

## ANALYTIC FOURIER-FEYNMAN TRANSFORMS ON ABSTRACT WIENER SPACE

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ABSTRACT. In this paper, we introduce an  $L_p$  analytic Fourier-Feynman transformation, show the existence of the  $L_p$  analytic Fourier-Feynman transforms for a certain class of cylinder functionals on an abstract Wiener space, and investigate its interesting properties. Moreover, we define a convolution product for two functionals on the abstract Wiener space and establish the relationships between the Fourier-Feynman transform for the convolution product of two cylinder functionals and the Fourier-Feynman transform for each functional.

### 1. Introduction

The study of an  $L_1$  analytic Fourier-Feynman transformation on a classical Wiener space was initiated by Brue in [1]. In [2] Cameron and Storvick introduced an  $L_2$  analytic Fourier-Feynman transformation on a classical Wiener space. In [7] Johnson and Skoug developed an  $L_p$  analytic Fourier-Feynman transformation theory for  $1 \leq p \leq 2$  which extended the results in [2] and established several relationships between the  $L_1$  and  $L_2$  Fourier-Feynman transformation theories. In [5] Huffman, Park and Skoug developed an  $L_p$  analytic Fourier-Feynman transformation theory on a certain class of cylinder functionals on a classical Wiener space and they defined a convolution product for two cylinder functionals on the classical Wiener space and then verified that the Fourier-Feynman transform of the convolution product of two functionals is the product of Fourier-Feynman transforms of each functional.

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In this paper, we develop an  $L_p$  analytic Fourier-Feynman transformation theory on a certain class of cylinder functionals on an abstract Wiener space. In section 3, we verify the existence of the  $L_p$  analytic Fourier-Feynman transform for the class of cylinder functionals and investigate several interesting properties for the  $L_p$  analytic Fourier-Feynman transformation. In section 4, we define a convolution product for two functionals on the abstract Wiener space and then establish the relationships between the Fourier-Feynman transform for the convolution product of two functionals and the Fourier-Feynman transform for each functional.

## 2. Preliminaries and Notation

Let  $H$  be a real separable infinite dimensional Hilbert space with norm  $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$ , where  $\langle \cdot, \cdot \rangle$  is an inner product on  $H$ . Let  $\|\cdot\|_o$  be a fixed measurable norm on  $H$  (for definition, see [11]) and  $B$  the completion of  $H$  with respect to  $\|\cdot\|_o$ . If  $\nu$  is a Gauss measure on  $H$ , then  $\nu$  induces a cylinder set measure  $\tilde{\nu}$  on  $B$  which in turn extends to a countably additive measure  $\mu$  on  $(B, \mathcal{B}(B))$ , where  $\mathcal{B}(B)$  is the Borel  $\sigma$ -algebra of sets in  $B$ . Then  $(B, H, \mu)$  is called an *abstract Wiener space* and  $\mu$  is called an *abstract Wiener measure*. For more details, see [11]. We denote the Wiener integral of a functional  $F$  with respect to  $\mu$  by

$$\int_B F(x) d\mu(x).$$

Let  $\{e_j, j = 1, 2, 3, \dots\}$  be a complete orthonormal set in  $H$  such that the  $e_j$ 's are in  $B^*$ , the topological dual space of  $B$ . For each  $h$  in  $H$  and  $x$  in  $B$ , we define the *stochastic inner product*  $(h, x)^\sim$  as follows:

$$(2.1) \quad (h, x)^\sim = \begin{cases} \lim_{n \rightarrow \infty} \sum_{j=1}^n \langle h, e_j \rangle (e_j, x), & \text{if the limit exists} \\ 0, & \text{otherwise} \end{cases}$$

where  $(\cdot, \cdot)$  is the natural dual pairing between  $B^*$  and  $B$ .

The following lemma is quoted from [9,10].

LEMMA 2.1. *Let  $(B, H, \mu)$  be an abstract Wiener space. Then*

- (i)  $(h, x)^\sim$  is essentially independent of the choice of the complete orthonormal set used in its definition.
- (ii) for every real number  $\alpha$ ,  $(\alpha h, x)^\sim = \alpha(h, x)^\sim = (h, \alpha x)^\sim$  hold for  $h \in H$  and  $x \in B$ .
- (iii) for each  $h \in H (h \neq 0)$ , the random variable  $x \mapsto (h, x)^\sim$  is Gaussian with mean zero and variance  $|h|^2$ , that is,

$$\int_B \exp\{i(h, x)^\sim\} d\mu(x) = \exp\left\{-\frac{1}{2}|h|^2\right\}.$$

- (iv) if  $\{h_1, \dots, h_n\}$  is an orthogonal set in  $H$ , then the random variables  $(h_i, x)^\sim$ 's are independent.

