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FUZZY r-SEMICONTINUOUS, r-SEMIOPEN AND r-SEMICLOSED MAPS

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Abstract. In this paper, we investigate some conditions which are equivalent to fuzzy r-homeomorphisms and give some characterizing theorems for fuzzy r-semicontinuous, r-semiopen and r-semiclosed maps.

1. Introduction

Chang [2] introduced fuzzy topological spaces and some authors [4, 5, 7] introduced new definitions of fuzzy topology as a generalization of Chang's fuzzy topology. We introduced fuzzy r-semiopen sets and fuzzy r-semicontinuous maps which are generalization of fuzzy semiopen sets and fuzzy semicontinuous maps in Chang's fuzzy topology, respectively [6]. In this paper, we investigate some conditions which are equivalent to fuzzy r-homeomorphisms and give some characterizing theorems for fuzzy r-semicontinuous, r-semiopen and r-semiclosed maps.

2. Preliminaries

In this paper, I denotes the unit interval $[0, 1]$ of the real line and $I_0 = (0, 1]$. A member μ of I^X is called a *fuzzy set* of X. For any $\mu \in I^X$, μ^c denotes the complement $1-\mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on X with value 0 and 1, respectively. All other notations are standard notations of fuzzy set theory.

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Definition 2.1. [3, 6] Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-closure* is defined by

$$\mathrm{cl}(\mu, r) = \bigwedge\{\rho \in I^X \mid \mu \leq \rho, \mathcal{F}_{\mathcal{T}}(\rho) \geq r\}$$

and the *fuzzy r-interior* is defined by

$$\mathrm{int}(\mu, r) = \bigvee\{\rho \in I^X \mid \mu \geq \rho, \mathcal{T}(\rho) \geq r\}.$$

¿From now on, for $r \in I_0$ we will call μ a *fuzzy r-open set* of X if $\mathcal{T}(\mu) \geq r$, μ a *fuzzy r-closed set* of X if $\mathcal{F}(\mu) \geq r$. Note that μ is fuzzy r-closed if and only if $\mu = \mathrm{cl}(\mu, r)$ and μ is fuzzy r-open if and only if $\mu = \mathrm{int}(\mu, r)$.

Definition 2.2. [6] Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy r-semiopen* if there is a fuzzy r-open set ρ in X such that $\rho \leq \mu \leq \mathrm{cl}(\rho, r)$,

(2) *fuzzy r-semiclosed* if there is a fuzzy r-closed set ρ in X such that $\mathrm{int}(\rho, r) \leq \mu \leq \rho$.

Definition 2.3. [6] Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-semiclosure* is defined by

$$\mathrm{scl}(\mu, r) = \bigwedge\{\rho \in I^X \mid \mu \leq \rho,\ \rho \text{ is fuzzy } r\text{-semiclosed}\}$$

and the *fuzzy r-semiinterior* is defined by

$$\mathrm{sint}(\mu, r) = \bigvee\{\rho \in I^X \mid \mu \geq \rho,\ \rho \text{ is fuzzy } r\text{-semiopen}\}.$$

Obviously $\mathrm{scl}(\mu, r)$ is the smallest fuzzy r-semiclosed set which contains μ and $\mathrm{sint}(\mu, r)$ is the greatest fuzzy r-semiopen set which contained in μ. Also, $\mathrm{scl}(\mu, r) = \mu$ for any fuzzy r-semiclosed set μ and $\mathrm{sint}(\mu, r) = \mu$ for any fuzzy r-semiopen set μ. Moreover, we have

$$\mathrm{int}(\mu, r) \leq \mathrm{sint}(\mu, r) \leq \mu \leq \mathrm{scl}(\mu, r) \leq \mathrm{cl}(\mu, r).$$

It is obvious that any fuzzy r-open (r-closed) set is fuzzy r-semiopen (r-semiclosed). But the converse need not be true. The intersection (union) of any two fuzzy r-semiopen (r-semiclosed) sets need not be fuzzy r-semiopen (r-semiclosed).

3. Fuzzy r-semicontinuous maps

DEFINITION 3.1. [6] Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy r-continuous* map if $f^{-1}(\mu)$ is a fuzzy r-open set of X for each fuzzy r-open set μ of Y, or equivalently, $f^{-1}(\mu)$ is a fuzzy r-closed set of X for each fuzzy r-closed set μ of Y,
(2) a *fuzzy r-open* map if $f(\mu)$ is a fuzzy r-open set of Y for each fuzzy r-open set μ of X,
(3) a *fuzzy r-closed* map if $f(\mu)$ is a fuzzy r-closed set of Y for each fuzzy r-closed set μ of X,
(4) a *fuzzy r-homeomorphism* if f is bijective, fuzzy r-continuous and fuzzy r-open.

