lambda, \mu \in (0, 1)$, it is easy to show that for any $\alpha, \beta \in (0, 1]$,

$$x_\alpha = \bigcup_{0 < \lambda < \alpha} \left(\bigcap_{0 < \mu < 1} A_{\lambda\mu} \right) = \bigcup_{0 < \lambda < \alpha} \left(\bigcap_{0 < \mu < 1} Cl(A_{\lambda\mu}) \right)$$

and

$$y_\beta = \bigcup_{0 < \mu < \beta} \left(\bigcap_{0 < \lambda < 1} A_{\lambda\mu} \right) = \bigcup_{0 < \mu < \beta} \left(\bigcap_{0 < \lambda < 1} Cl(A_{\lambda\mu}) \right).$$

Thus, X is fuzzy HR-T_2.

THEOREM 2.8. *For an fts X, the following are equivalent:*

1. *X is fuzzy GS-T_2.*
2. *for any two distinct fuzzy points x_α and y_β: (1) if $x \neq y$, then there exist fuzzy open nbds U and V of x_α and y_β, respectively, such that $Cl(V) \leq 1_X - U$ and $Cl(U) \leq 1_X - V$; (2) if $x = y$ and $\alpha < \beta$ (say), then there exists a fuzzy open nbd U of x_α such that $y_\beta \notin Cl(U)$.*

Proof. Similar to the proof of Theorem 2.5. ☐

THEOREM 2.9. *For an fts X, the following are equivalent:*

1. *X is fuzzy HR-T_2.*
2. *if $\alpha \in (0, 1)$ and $\mathcal{U}$ is the collection of all fuzzy open nbds of x_α, then $x_\alpha = \cap_{U \in \mathcal{U}} Cl(U)$.*

3. for any $x, y \in X$ and any $\alpha, \beta \in (0, 1)$: (1) if $x \neq y$, then there exist fuzzy open nbds U and V of x_α and y_β, respectively, such that $y_\beta \notin Cl(U)$ and $x_\alpha \notin Cl(V)$; (2) if $x = y$ and $\alpha < \beta$ (say), then there exists a fuzzy open nbd U of x_α such that $y_\beta \notin Cl(U)$.

4. for any two distinct fuzzy points x_α and y_β with $x \neq y$ or else $x = y$ and $\frac{1}{2} < \alpha < \beta$, there exist fuzzy open sets U and V such that $x_{\alpha q}U$, $y_{\beta q}V$ and $U \cap V \leq \frac{1}{2}$.

5. $\Delta = 1_X - sup\{A \times B | A \text{ and } B \text{ are fuzzy sets and } A \leq 1_X - B\}$ is fuzzy closed in $X \times X$.

Proof. (1⇒2) By hypothesis, there exists a collection $\{V_{lk} | l \in L, k \in K_l\}$ of fuzzy open sets in X such that

$$1_X - x_\alpha = \bigcup_{l \in L}\left(\bigcap_{k \in K_l} Cl(V_{lk})\right).$$

For every $l \in L$ and every $k \in K_l$, let $U_{lk} = 1_X - Cl(V_{lk})$. Then

$$x_\alpha = \bigcap_{l \in L}\left(\bigcup_{k \in K_l} U_{lk}\right).$$

Since for each $l \in L$, $\cup_{k \in K_l} U_{lk}$ is a fuzzy open neighborhood of x_α, we have

$$x_\alpha \in \bigcap_{U \in \mathcal{U}} U \leq \bigcap_{l \in L}\left(\bigcup_{k \in K_l} U_{lk}\right) = x_\alpha.$$