Let  $(B, H, \mu)$  be an abstract Wiener space. For every  $\lambda > 0$ , let  $T_\lambda : B \rightarrow B$  be the transformation defined by  $T_\lambda x = \sqrt{\lambda}x$  and  $\mu_\lambda$  the Borel measure on  $B$  defined by  $\mu_\lambda = \mu \circ T_\lambda^{-1}$ . Then  $\mu_1 = \mu$  and  $\mu_\lambda(A) = \mu_1(\frac{1}{\sqrt{\lambda}}A)$  for every  $A$  in  $\mathcal{B}(B)$ . Let  $\mathcal{S}_\lambda(B)$  be the completion of  $\mathcal{B}(B)$  with respect to  $\mu_\lambda$ , and let  $\mathcal{N}_\lambda(B) = \{A \in \mathcal{S}_\lambda(B) : \mu_\lambda(A) = 0\}$ . Let  $\mathcal{S}(B) = \bigcap_{\lambda>0} \mathcal{S}_\lambda(B)$ , and  $\mathcal{N}(B) = \bigcap_{\lambda>0} \mathcal{N}_\lambda(B)$ . Every set in  $\mathcal{S}(B)$ (or  $\mathcal{N}(B)$ ) is called a *scale-invariant measurable* (or *scale-invariant null*) set. A real (or complex)-valued functional  $F$  on  $B$  is *scale-invariant measurable* if  $F$  is measurable with respect to the  $\sigma$ -algebra  $\mathcal{S}(B)$ . A property that holds except on a scale-invariant null set is said to hold *scale-invariant almost everywhere* (briefly, *s-a.e.*). If two functionals  $F$  and  $G$  are equal *s-a.e.*, then we write  $F \approx G$ .

Let  $\mathbb{R}^n$  denote the  $n$ -dimensional Euclidean space and let  $\mathbb{C}, \mathbb{C}_+$  and  $\mathbb{C}_+^\sim$  denote respectively the complex numbers, the complex numbers with positive real part, and the non-zero complex numbers with nonnegative real part. Let  $F$  be a complex-valued scale-invariant measurable functional on  $B$  such that

$$(2.2) \quad J(F; \lambda) = \int_B F(\lambda^{-\frac{1}{2}}x) d\mu(x)$$

exists as a finite number for all real  $\lambda > 0$ . If there exists a function  $J^*(F; z)$  analytic in  $\mathbb{C}_+$  such that  $J^*(F; \lambda) = J(F; \lambda)$  for all real  $\lambda > 0$ ,

then we say that  $J^*(F; z)$  is the *analytic Wiener integral* of  $F$  over  $B$  with parameter  $z$ , and for each  $z \in \mathbb{C}_+$ , we write

$$(2.3) \quad I^{anw}(F; z) = J^*(F; z).$$

Let  $q$  be a non-zero real number and  $F$  a functional such that  $I^{anw}(F; z)$  exists for all  $z \in \mathbb{C}_+$ . If the following limit exists, then we call it the *analytic Feynman integral* of  $F$  over  $B$  with parameter  $q$ , and we write

$$(2.4) \quad I^{anf}(F; q) = \lim_{z \rightarrow -iq} I^{anw}(F; z),$$

where  $z$  approaches  $-iq$  through  $\mathbb{C}_+$ .

NOTATION.

(i) For each  $z \in \mathbb{C}_+$  and  $y \in B$ , let

$$(2.5) \quad (\mathcal{T}_z(F))(y) = I^{anw}(F(\cdot + y); z).$$

(ii) Given a number  $p$  with  $1 \leq p < \infty$ ,  $p$  and  $p'$  will always be related by  $1/p + 1/p' = 1$ .

(iii) Let  $1 < p \leq 2$  and let  $\{F_n\}$  and  $F$  be scale-invariant measurable functionals such that for each  $\rho > 0$ ,

$$(2.6) \quad \lim_{n \rightarrow \infty} \int_B |F_n(\rho x) - F(\rho x)|^{p'} d\mu(x) = 0.$$

Then we write

$$(2.7) \quad \text{l. i. m.}_{n \rightarrow \infty} (w_s^{p'}) (F_n) \approx F,$$

and we call  $F$  the *scale-invariant limit in the mean of order  $p'$* .

A similar definition is understood when  $n$  is replaced by the continuously varying parameter  $z$ .

DEFINITION 2.1. Let  $q \neq 0$  be a real number. For  $1 < p \leq 2$ , we define the  $L_p$  analytic Fourier-Feynman transform  $\mathcal{T}_q^{(p)}(F)$  for a functional  $F$  on  $(B, H, \mu)$  by

$$(2.8) \quad (\mathcal{T}_q^{(p)}(F))(y) = \text{l. i. m.}_{z \rightarrow -iq} (w_s^{p'}) (\mathcal{T}_z(F))(y)$$

for  $s$ -a.e.  $y \in B$ , whenever this limit exists, where  $z$  approaches  $-iq$  through  $\mathbb{C}_+$ .

We define the  $L_1$  analytic Fourier-Feynman transform  $\mathcal{T}_q^{(1)}(F)$  of  $F$  by

$$(2.9) \quad (\mathcal{T}_q^{(1)}(F))(y) = \lim_{z \rightarrow -iq} (\mathcal{T}_z(F))(y),$$

for  $s$ -a.e.  $y \in B$ , where  $z$  approaches  $-iq$  through  $\mathbb{C}_+$ .

