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ON THE LEFT REGULAR *po*-Γ-SEMIGROUPS

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Abstract. We consider the ordered Γ-semigroups in which $x\gamma x$ $(x \in M, \gamma \in \Gamma)$ are left elements. We show that this *po*-Γ-semigroup is left regular if and only if M is a union of left simple sub-Γ-semigroups of M.

The concept of left regular *poe*-semigroups has been introduced in [2] and extends the concept of left regular *po*-semigroups not having the greatest element "e" in [1]. In [5], Lee and Jung showed that a *poe*-semigroup S in which every $x^2 (x \in S)$ is a left ideal element is left regular if and only if there exists a family $\{S_\alpha | \alpha \in Y\}$ of left simple subsemigroups of S such that $S = \cup\{S_\alpha | \alpha \in Y\}$. Recently, Kwon([3]) showed that a *poe*-Γ-semigroup is left regular if and only if M is a union of left simple sub-Γ-semigroups of M.

Now we consider a *po*-Γ-semigroups which does not necessarily have a greatest element "e". In this paper we prove that a *po*-Γ-semigroup M in which every $x\gamma x(x \in M, \gamma \in \Gamma)$ is a left ideal element is left regular if and only if M is a union of left simple sub-Γ-semigroups of M.

M. K. Sen ([6]) introduced Γ-semigroups in 1981. M. K. Sen and N. K. Saha ([7],[8]) introduced Γ-semigroups different from the first definition of Γ-semigroups in the sense of Sen (1981). From Sen ([6]) we recall the following definition of Γ-semigroup.

Let M and Γ be any two non-empty sets. M is called a Γ-*semigroup* if

(1) $M\Gamma M \subseteq M, \Gamma M \Gamma \subseteq \Gamma$.

(2) $(a\gamma b)\mu c = a(\gamma b \mu)c = a\gamma(b\mu c)$

for all $a, b, c \in M$ and $\gamma, \mu \in \Gamma$.

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EXAMPLE 1. Let M be the set of all integers of the form $4n+1$ where n is an integer and Γ denote the set of all integers of the form $4n+3$. If $a\gamma b$ is $a+\gamma+b$, $\gamma a\mu$ is $\gamma+a+\mu$ (usual sum of the integers) for all $a, b \in M$ and $\gamma, \mu \in \Gamma$, then M is a Γ-semigroup.

A *po*-Γ-semigroup(: partially ordered Γ-semigroup)([5]) is an ordered set M at the same time a Γ-semigroup such that:

$$a \le b \Longrightarrow a\gamma c \le b\gamma c \text{ and } c\mu a \le c\mu b$$

$\forall a, b, c \in M$ and $\forall \gamma, \mu \in \Gamma$.

Notation. For subsets A, B of M, let

$$A\Gamma B := \{a\gamma b | a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

DEFINITION 1. Let M be a *po*-Γ-semigroup and A a nonempty subset of M. A is called a *left ideal* of M if
(1) $M\Gamma A \subseteq A$.
(2) $a \in A, b \le a (b \in M) \Longrightarrow b \in A$.

DEFINITION 2. A *po*-Γ-semigroup M is called *left*(*right*) *regular* if for every $a \in M$ there exists $x \in M$ such that $a \le x\gamma(a\mu a)$(resp.$a \le (a\gamma a)\mu x$) for some $\gamma, \mu \in \Gamma$.

DEFINITION 3. Let M be a *po*-Γ-semigroup and T a nonempty subset of M. T is called *semiprime* if $a \in M, a\gamma a \in T (\gamma \in \Gamma) \Longrightarrow a \in T$.

DEFINITION 4. Let M be a *po*-Γ-semigroup. An element t of M is called *semiprime* if $a \in M, a\gamma a \le t (\gamma \in \Gamma)$ implies $a \le t$.

DEFINITION 5. Let M be a *po*-Γ-semigroup. A sub-Γ-semigroup T of M is called *left simple* if for every left ideal L of T we have $L = T$.

DEFINITION 6. An element t of M is called a *left ideal element* if $x\gamma x \le t$ for all $x \in M$ and $\gamma \in \Gamma$.

Notation. Let M be a *po*-Γ-semigroup. For $H \subseteq M$,

$$(H] = \{t \in M | t \le h \text{ for some } h \in H\}.$$

We denote by $L(x)$ the left ideal of M generated by $x(x \in M)$. For a *po*-Γ-semigroup M we can easily prove that :

$$\begin{aligned} L(x) &= \{t \in M | t \leq x \text{ or } t \leq a\gamma x \text{ for some } a \in M \text{ and } \gamma \in \Gamma\} \\ &= (x \cup M\Gamma x], \forall x \in M. \end{aligned}$$

We define a relation "$\mathcal{L}$" on M as follows:

$$a\mathcal{L}b \iff L(a) = L(b).$$

Then $\mathcal{L}$ is a right congruence on M i.e. it is an equivalence relation on M such that

$$a\mathcal{L}b \Longrightarrow (a\gamma c)\mathcal{L}(b\gamma c), \forall c \in M, \forall \gamma \in \Gamma.$$