Thus,

$$x_\alpha = \bigcap_{U \in \mathcal{U}} U.$$

First, we shall prove that $(\cap_{U \in \mathcal{U}} Cl(U))(x) = \alpha$. To show this, assume to the contrary that $(\cap_{U \in \mathcal{U}} Cl(U))(x) = \beta > \alpha$. Then for all $U \in \mathcal{U}$, $(Cl(U))(x) \geq \beta$. Choose $\gamma \in (\alpha, \beta)$. By hypothesis, there exists a collection $\{V_{ij} | i \in I, j \in J_i\}$ of fuzzy open sets in X such that $x_\gamma = \cup_{i \in I}(\cap_{j \in J_i} V_{ij}) = \cup_{i \in I}(\cap_{j \in J_i} Cl(V_{ij}))$. Then there exists $i_0 \in I$ such that $\alpha \leq (\cap_{j \in J_{i_0}} V_{i_0 j})(x)$. This means that $\{V_{i_0 j} | j \in J_{i_0}\} \subset \mathcal{U}$. Since $(\cap_{j \in J_{i_0}} Cl(V_{i_0 j}))(x) \leq \gamma$, there exists $j_0 \in J_{i_0}$ such

that $(Cl(V_{i_0 j_0}))(x) < \beta$. Consequently, we have an obvious contradiction that

$$\beta = \left(\bigcap_{U \in \mathcal{U}} Cl(U) \right)(x) \leq \left(Cl(V_{i_0 j_0}) \right)(x) < \beta.$$

Thus

$$\left(\bigcap_{U \in \mathcal{U}} Cl(U) \right)(x) = \alpha.$$

Now, we shall prove $(\cap_{U \in \mathcal{U}} Cl(U))(y) = 0$ for all $y \in X - \{x\}$. By hypothesis, there exists a collection $\{U_{ij} | i \in I, j \in J_i\}$ of fuzzy open sets in X such that

$$x_1 = \bigcup_{i \in I} \left(\bigcap_{j \in J_i} U_{ij} \right) = \bigcup_{i \in I} \left(\bigcap_{j \in J_i} Cl(U_{ij}) \right).$$

This means that $(\cap_{j \in J_i} Cl(U_{ij}))(y) = 0$ for all $i \in I$ and all $y \in X - \{x\}$ and that, for every $\beta \in [\alpha, 1)$, there exists $i_\beta \in I$ such that $\beta < (\cap_{j \in J_{i_\beta}} U_{i_\beta j})(x) \leq 1$, i.e, $\{U_{i_\beta j} | j \in J_{i_\beta}\} \subset \mathcal{U}$. Let y be a point in $X - \{x\}$. For each positive integer n, choose $j_n \in J_{i_\beta}$ such that $(Cl(U_{i_\beta j_n}))(y) < 1/n$. Notice that each $U_{i_\beta j_n}$ is a member of $\mathcal{U}$ and $(\cap_n Cl(U_{i_\beta j_n}))(y) = 0$. Since y is arbitrary in $X - \{x\}$ and $\{U_{i_\beta j_n}\}$ is a subcollection of $\mathcal{U}$, we have

$$\left(\bigcap_{U \in \mathcal{U}} Cl(U) \right)(y) = 0 \text{ for all } y \in X - \{x\}.$$

(2⇒3) Clear.

(3⇒4) Assume that $x \neq y$. By hypothesis (1), there exist fuzzy open nbds W and V of $x_{\frac{\alpha}{2}}$ and $y_{1-\frac{\beta}{2}}$, respectively, such that $(Cl(V))(x) < \frac{\alpha}{2}$ and $(Cl(W))(y) < 1 - \frac{\beta}{2}$. Let $U = 1_X - Cl(V)$. Then $x_{\alpha}qU$ and $y_{\beta}qV$ such that U_qV and hence $U \cap V \leq \frac{1}{2}$. Now, assume that $x = y$ and $\frac{1}{2} < \alpha < \beta$. Choose $\epsilon > 0$ such that $\alpha < \beta - \epsilon$. By hypothesis (2), there exists a fuzzy open nbd U of x_{α} such that $(Cl(U))(x) < \beta$. Let $V = 1_X - Cl(U)$. Then $x_{\alpha}qU$ and $y_{\beta}qV$ such that U_qV and hence $U \cap V \leq \frac{1}{2}$.