DEFINITION 3.2. [6] Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy r-semicontinuous* map if $f^{-1}(\mu)$ is a fuzzy r-semiopen set of X for each fuzzy r-open set μ of Y, or equivalently, $f^{-1}(\mu)$ is a fuzzy r-semiclosed set of X for each fuzzy r-closed set μ of Y,
(2) a *fuzzy r-semiopen* map if $f(\mu)$ is a fuzzy r-semiopen set of Y for each fuzzy r-open set μ of X,
(3) a *fuzzy r-semiclosed* map if $f(\mu)$ is a fuzzy r-semiclosed set of Y for each fuzzy r-closed set μ of X.

It is clear that every fuzzy r-continuous(r-open, r-closed) map is a fuzzy r-semicontinuous(r-semiopen, r-semiclosed) map for each $r \in I_0$. However the converse need not be true.

THEOREM 3.3. [6] *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then f is fuzzy r-continuous if and only if $f(\mathrm{cl}(\rho, r)) \le \mathrm{cl}(f(\rho), r)$ for each fuzzy set ρ of X.*

THEOREM 3.4. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then f is fuzzy r-closed if and only if $f(\mathrm{cl}(\rho, r)) \ge \mathrm{cl}(f(\rho), r)$ for each fuzzy set ρ of X.*

Proof. Let f be a fuzzy r-closed map and ρ any fuzzy set of X. Note $\mathrm{cl}(\rho, r)$ is a fuzzy r-closed set of X. Since f is fuzzy r-closed, $f(\mathrm{cl}(\rho, r))$ is a fuzzy r-closed set of Y. Thus

$$f(\mathrm{cl}(\rho, r)) = \mathrm{cl}(f(\mathrm{cl}(\rho, r)), r) \geq \mathrm{cl}(f(\rho), r).$$

Conversely, let ρ be fuzzy r-closed in X. Then $\mathrm{cl}(\rho, r) = \rho$. By hypothesis,

$$\mathrm{cl}(f(\rho), r) \leq f(\mathrm{cl}(\rho, r)) = f(\rho) \leq \mathrm{cl}(f(\rho), r).$$

Thus $\mathrm{cl}(f(\rho), r) = f(\rho)$ and hence $f(\rho)$ is fuzzy r-closed in Y. Therefore f is fuzzy r-closed. □

¿From Theorem 3.3 and Theorem 3.4 we have the following result.

THEOREM 3.5. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a bijection and $r \in I_0$. Then the following statements are equivalent :*

(1) *f is a fuzzy r-homeomorphism.*
(2) *f is fuzzy r-continuous and fuzzy r-closed.*
(3) *$f(\mathrm{cl}(\rho, r)) = \mathrm{cl}(f(\rho), r)$ for each fuzzy set ρ of X.*

The notion of fuzzy r-semicontinuity can be restated in terms of fuzzy r-closure and fuzzy r-interior.

THEOREM 3.6. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent :*

(1) *f is a fuzzy r-semicontinuous map.*
(2) *$\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r) \leq f^{-1}(\mathrm{cl}(\mu, r))$ for each fuzzy set μ of Y.*
(3) *$f(\mathrm{int}(\mathrm{cl}(\rho, r), r)) \leq \mathrm{cl}(f(\rho), r)$ for each fuzzy set ρ of X.*

Proof. (1) ⇒ (2) Let f be a fuzzy r-semicontinuous map and μ any fuzzy set of Y. Then $\mathrm{cl}(\mu, r)$ is a fuzzy r-closed set of Y. Since f is fuzzy r-semicontinuous, $f^{-1}(\mathrm{cl}(\mu, r))$ is a fuzzy r-semiclosed set of X. Thus

$$f^{-1}(\mathrm{cl}(\mu, r)) \geq \mathrm{int}(\mathrm{cl}(f^{-1}(\mathrm{cl}(\mu, r)), r), r) \geq \mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r).$$

(2) $\Rightarrow$ (3) Let ρ be a fuzzy set of X. Then $f(\rho)$ is a fuzzy set of Y. By (2),

$$f^{-1}(\mathrm{cl}(f(\rho), r)) \geq \mathrm{int}(\mathrm{cl}(f^{-1}f(\rho), r), r) \geq \mathrm{int}(\mathrm{cl}(\rho, r), r).$$