Indeed: Let $a\mathcal{L}b$. If $t \in L(a\gamma c)$, then $t \leq a\gamma c$ or $t \leq x\mu(a\gamma c)$ for some $x \in M$ and $\mu \in \Gamma$. Since $a \in L(a) = L(b)$, we have $a \leq b$ or $a \leq y\delta b$ for some $y \in M$ and $\delta \in \Gamma$. If $a \leq b$, then $t \leq b\gamma c$ or $t \leq x\mu(b\gamma c)$ i.e. $t \in L(b\gamma c)$. If $a \leq y\delta b(y \in M, \delta \in \Gamma)$, then $t \leq (y\delta b)\gamma c = y\delta(b\gamma c)$ or $t \leq (x\mu y)\delta(b\gamma c)$ i.e. $t \in L(b\gamma c)$. Thus we have $L(a\gamma c) \subseteq L(b\gamma c)$. By symmetry, $L(b\gamma c) \subseteq L(a\gamma c)$.

THEOREM 1. *Let M be a po-Γ-semigroup. The following are equivalent:*

(1) *M is left regular.*
(2) $L(a) \subseteq L(a\gamma a), \forall a \in M, \forall \gamma \in \Gamma$.
(3) $a\mathcal{L}(a\gamma a),\ \forall a \in M, \forall \gamma \in \Gamma$.

Proof. (1) $\Longrightarrow$ (2). Let M be left regular. If $t \in L(a)$, then

$$t \leq a \text{ or } t \leq x\gamma a$$

for some $x \in M$ and $\gamma \in \Gamma$. Since M is left regular, $a \leq y\mu(a\gamma a)$ for some $y \in M$ and $\mu, \gamma \in \Gamma$.

If $t \leq a$, then $t \leq a \leq y\mu(a\gamma a)(y \in M, \mu, \gamma \in \Gamma)$.

If $t \leq x\gamma a$, then $t \leq x\gamma a \leq x\gamma(y\mu a\gamma a) = (x\gamma y)\mu(a\gamma a)$.

In any case, $t \leq z\mu(a\gamma a)$ for some $z \in M$. Hence $t \in L(a\gamma a)$, and so $L(a) \subseteq L(a\gamma a)$.

(2) $\Longrightarrow$ (3). Let $a \in M$. Then

$$\begin{aligned} t \in L(a\gamma a) &\Rightarrow t \leq a\gamma a(\forall\gamma \in \Gamma) \text{ or } t \leq x\mu(a\gamma a)(x \in M, \mu \in \Gamma) \\ &\Rightarrow t \leq z\gamma a \text{ for some } z \in M. \\ &\Rightarrow t \in L(a). \end{aligned}$$

By (2), $L(a) = L(a\gamma a)$. Thus we have $a\mathcal{L}(a\gamma a)(\forall a \in M, \forall\gamma \in \Gamma)$.
(3) $\Longrightarrow$ (1). Let $a \in M$. Since $a\mathcal{L}(a\gamma a)(\forall\gamma \in \Gamma)$, we have

$$a \in L(a) = L(a\gamma a) \Rightarrow a \leq a\gamma a \text{ or } a \leq x\mu(a\gamma a)(\mu \in \Gamma, x \in M).$$

If $a \leq a\gamma a(\gamma \in \Gamma)$, then $a\gamma a \leq a\gamma(a\gamma a)$ and so $a \leq a\gamma(a\gamma a)$. In any case a is left regular, and so M is left regular. □

THEOREM 2. *Let M be a po-Γ-semigroup in which every $x\gamma x(x \in M, \gamma \in \Gamma)$ is a left ideal element. The following are equivalent:*

(1) *M is left regular.*
(2) *Every left ideal element of M is semiprime.*
(3) *Every left ideal of M is semiprime.*

Proof. (1) $\Longrightarrow$ (2). Let t be a left ideal element of $M, a \in M$ and $a\gamma a \leq t(\gamma \in \Gamma)$. Since M is left regular, $a \leq x\mu(a\gamma a) \leq x\mu t \leq t(x \in M, \mu \in \Gamma)$. Thus t is semiprime.

(2) $\Longrightarrow$ (3). Let L be a left ideal of $M, a \in M$ and $a\gamma a \in L(\gamma \in \Gamma)$. Since $a\gamma a \leq a\gamma a$, and $a\gamma a$ is a left ideal element and so it is semiprime, we have $a \leq a\gamma a(\gamma \in \Gamma)$. And since L is a left ideal, $a \in L$. Hence L is semiprime.