We note that for  $1 \leq p \leq 2$ ,  $\mathcal{T}_q^{(p)}(F)$  is defined only  $s$ -a.e.. We also note that if  $\mathcal{T}_q^{(p)}(F)$  exists and if  $F \approx G$ , then  $\mathcal{T}_q^{(p)}(G)$  exists and  $\mathcal{T}_q^{(p)}(F) \approx \mathcal{T}_q^{(p)}(G)$ .

DEFINITION 2.2. Let  $(B, H, \mu)$  be an abstract Wiener space. Let  $n$  be a positive integer, and  $\{h_1, \dots, h_n\}$  an orthonormal set in  $H$ . For every  $p$  with  $1 \leq p < \infty$ , let  $\mathcal{F}(n; p)$  denote the class of cylinder functionals  $F$  on  $B$  with the following form :

$$(2.10) \quad F(x) = f((h_1, x)^\sim, \dots, (h_n, x)^\sim), \quad x \in B,$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  is in  $L_p(\mathbb{R}^n)$ , the space of functions whose  $p$ -th powers are Lebesgue integrable on  $\mathbb{R}^n$  and  $(h_j, x)^\sim$ 's are defined as in the formula (2.1).

Let  $\mathcal{F}(n; \infty)$  denote the class of cylinder functionals  $F$  on  $B$  with the form (2.10), where  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  is in  $C_o(\mathbb{R}^n)$ , the space of continuous functions on  $\mathbb{R}^n$  that vanish at infinity.

Note that if  $F$  is in  $\mathcal{F}(n; p)$ , then  $F$  is scale-invariant measurable.

DEFINITION 2.3. Let  $F$  and  $G$  be two complex-valued functionals on the abstract Wiener space  $(B, H, \mu)$ . For each  $z \in \mathbb{C}_+^\sim$ , we define their convolution product  $(F * G)_z$  as follows :

When  $z$  belongs to  $\mathbb{C}_+$ ,

$$(2.11(a)) \quad (F * G)_z(y) = \mathcal{I}^{anw} \left[ F\left(\frac{1}{\sqrt{2}}(y + \cdot)\right) G\left(\frac{1}{\sqrt{2}}(y - \cdot)\right); z \right]$$

for  $y \in B$ , if it exists.

When  $z = -iq$  ( $q \in \mathbb{R} - \{0\}$ ),

$$(2.11(b)) \quad (F * G)_q(y) = \mathcal{I}^{anf} \left[ F\left(\frac{1}{\sqrt{2}}(y + \cdot)\right) G\left(\frac{1}{\sqrt{2}}(y - \cdot)\right); q \right]$$

for  $y \in B$ , if it exists.

The following theorem is the well-known Wiener Integration Formula :

**THEOREM 2.4.** *Let  $(B, H, \mu)$  be an abstract Wiener space and let  $\{h_1, \dots, h_n\}$  be an orthonormal set in  $H$ . Define a functional  $F$  on  $B$  by*

$$F(x) = f((h_1, x)^\sim, \dots, (h_n, x)^\sim), \quad x \in B,$$

where  $f$  is a complex-valued Lebesgue measurable function defined on  $\mathbb{R}^n$ . Then  $F$  is an Wiener measurable functional on  $B$ , and

$$(2.12) \quad \int_B F(x) d\mu(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{-\frac{1}{2} \sum_{j=1}^n u_j^2\right\} d\vec{u},$$

where  $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$  and  $d\vec{u} = du_1 du_2 \dots du_n$ .

Finally we will close this section by introducing the following well-known integration formula :

$$(2.13) \quad \int_{\mathbb{R}} \exp\{-au^2 + bu\} du = \sqrt{\frac{\pi}{a}} \exp\left\{\frac{b^2}{4a}\right\},$$

where  $a$  is a complex number with  $Re(a) \geq 0$  and  $b$  is a complex number.

**3. The  $L_p$  analytic Fourier-Feynman Transformation on the class  $\mathcal{F}(n; p)$**

We begin this section by obtaining the following theorem.

**THEOREM 3.1.** *Let  $F \in \mathcal{F}(n; p)$  be given by the formula (2.10), where  $1 \leq p \leq \infty$ . Then the transform  $(\mathcal{T}_z(F))(y)$  exists for every  $z \in \mathbb{C}_+$ , and is expressed by*

$$(3.1) \quad (\mathcal{T}_z(F))(y) \approx (G_z f)((h_1, y)^\sim, \dots, (h_n, y)^\sim),$$

where the set  $\{h_1, \dots, h_n\}$  is given as in Definition 2.2, and  $(G_z f)(\cdot)$  is given by

$$(3.2) \quad \begin{aligned} (G_z f)(w_1, \dots, w_n) &= (G_z f)(\vec{w}) \\ &= \left(\frac{z}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{-\frac{z}{2} \sum_{j=1}^n (u_j - w_j)^2\right\} d\vec{u}, \end{aligned}$$

where  $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$  and  $d\vec{u} = du_1 du_2 \dots du_n$ .