Hence

$$\mathrm{cl}(f(\rho), r) \geq ff^{-1}(\mathrm{cl}(f(\rho), r)) \geq f(\mathrm{int}(\mathrm{cl}(\rho, r), r)).$$

(3) $\Rightarrow$ (1) Let μ be a fuzzy r-closed set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. By (3),

$$f(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r)) \leq \mathrm{cl}(ff^{-1}(\mu), r) \leq \mathrm{cl}(\mu, r) = \mu.$$

So

$$\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r) \leq f^{-1}f(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r)) \leq f^{-1}(\mu).$$

Thus $f^{-1}(\mu)$ is a fuzzy r-semiclosed set of X and hence f is a fuzzy r-semicontinuous map. □

We already knew the following theorem.

THEOREM 3.7. [6] *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent :*

(1) *f is a fuzzy r-semicontinuous map.*
(2) *$f(\mathrm{scl}(\rho, r)) \leq \mathrm{cl}(f(\rho), r)$ for each fuzzy set ρ of X.*
(3) *$\mathrm{scl}(f^{-1}(\mu), r) \leq f^{-1}(\mathrm{cl}(\mu, r))$ for each fuzzy set μ of Y.*
(4) *$f^{-1}(\mathrm{int}(\mu, r)) \leq \mathrm{sint}(f^{-1}(\mu), r)$ for each fuzzy set μ of Y.*

If the map f is a bijection, we have:

THEOREM 3.8. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a bijection and $r \in I_0$. Then f is a fuzzy r-semicontinuous map if and only if $\mathrm{int}(f(\rho), r) \leq f(\mathrm{sint}(\rho, r))$ for each fuzzy set ρ of X.*

Proof. Let f be a fuzzy r-semicontinuous map and ρ any fuzzy set of X. Since $\text{int}(f(\rho), r)$ is fuzzy r-open in Y, $f^{-1}(\text{int}(f(\rho), r))$ is fuzzy r-semiopen in X. Since f is one-to-one, we have

$$f^{-1}(\text{int}(f(\rho), r)) \leq \text{sint}(f^{-1}f(\rho), r) = \text{sint}(\rho, r).$$

Since f is onto,

$$\text{int}(f(\rho), r) = ff^{-1}(\text{int}(f(\rho), r)) \leq f(\text{sint}(\rho, r)).$$

Conversely, let μ be fuzzy r-open of Y. Then $\text{int}(\mu, r) = \mu$. Since f is onto,

$$f(\text{sint}(f^{-1}(\mu), r)) \geq \text{int}(ff^{-1}(\mu), r) = \text{int}(\mu, r) = \mu.$$

Since f is one-to-one, we have

$$f^{-1}(\mu) \leq f^{-1}f(\text{sint}(f^{-1}(\mu), r)) = \text{sint}(f^{-1}(\mu), r) \leq f^{-1}(\mu).$$

Thus $f^{-1}(\mu) = \text{sint}(f^{-1}(\mu), r)$ and hence $f^{-1}(\mu)$ is fuzzy r-semiopen in X. Therefore f is fuzzy r-semicontinuous. □

THEOREM 3.9. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent :*

(1) *f is a fuzzy r-semiopen map.*
(2) *$f(\text{int}(\rho, r)) \leq \text{sint}(f(\rho), r)$ for each fuzzy set ρ of X.*
(3) *$\text{int}(f^{-1}(\mu), r) \leq f^{-1}(\text{sint}(\mu, r))$ for each fuzzy set μ of Y.*

Proof. (1) $\Rightarrow$ (2) Let ρ be any fuzzy set of X. Clearly $\text{int}(\rho, r)$ is fuzzy r-open in X. Since f is fuzzy r-semiopen, $f(\text{int}(\rho, r))$ is fuzzy r-semiopen in Y. Thus

$$f(\text{int}(\rho, r)) = \text{sint}(f(\text{int}(\rho, r)), r) \leq \text{sint}(f(\rho), r).$$

(2) $\Rightarrow$ (3) Let μ be any fuzzy set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. By (2),

$$f(\text{int}(f^{-1}(\mu), r)) \leq \text{sint}(ff^{-1}(\mu), r) \leq \text{sint}(\mu, r).$$

Thus we have

$$\mathrm{int}(f^{-1}(\mu), r) \leq f^{-1}f(\mathrm{int}(f^{-1}(\mu), r)) \leq f^{-1}(\mathrm{sint}(\mu, r)).$$