(3) $\Longrightarrow$ (1). Let $a \in M$. Since a left ideal $L(a\gamma a)(\gamma \in \Gamma)$ is semiprime and $a\gamma a \in L(a\gamma a)$, we have $a \in L(a\gamma a)$. Thus $L(a) \subseteq L(a\gamma a)(\gamma \in \Gamma)$. By Theorem 1, M is left regular. □

THEOREM 3. *Let M be a po-Γ-semigroup in which every $a\gamma a(a \in M, \gamma \in \Gamma)$ is a left ideal element. Then we have that M is left regular if and only if there exists a family $\{M_\alpha | \alpha \in Y\}$ of left simple sub-Γ-semigroups of M such that $M = \cup\{M_\alpha | \alpha \in Y\}$.*

Proof. Assume that M is left regular. We denote by $\mathcal{L}(x)$ the $\mathcal{L}$-class of M containing $x(x \in M)$.

Then $\mathcal{L}(x)$ is a left simple sub-Γ-semigroup of $M, \forall x \in M$.
In fact, since $x \in \mathcal{L}(x)$, $\mathcal{L}(x)$ is nonempty.

Let $a, b \in \mathcal{L}(x)$. Then $a\mathcal{L}x$ and $x\mathcal{L}b$. Since $\mathcal{L}$ is a right congruence on M, we have $(a\gamma b)\mathcal{L}(x\gamma b)$ and $(x\gamma b)\mathcal{L}(b\gamma b)$ $\forall\gamma \in \Gamma$. Since M is left regular, by Theorem 1, $(b\gamma b)\mathcal{L}b$. Hence we have $(a\gamma b)\mathcal{L}b$ and so $a\gamma b \in \mathcal{L}(b) = \mathcal{L}(x)(\forall\gamma \in \Gamma)$. Thus $\mathcal{L}(x)$ is sub-Γ-semigroup of M.

Let L be a left ideal of $\mathcal{L}(x)$ and $z \in L$. If $y \in \mathcal{L}(x)$, then $z \in L \subseteq \mathcal{L}(x) = \mathcal{L}(y)$. Since M is left regular, by Theorem 1, we have $y \in L(y) = L(z) = L(z\gamma z)(\forall\gamma \in \Gamma)$. Then $y \leq z\gamma z$ or $y \leq t\mu(z\gamma z)(t \in M, \gamma, \mu \in \Gamma)$.

If $y \leq z\gamma z$ then, since L is a left ideal of $\mathcal{L}(x)$, we have $y \leq z\gamma z \in \mathcal{L}(x)\Gamma L \subseteq L$, and $y \in L$. And if $y \leq t\mu(z\gamma z)$, then $y \leq z\gamma z$ since every $z\gamma z(z \in M, \gamma \in \Gamma)$ is a left ideal element. In any case, $y \in L$ and so $L = \mathcal{L}(x)$. Hence every $\mathcal{L}(x)$ is a left simple sub-Γ-semigroup of M. Now $M = \cup\{\mathcal{L}(x) | x \in M\}$.

Conversely, suppose that $M = \cup\{M_\alpha | \alpha \in Y\}$ where M_α is a left simple sub-Γ-semigroup of $M, \forall\alpha \in Y$. Let L be a left ideal of $M, a \in M$ and $a\gamma a \in L(\gamma \in \Gamma)$. Then $a \in M_\alpha$ for some $\alpha \in Y$ and $L \cap M_\alpha$ is a left ideal of M_α.

Indeed: Since $a\gamma a \in L$ and $a\gamma a \in M_\alpha$, $L \cap M_\alpha$ is nonempty and $L \cap M_\alpha \subseteq M_\alpha$. Furthermore

$$M_\alpha\Gamma(L \cap M_\alpha) \subseteq M_\alpha\Gamma L \cap M_\alpha\Gamma M_\alpha \subseteq M\Gamma L \cap M_\alpha \subseteq L \cap M_\alpha.$$

Let $x \in I \cap M_\alpha$ and $y \leq x(y \in M_\alpha)$. Since $x \in L$ and $y \leq x$, $y \in L$. Thus $y \in L \cap M_\alpha$. Hence $L \cap M_\alpha$ is a left ideal of M_α. Since M_α is left simple, we have $I \cap M_\alpha = M_\alpha$, and $a \in L$. Hence L is semiprime. By Theorem 2, M is left regular. □

Remark. If $\mathcal{L}(x)$ is a sub-Γ-semigroup of $M, \forall x \in M$, then M is left regular.

In fact: Since $x\gamma x \in \mathcal{L}(x)(\gamma \in \Gamma)$, we have $(x\gamma x)\mathcal{L}x, \forall x \in M$. By Lemma, M is left regular.

THEOREM 4. *For a po-*Γ*-semigroup* M*, the following conditions are equivalent:*

(1) M *is left regular.*
(2) *Every* $\mathcal{L}$*-class of* M *is a left simple sub-*Γ*-semigroup of* M*.*
(3) *Every* $\mathcal{L}$*-class of* M *is a sub-*Γ*-semigroup of* M*.*
(4) M *is a union of disjoint left simple sub-*Γ*-semigroups of* M*.*
(5) M *is a union of left simple sub-*Γ*-semigroups of* M*.*

Proof. From the proof of the Theorem 3, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1). On the other hand, (2) $\Rightarrow$ (3) is obvious and (3) $\Rightarrow$ (1) by the Remark. □

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