*Proof.* We first show that the formula (3.1) holds for every  $\lambda > 0$  and  $s$ -a.e.  $y \in B$ . Using the formula (2.12) of Theorem 2.4, we have, for every  $\lambda > 0$  and  $s$ -a.e.  $y \in B$ ,

$$\begin{aligned} &(\mathcal{T}_\lambda(F))(y) \\ &= \int_B F(\lambda^{-\frac{1}{2}}x + y) d\mu(x) \\ &= \int_B f(\lambda^{-\frac{1}{2}}(h_1, x)^\sim + (h_1, y)^\sim, \dots, \lambda^{-\frac{1}{2}}(h_n, x)^\sim + (h_n, y)^\sim) d\mu(x) \\ &= \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n (u_j - (h_j, y)^\sim)^2\right\} d\vec{u} \\ &= (G_\lambda f)((h_1, y)^\sim, \dots, (h_n, y)^\sim), \end{aligned}$$

where  $(G_\lambda f)(\vec{w})$  is given by the formula (3.2).

Next with the help of the Morera's Theorem, we can verify that  $(\mathcal{T}_z(F))(y)$  is an analytic function of  $z$  over  $\mathbb{C}_+$ , and so the formula (3.1) holds throughout  $\mathbb{C}_+$ . □

LEMMA 3.2.

- (i) Let  $z \in \mathbb{C}_+^\sim$  and let  $F \in \mathcal{F}(n; 1)$  be given by the formula (2.10). Then  $(\mathcal{T}_z(F))(\cdot)$  belongs to  $\mathcal{F}(n; \infty)$ .
- (ii) Let  $z \in \mathbb{C}_+^\sim$  and let  $F \in \mathcal{F}(n; p)$  ( $1 < p \leq 2$ ) be given by the formula (2.10). Then  $(\mathcal{T}_z(F))(\cdot)$  belongs to  $\mathcal{F}(n; p')$ , where  $1/p + 1/p' = 1$ .

*Proof.* (i) Let  $F \in \mathcal{F}(n; 1)$ . Then we can show that  $(G_z f)(\vec{w})$  belongs to  $C_o(\mathbb{R}^n)$  as a function of  $\vec{w} \in \mathbb{R}^n$ , where  $(G_z f)(\vec{w})$  is given by the formula (3.2). Hence  $(\mathcal{T}_z(F))(\cdot)$  belongs to  $\mathcal{F}(n; \infty)$ .

(ii) Let  $F \in \mathcal{F}(n; p)$ , where  $1 < p \leq 2$ . Then it follows from [6; Lemma 1.1, p.98] that for each  $z \in \mathbb{C}_+^\sim$ ,  $G_z$  is in  $\mathcal{L}(L_p(\mathbb{R}^n), L_{p'}(\mathbb{R}^n))$ , the space of continuous linear operators from  $L_p(\mathbb{R}^n)$  to  $L_{p'}(\mathbb{R}^n)$ . Hence  $(\mathcal{T}_z(F))(\cdot)$  belongs to  $\mathcal{F}(n; p')$ .  $\square$

LEMMA 3.3. Let  $F \in \mathcal{F}(n; 1)$  be given by the formula (2.10) and let  $(G_z f)(\cdot)$  be given by the formula (3.2). Then

- (i)  $(G_z f)(\cdot)$  converges pointwise to  $(G_{-iq} f)(\cdot)$ , whenever  $z$  approaches  $-iq$  through  $\mathbb{C}_+$ , and
- (ii) as elements of  $C_o(\mathbb{R}^n)$ ,  $(G_z f)(\cdot)$  converges weakly to  $(G_{-iq} f)(\cdot)$ , whenever  $z$  approaches  $-iq$  through  $\mathbb{C}_+$ .

*Proof.* (i) follows immediately by the dominated convergence theorem. To prove (ii), let  $\nu \in \mathcal{M}(\mathbb{R}^n)$ , the space of complex-valued Borel measures on  $\mathcal{B}(\mathbb{R}^n)$ , the Borel  $\sigma$ -algebra for  $\mathbb{R}^n$ . Since  $\mathcal{M}(\mathbb{R}^n)$  is the dual space of  $C_o(\mathbb{R}^n)$  and  $|(G_z f)(\vec{w})| \leq |\frac{z}{2\pi}|^{\frac{n}{2}} \|f\|_1$ , we have

$$\lim_{z \rightarrow -iq} \int_{\mathbb{R}^n} (G_z f)(\vec{w}) d\nu(\vec{w}) = \int_{\mathbb{R}^n} (G_{-iq} f)(\vec{w}) d\nu(\vec{w})$$

by the dominated convergence theorem.  $\square$

Now we shall verify that the  $L_p$  analytic Fourier-Feynman transform exists for all functionals  $F \in \mathcal{F}(n; p)$ , where  $1 \leq p \leq 2$ .



**THEOREM 3.4.** *Let  $F \in \mathcal{F}(n;1)$  be given by the formula (2.10). Then for all  $q \in \mathbb{R} - \{0\}$ , the  $L_1$  analytic Fourier-Feynman transform  $\mathcal{T}_q^{(1)}(F)$  exists as an element of  $\mathcal{F}(n; \infty)$ , and is expressed by*

$$(3.3) \quad \mathcal{T}_q^{(1)}(F)(y) \approx (G_{-iq}f)((h_1, y)^\sim, \dots, (h_n, y)^\sim),$$

where  $\{h_1, \dots, h_n\}$  is given as in Definition 2.2 and  $(G_{-iq}f)(\cdot)$  is given by the formula (3.2).