(3) $\Rightarrow$ (1) Let ρ be any fuzzy r-open set of X. Then $\mathrm{int}(\rho, r) = \rho$ and $f(\rho)$ is a fuzzy set of Y. By (3),

$$\rho = \mathrm{int}(\rho, r) \leq \mathrm{int}(f^{-1}f(\rho), r) \leq f^{-1}(\mathrm{sint}(f(\rho), r)).$$

Hence we have

$$f(\rho) \leq ff^{-1}(\mathrm{sint}(f(\rho), r)) \leq \mathrm{sint}(f(\rho), r) \leq f(\rho).$$

Thus $f(\rho) = \mathrm{sint}(f(\rho), r)$ and hence $f(\rho)$ is fuzzy r-semiopen in Y. Therefore f is fuzzy r-semiopen. □

THEOREM 3.10. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent :*

(1) *f is a fuzzy r-semiclosed map.*
(2) $\mathrm{scl}(f(\rho), r) \leq f(\mathrm{cl}(\rho, r))$ *for each fuzzy set ρ of X.*

Proof. (1) $\Rightarrow$ (2) Let ρ be any fuzzy set of X. Clearly $\mathrm{cl}(\rho, r)$ is fuzzy r-closed in X. Since f is fuzzy r-semiclosed, $f(\mathrm{cl}(\rho, r))$ is fuzzy r-semiclosed in Y. Thus we have

$$\mathrm{scl}(f(\rho), r) \leq \mathrm{scl}(f(\mathrm{cl}(\rho, r)), r) = f(\mathrm{cl}(\rho, r)).$$

(2) $\Rightarrow$ (1) Let ρ be any fuzzy r-closed of X. Then $\mathrm{cl}(\rho, r) = \rho$. By (2),

$$\mathrm{scl}(f(\rho), r) \leq f(\mathrm{cl}(\rho, r)) = f(\rho) \leq \mathrm{scl}(f(\rho), r).$$

Thus $f(\rho) = \mathrm{scl}(f(\rho), r)$ and hence $f(\rho)$ is fuzzy r-semiclosed in Y. Therefore f is fuzzy r-semiclosed. □

THEOREM 3.11. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a bijection and $r \in I_0$. Then f is a fuzzy r-semiclosed map if and only if $f^{-1}(\mathrm{scl}(\mu, r)) \leq \mathrm{cl}(f^{-1}(\mu), r)$ for each fuzzy set μ of Y.*

Proof. Let f be a fuzzy r-semiclosed map and μ any fuzzy set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. Since f is onto, we have

$$\mathrm{scl}(\mu, r) = \mathrm{scl}(ff^{-1}(\mu), r) \le f(\mathrm{cl}(f^{-1}(\mu), r)).$$

Since f is one-to-one, we have

$$f^{-1}(\mathrm{scl}(\mu, r)) \le f^{-1}f(\mathrm{cl}(f^{-1}(\mu), r)) = \mathrm{cl}(f^{-1}(\mu), r).$$

Conversely, let ρ be fuzzy r- closed of X. Then $\mathrm{cl}(\rho, r) = \rho$. Since f is one-to-one,

$$f^{-1}(\mathrm{scl}(f(\rho), r)) \le \mathrm{cl}(f^{-1}f(\rho), r) = \mathrm{cl}(\rho, r) = \rho.$$

Since f is onto, we have

$$\mathrm{scl}(f(\rho), r) = ff^{-1}(\mathrm{scl}(f(\rho), r)) \le f(\rho) \le \mathrm{scl}(f(\rho), r).$$

Thus $f(\rho) = \mathrm{scl}(f(\rho), r)$ and hence $f(\rho)$ is fuzzy r-semiclosed in Y. Therefore f is fuzzy r-semiclosed. □

THEOREM 3.12. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ and $g : (Y, \mathcal{U}) \to (Z, \mathcal{V})$ be maps and $r \in I_0$. Then the following statements are true.*

(1) *If f is fuzzy r-semicontinuous and g is fuzzy r-continuous then $g \circ f$ is fuzzy r-semicontinuous.*
(2) *If f is fuzzy r-open and g is fuzzy r-semiopen then $g \circ f$ is fuzzy r-semiopen.*
(3) *If f is fuzzy r-closed and g is fuzzy r-semiclosed then $g \circ f$ is fuzzy r-semiclosed.*

Proof. Straightforward. □

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