(4⇒5) Theorem 2.14 of [5].

(5⇒1) Proposition 7 of [4].   □

THEOREM 2.10. *For any fts X, the following are equivalent:*

1. *X is fuzzy GS-T_2.*
2. *X is fuzzy S-T_2 and for $x, y \in X$ with $x \neq y$, there exists a fuzzy open nbd U of x_1 such that $Cl(U)(y) = 0$.*
3. *X is fuzzy HR-T_2 and for $x, y \in X$ with $x \neq y$, there exists a fuzzy open nbd U of x_1 such that $Cl(U)(y) = 0$.*
4. *for $x, y \in X$: (1) if $x \neq y$, then there exists a fuzzy open nbd U of x_1 such that $Cl(U)(y) = 0$; (2) if $x = y$ and $\alpha < \beta$, then there exists a fuzzy open nbd U of x_α such that $y_\beta \notin Cl(U)$.*

Proof. (1$\Leftrightarrow$2) Theorem 2.11 of [5].

(2$\Rightarrow$3) Theorem 2.15 of [5].

(3$\Rightarrow$2) Let $x \in X$ and let $\mathcal{U}$ be the collection of all fuzzy open nbds of x_1. For each $y \in X - \{x\}$, choose a fuzzy open nbd U_y of x_1 such that $Cl(U_y)(y) = 0$. Then $x_1 = \cap_{y \in X - \{x\}} Cl(U_y)$. Since $\{U_y | y \in X - \{x\}\} \subset \mathcal{U}$, we have

$$\bigcap_{y \in X - \{x\}} Cl(U_y) = x_1 \leq \bigcap_{U \in \mathcal{U}} Cl(U) \leq \bigcap_{y \in X - \{x\}} Cl(U_y).$$

Therefore, $x_1 = \cap_{U \in \mathcal{U}} Cl(U)$. By combining this with Theorem 2.9, we obtain X is fuzzy S-T_2.

(3$\Leftrightarrow$4) Obvious by the definition of fuzzy HR-T_2 spaces. $\square$

3. Fuzzy compactness

DEFINITION 3.1. ([2]) A collection $\mathcal{U} = \{U_i | i \in I\}$ of fuzzy open sets in X is called a *fuzzy open covering* of a fuzzy set A in X if

$$\left(\bigcup_{i \in I} U_i \right)(x) = 1 \text{ for all } x \in A_0.$$

DEFINITION 3.2. ([2]) A fuzzy set A in an fts X *fuzzy GS-compact* if every fuzzy open covering $\mathcal{U}$ of A has a finite subcollection $\{U_j | j = 1, \cdots, n\}$ of $\mathcal{U}$ such that $\cup_{j=1}^{n} U_j \geq A$.

REMARK 3.3. If A is a fuzzy set in a fuzzy S-T_2 space X such that $A(x) = 1$ for some $x \in A_0$, then A is not fuzzy GS-compact. To prove this, for each $y \in A_0 - \{x\}$, choose fuzzy open nbds U_y and V_y of x_1 and y_1, respectively, such that $x_1 \notin Cl(V_y)$ and $y_1 \notin Cl(U_y)$. Also, for each $\alpha \in (0, 1)$, choose a fuzzy open nbd W_α of x_α such that $x_1 \notin Cl(W_\alpha)$. Clearly, the collection $\{U_y | y \in A_0 - \{x\}\} \cup \{W_\alpha | \alpha \in (0, 1)\}$ is a fuzzy open covering of A that has no finite subcollection whose union contains A. Thus, A is not fuzzy GS-compact. Since every fuzzy GS-T_2 space is fuzzy S-T_2, we have that every fuzzy GS-T_2 space is not fuzzy GS-compact.