*Proof.* By Lemmas 3.2 and 3.3,  $\mathcal{T}_q^{(1)}(F)$  exists as an element of  $\mathcal{F}(n; \infty)$ , and the formula (3.3) is established. □

**REMARK.** When  $1 < p \leq 2$  and  $Re z = 0$ , the integral in the formula (3.2) should be interpreted in the mean just as in the theory of the  $L_p$  Fourier transformation [12].

**THEOREM 3.5.** *Let  $F \in \mathcal{F}(n;p)$  ( $1 < p \leq 2$ ) be given by the formula (2.10). Then for all non-zero real number  $q$ , the  $L_p$  analytic Fourier-Feynman transform  $\mathcal{T}_q^{(p)}(F)$  exists as an element of  $\mathcal{F}(n;p')$ , and is expressed by*

$$(3.4) \quad \mathcal{T}_q^{(p)}(F)(y) \approx (G_{-iq}f)((h_1, y)^\sim, \dots, (h_n, y)^\sim),$$

where the set  $\{h_1, \dots, h_n\}$  is given as in Definition 2.2,  $(G_{-iq}f)(\cdot)$  is given by the formula (3.2), and  $1/p + 1/p' = 1$ .

*Proof.* With the help of [6; Lemma 1.2, p.100], we conclude that for each  $f \in L_p(\mathbb{R}^n)$ ,  $(G_z f)(\cdot)$  belongs to  $L_{p'}(\mathbb{R}^n)$ , and  $\|(G_z f)(\cdot) - (G_{-iq}f)(\cdot)\|_{p'}$  comes close zero, whenever  $z$  approaches  $-iq$  through  $\mathbb{C}_+$ , where  $1/p + 1/p' = 1$ .

Using the formula (2.12) of Theorem 2.4, we have, for each  $\rho > 0$ ,

$$\begin{aligned} & \int_B |(G_z f)(\rho(h_1, y)^\sim, \dots, \rho(h_n, y)^\sim) - \\ & \quad (G_{-iq}f)(\rho(h_1, y)^\sim, \dots, \rho(h_n, y)^\sim)|^{p'} d\mu(y) \\ &= (2\pi\rho^2)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |(G_z f)(\vec{u}) - (G_{-iq}f)(\vec{u})|^{p'} \exp\left\{-\frac{1}{2\rho^2} \sum_{j=1}^n u_j^2\right\} d\vec{u} \\ &\leq (2\pi\rho^2)^{-\frac{n}{2}} \|(G_z f)(\cdot) - (G_{-iq}f)(\cdot)\|_{p'}^{p'}. \end{aligned}$$

Because the last expression comes close zero, whenever  $z$  approaches  $-iq$  through  $\mathbb{C}_+$ , we conclude that  $\mathcal{T}_q^{(p)}(F)$  exists as an element of  $\mathcal{F}(n; p')$ , and the formula (3.4) is established.  $\square$

EXAMPLE 1. Let  $(B, H, \mu)$  be an abstract Wiener space and let  $\{h_1, \dots, h_n\}$  be an orthonormal set in  $H$ . Consider a functional  $F : B \rightarrow \mathbb{R}$  defined by

$$(3.5) \quad F(x) = \exp \left\{ - \sum_{j=1}^n a_j ((h_j, x)^\sim)^2 \right\}, \quad x \in B,$$

where  $a_j > 0$  for  $j = 1, \dots, n$ .

Then it is obvious that  $F$  belongs to  $\mathcal{F}(n; p)$ , where  $1 \leq p \leq \infty$ .

We first calculate immediately the  $L_1$  analytic Fourier-Feynman transform  $\mathcal{T}_q^{(1)}(F)$  of  $F$  given by the formula (3.5), where  $q$  is a non-zero real number. Using the formula (2.12) of Theorem 2.4, we have, for each  $\lambda > 0$  and  $s$ -a.e.  $y \in B$ ,

$$\begin{aligned} & (\mathcal{T}_\lambda(F))(y) \\ &= \int_B F(\lambda^{-\frac{1}{2}}x + y) d\mu(x) \\ &= \exp \left\{ - \sum_{j=1}^n a_j ((h_j, y)^\sim)^2 \right\} \\ & \cdot \int_B \exp \left\{ - \sum_{j=1}^n \left( \frac{a_j}{\lambda} ((h_j, x)^\sim)^2 + \frac{2a_j}{\sqrt{\lambda}} (h_j, y)^\sim (h_j, x)^\sim \right) \right\} d\mu(x) \\ &= \exp \left\{ - \sum_{j=1}^n a_j ((h_j, y)^\sim)^2 \right\} \\ & \cdot \left( \frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp \left\{ - \sum_{j=1}^n \left( \left( \frac{1}{2} + \frac{a_j}{\lambda} \right) u_j^2 + \frac{2a_j}{\sqrt{\lambda}} (h_j, y)^\sim u_j \right) \right\} d\vec{u} \\ &= \prod_{j=1}^n \left( \frac{\lambda}{\lambda + 2a_j} \right)^{\frac{1}{2}} \exp \left\{ \frac{-\lambda a_j}{\lambda + 2a_j} ((h_j, y)^\sim)^2 \right\}, \end{aligned}$$

where the last equality comes from the formula (2.13).

By analytic continuation for  $\lambda$  in the last expression, we have, for all  $z \in \mathbb{C}_+$  and  $s$ -a.e.  $y \in B$ ,

$$(\mathcal{T}_z(F))(y) = \prod_{j=1}^n \left( \frac{z}{z + 2a_j} \right)^{\frac{1}{2}} \exp \left\{ \frac{-za_j}{z + 2a_j} ((h_j, y)^\sim)^2 \right\}.$$

Hence

$$(3.6) \quad \begin{aligned} (\mathcal{T}_q^{(1)}(F))(y) &\approx \lim_{z \rightarrow -iq} (\mathcal{T}_z(F))(y) \\ &\approx \prod_{j=1}^n \left( \frac{-iq}{2a_j - iq} \right)^{\frac{1}{2}} \exp \left\{ \frac{iq a_j}{2a_j - iq} ((h_j, y)^\sim)^2 \right\}. \end{aligned}$$

But taking  $f(\vec{u}) = \exp\{-\sum_{j=1}^n a_j u_j^2\}$  and  $z = -iq$  in the formula (3.2), we have

$$(3.7) \quad \begin{aligned} &(G_{-iq}f)((h_1, y)^\sim, \dots, (h_n, y)^\sim) \\ &= \left( \frac{-iq}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp \left\{ -\sum_{j=1}^n a_j u_j^2 + \frac{iq}{2} \sum_{j=1}^n (u_j - (h_j, y)^\sim)^2 \right\} d\vec{u} \\ &= \prod_{j=1}^n \left( \frac{-iq}{2a_j - iq} \right)^{\frac{1}{2}} \exp \left\{ \frac{iq a_j}{2a_j - iq} ((h_j, y)^\sim)^2 \right\}, \end{aligned}$$

where the last equality comes from the formula (2.13).

Finally we can verify that the last expression belongs to  $\mathcal{F}(n; \infty)$ , and from the formulas (3.6) and (3.7) we have

$$(\mathcal{T}_q^{(1)}(F))(y) \approx (G_{-iq}f)((h_1, y)^\sim, \dots, (h_n, y)^\sim).$$

Thus we have verified that the formula (3.3) of Theorem 3.4 holds for the functional  $F$  given by the formula (3.5).

**EXAMPLE 2.** Let  $(B, H, \mu)$  and  $\{h_1, \dots, h_n\}$  be given as in Example 1. Consider a functional  $H : B \rightarrow \mathbb{C}$  defined by

$$(3.8) \quad H(x) = \exp \left\{ \sum_{j=1}^n a_j (h_j, x)^\sim \right\}, \quad x \in B,$$

where  $a_j$  is a complex number for  $j = 1, \dots, n$ .

Then it is obvious that  $H$  does not belong to  $\mathcal{F}(n; p)$ , where  $1 \leq p \leq \infty$ .

We first calculate immediately the  $L_1$  analytic Fourier-Feynman transform  $\mathcal{T}_q^{(1)}(H)$  of  $H$  given by (3.8), where  $q$  is a non-zero real number. Using the formula (2.12) of Theorem 2.4, we have, for each  $\lambda > 0$  and  $s$ -a.e.  $y \in B$ ,

$$\begin{aligned} & (\mathcal{T}_\lambda(H))(y) \\ &= \int_B H(\lambda^{-\frac{1}{2}}x + y) d\mu(x) \\ &= \exp\left\{\sum_{j=1}^n a_j(h_j, y)^\sim\right\} \int_B \exp\left\{\sum_{j=1}^n \frac{a_j}{\sqrt{\lambda}}(h_j, x)^\sim\right\} d\mu(x) \\ &= \exp\left\{\sum_{j=1}^n a_j(h_j, y)^\sim\right\} \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp\left\{\sum_{j=1}^n \left(-\frac{u_j^2}{2} + \frac{a_j}{\sqrt{\lambda}}u_j\right)\right\} d\vec{u} \\ &= \prod_{j=1}^n \exp\left\{\frac{1}{2\lambda}a_j^2 + a_j(h_j, y)^\sim\right\}, \end{aligned}$$

where the last equality comes from the formula (2.13).

By analytic continuation for  $\lambda$  in the last expression, we have, for all  $z \in \mathbb{C}_+$  and  $s$ -a.e.  $y \in B$ ,

$$(\mathcal{T}_z(H))(y) = \prod_{j=1}^n \exp\left\{\frac{1}{2z}a_j^2 + a_j(h_j, y)^\sim\right\}.$$

Hence

$$\begin{aligned} (3.9) \quad (\mathcal{T}_q^{(1)}(H))(y) &\approx \lim_{z \rightarrow -iq} (\mathcal{T}_z(H))(y) \\ &\approx \prod_{j=1}^n \exp\left\{\frac{i}{2q}a_j^2 + a_j(h_j, y)^\sim\right\}. \end{aligned}$$

But by taking  $h(\vec{u}) = \exp\{\sum_{j=1}^n a_j u_j\}$  and  $z = -iq$  in (3.2), we obtain

$$(3.10) \quad (G_{-iq}h)((h_1, y)^\sim, \dots, (h_n, y)^\sim)$$

$$\begin{aligned}
&= \left(\frac{-iq}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp\left\{\sum_{j=1}^n a_j u_j + \frac{iq}{2} \sum_{j=1}^n (u_j - (h_j, y)^\sim)^2\right\} d\vec{u} \\
&= \prod_{j=1}^n \exp\left\{\frac{i}{2q} a_j^2 + a_j (h_j, y)^\sim\right\},
\end{aligned}$$

where the last equality comes from the formula (2.13).