REMARK 3.4. If an infinite fts X is fuzzy GS-compact, then we claim that some fuzzy GS-compact sets in X is not fuzzy closed under any fuzzy T_2-axiom. To justify this, assume that all fuzzy GS-compact sets in X is fuzzy closed under a certain fuzzy T_2-axiom. Let A be a fuzzy set in X. If $A_0 = X$, it is clear that A is fuzzy GS-compact and hence fuzzy closed. Assume $A_0 \subset X$. For each $\epsilon \in (0, 1]$, define a fuzzy set A_ϵ as follows:
$$A_\epsilon(y) = \begin{cases} A(y), \ y \in A_0 \\ \epsilon, \ y \notin A_0. \end{cases}$$

By fuzzy GS-compactness of A_ϵ and hypothesis, $A = \cap_{\epsilon \in (0,1]} A_\epsilon$ is fuzzy closed. Since A is arbitrary, we have that X is a fuzzy discrete space. Hence X is not fuzzy GS-compact. Contradiction.

REMARK 3.5. A necessary condition for a fuzzy set A in a SS-T_2 space X to be GS-compact is that for any $x \in A_0$ with $A(x) = 1$, there exist $\epsilon \in (0, 1)$ depending on x such that for all $\gamma \geq \epsilon$ and all fuzzy open nbd U of x_γ, $U(x) = 1$. To show this, assume for each $\alpha < 1$ there exists a fuzzy open nbd V_α such that $V_\alpha(x) < 1$. Since X is fuzzy SS-T_2, there exists a fuzzy open nbd U_y of y_1 such that $U_y(x) = 0$ for each $y \in A_0 - \{x\}$. Clearly, $\{U_y | y \in A_0 - \{x\}\} \cup \{V_\alpha | \alpha \in (0, 1)\}$ is a fuzzy open covering of A that has no finite subcollection whose union contains A. Thus, A is not fuzzy GS-compact.

From the above remark, we know that every fuzzy SS-T_2 GS-compact space is not fuzzy GS-T_2.

We end this paper with a remark for fuzzy local compactness.

DEFINITION 3.6. ([7]) An fts X is said to be *fuzzy locally compact* if for any fuzzy point x_α in X, there exists a fuzzy open set U in X such that $x_\alpha \in U$ and U is fuzzy GS-compact.

Note that an fts X is fuzzy locally compact if for every $x \in X$, there exists a fuzzy open nbd U of x_1 such that U is fuzzy GS-compact.

S. Dang and A. Behera quoted the result, due to Ganguly and Saha, that "every fuzzy GS-compact set in a fuzzy GS-T_2 space is fuzzy closed" in order to characterize the concept of fuzzy local compactness in fuzzy SS-T_2 spaces. But this result is not true in the case that the given fts is SS-T_2 as is seen from the following example.

EXAMPLE 3.7. Let $X = [0, 1]$. For each $x \in (0, 1]$, consider the fuzzy set A_x defined by

$$A_x(y) = \begin{cases} 1, & y = 0 \\ 0, & y = x \\ \frac{1}{4}, & y \notin \{0, x\}. \end{cases}$$

Let τ be the fuzzy topology on X having $\{A_x | x \in (0, 1]\} \cup \{x_1 | x \in (0, 1]\}$ as a subbasis. Clearly, (X, τ) is fuzzy SS-T_2 and fuzzy locally compact. Now, let U be a fuzzy open nbd of 0_1. If U is fuzzy GS-compact, then the fuzzy open covering $\{A_1\} \cup \{x_1 | x \in (0, 1]\}$ has a finite subcollection that covers U. Since $U(x) > 0$ for all but finitely many points of X, there exists a point $x \in (0, 1]$ such that $U(x) = \frac{1}{4}$. Thus, from the fact that every fuzzy set having $\frac{3}{4}$ as value at some points in X is not fuzzy open, we conclude that U is not fuzzy closed.

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