Finally, combining the formulas (3.9) and (3.10), we have

$$(\mathcal{T}_q^{(1)}(H))(y) \approx (G_{-iq}h)((h_1, y)^\sim, \dots, (h_n, y)^\sim).$$

In this example, we have found a functional  $H$  for which the formula (3.3) of Theorem 3.4 holds, but  $H$  does not belong to  $\mathcal{F}(n; p)$ , where  $1 \leq p \leq \infty$ .

#### 4. Convolutions and Fourier-Feynman Transforms of Convolution Products

We begin with this section by giving an expression for the convolution product  $(F * G)_z$  with  $z \in \mathbb{C}_+$ , where  $F$  and  $G$  belong to  $\mathcal{F}(n; p)$  and  $\mathcal{F}(n; p')$ , respectively, and  $1/p + 1/p' = 1$ . Moreover, we shall present an expression of the  $L_p$  analytic Fourier-Feynman transform for the convolution product of two cylinder functionals on an abstract Wiener space  $(B, H, \mu)$ .

**THEOREM 4.1.** *Let  $F$  and  $G$  be given by the formula (2.10), and belong to  $\mathcal{F}(n; p)$  and  $\mathcal{F}(n; p')$ , respectively, where  $1 \leq p \leq 2$  and  $1/p + 1/p' = 1$ . Then for every  $z \in \mathbb{C}_+$ , the convolution product  $(F * G)_z$  exists as an element of  $\mathcal{F}(n; \infty)$ , and the following equation (4.1) holds for every  $z \in \mathbb{C}_+$  and s-a.e.  $y \in B$ :*

$$(4.1) \quad (F * G)_z(y) = H_z((h_1, y)^\sim, \dots, (h_n, y)^\sim),$$

where the set  $\{h_1, \dots, h_n\}$  is given as in Definition 2.2, and

$$\begin{aligned}
(4.2) \quad &H_z(w_1, \dots, w_n) = H_z(\vec{w}) \\
&= \left(\frac{z}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f\left(\frac{\vec{w} + \vec{u}}{\sqrt{2}}\right) g\left(\frac{\vec{w} - \vec{u}}{\sqrt{2}}\right) \exp\left\{-\frac{z}{2} \sum_{j=1}^n u_j^2\right\} d\vec{u},
\end{aligned}$$

where  $f$  and  $g$  correspond to  $F$  and  $G$  as in the formula (2.10), respectively.

*Proof.* The proof of the first part of this lemma will be given in that of Theorem 4.3 (ii). Now let us prove the second part of this lemma. Using the formula (2.12) of Theorem 2.4, we obtain, for each  $\lambda > 0$  and  $s$ -a.e.  $y \in B$

$$\begin{aligned}
& (F * G)_\lambda(y) \\
&= \int_B F\left(\frac{1}{\sqrt{2}}(y + \lambda^{-\frac{1}{2}}x)\right) G\left(\frac{1}{\sqrt{2}}(y - \lambda^{-\frac{1}{2}}x)\right) d\mu(x) \\
&= \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f\left\{\frac{((h_1, y)^\sim + u_1)}{\sqrt{2}}, \dots, \frac{((h_n, y)^\sim + u_n)}{\sqrt{2}}\right\} \\
&\quad \cdot g\left\{\frac{((h_1, y)^\sim - u_1)}{\sqrt{2}}, \dots, \frac{((h_n, y)^\sim - u_n)}{\sqrt{2}}\right\} \cdot \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n u_j^2\right\} d\vec{u} \\
&= H_\lambda((h_1, y)^\sim, \dots, (h_n, y)^\sim),
\end{aligned}$$

where  $H_\lambda(\cdot)$  is given by the formula (4.2). Hence the formula (4.1) holds for all  $\lambda > 0$  and  $s$ -a.e.  $y \in B$ . Now by analytic continuation for  $\lambda$ , we can verify that the formula (4.1) holds for all  $z \in \mathbb{C}_+$  and  $s$ -a.e.  $y \in B$ .  $\square$

REMARK. Note that the convolution product  $(F * G)_z$  exists for all  $z \in \mathbb{C}_+$  and is given by the formula (4.1), even if  $F$  and  $G$  belong to  $\bigcup_{1 \leq p \leq \infty} \mathcal{F}(n; p)$ .

The following theorem establishes an interesting relationship involving the convolution product and analytic Wiener integrals on an abstract Wiener space.

THEOREM 4.2. Let  $F$  and  $G$  belong to  $\bigcup_{1 \leq p \leq \infty} \mathcal{F}(n; p)$ . Then the following formula (4.3) holds for every  $z \in \mathbb{C}_+$  and  $s$ -a.e.  $y \in B$  :

$$(4.3) \quad (\mathcal{T}_z(F * G)_z)(y) = (\mathcal{T}_z(F))\left(\frac{y}{\sqrt{2}}\right) \cdot (\mathcal{T}_z(G))\left(\frac{y}{\sqrt{2}}\right).$$

*Proof.* We first calculate  $(\mathcal{T}_\lambda(F * G)_\lambda)$  for all  $\lambda > 0$ . Using the formulas (4.1) and (4.2), we obtain, for each  $\lambda > 0$  and  $s$ -a.e.  $y \in B$ ,

$$\begin{aligned}
 & (\mathcal{T}_\lambda(F * G)_\lambda)(y) \\
 &= \int_B (F * G)_\lambda(\lambda^{-\frac{1}{2}}x + y) d\mu(x) \\
 &= \int_B H_\lambda((h_1, \lambda^{-\frac{1}{2}}x + y)^\sim, \dots, (h_n, \lambda^{-\frac{1}{2}}x + y)^\sim) d\mu(x) \\
 &= \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^{2n}} H_\lambda(v_1 + (h_1, y)^\sim, \dots, v_n + (h_n, y)^\sim) \\
 &\quad \cdot \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n v_j^2\right\} d\vec{v} \\
 &= \left(\frac{\lambda}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} f\left\{\frac{(v_1 + u_1 + (h_1, y)^\sim)}{\sqrt{2}}, \dots, \frac{(v_n + u_n + (h_n, y)^\sim)}{\sqrt{2}}\right\} \\
 &\quad \cdot g\left\{\frac{(v_1 - u_1 + (h_1, y)^\sim)}{\sqrt{2}}, \dots, \frac{(v_n - u_n + (h_n, y)^\sim)}{\sqrt{2}}\right\} \\
 &\quad \cdot \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n (u_j^2 + v_j^2)\right\} d\vec{u} d\vec{v},
 \end{aligned}$$

where the third equality follows from the formula (2.12) of Theorem 2.4.

Now we will use the following transformation

$$w_j = \frac{1}{\sqrt{2}}(v_j + u_j) \quad \text{and} \quad r_j = \frac{1}{\sqrt{2}}(v_j - u_j)$$

for  $j = 1, \dots, n$ . Note that the Jacobian of this transformation is one, and

$$\sum_{j=1}^n (w_j^2 + r_j^2) = \sum_{j=1}^n (u_j^2 + v_j^2).$$

Now using the formulas (3.1) and (3.2), we obtain, for each  $\lambda > 0$

and  $s$ -a.e.  $y \in B$ ,

$$\begin{aligned}
& (\mathcal{T}_\lambda(F * G)_\lambda)(y) \\
&= \left(\frac{\lambda}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} f\left(w_1 + \frac{1}{\sqrt{2}}(h_1, y)^\sim, \dots, w_n + \frac{1}{\sqrt{2}}(h_n, y)^\sim\right) \\
&\cdot \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n w_j^2\right\} \cdot g\left(r_1 + \frac{1}{\sqrt{2}}(h_1, y)^\sim, \dots, r_n + \frac{1}{\sqrt{2}}(h_n, y)^\sim\right) \\
&\cdot \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n r_j^2\right\} d\vec{w} d\vec{r} \\
&= \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f\left(w_1 + \frac{1}{\sqrt{2}}(h_1, y)^\sim, \dots, w_n + \frac{1}{\sqrt{2}}(h_n, y)^\sim\right) \\
&\cdot \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n w_j^2\right\} d\vec{w} \\
&\cdot \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} g\left(r_1 + \frac{1}{\sqrt{2}}(h_1, y)^\sim, \dots, r_n + \frac{1}{\sqrt{2}}(h_n, y)^\sim\right) \\
&\cdot \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n r_j^2\right\} d\vec{r} \\
&= \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{w}) \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n \left(w_j - \frac{1}{\sqrt{2}}(h_j, y)^\sim\right)^2\right\} d\vec{w} \\
&\cdot \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} g(\vec{r}) \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n \left(r_j - \frac{1}{\sqrt{2}}(h_j, y)^\sim\right)^2\right\} d\vec{r} \\
&= (\mathcal{T}_\lambda(F))\left(\frac{y}{\sqrt{2}}\right) \cdot (\mathcal{T}_\lambda(G))\left(\frac{y}{\sqrt{2}}\right).
\end{aligned}$$

Since both  $\mathcal{T}_\lambda(F)$  and  $\mathcal{T}_\lambda(G)$  have analytic extensions for  $\lambda$  throughout  $\mathbb{C}_+$ ,  $\mathcal{T}_\lambda(F * G)_\lambda$  has also an analytic extension for  $\lambda$  throughout  $\mathbb{C}_+$ . Therefore we conclude that the formula (4.3) holds for all  $z \in \mathbb{C}_+$  and  $s$ -a.e.  $y \in B$ .  $\square$

**THEOREM 4.3.** *The following statements are true for all  $z \in \mathbb{C}_+^\sim$ .*

- (i) *Let  $F \in \mathcal{F}(n; 1)$  and  $G \in \mathcal{F}(n; p)$ , where  $1 \leq p < \infty$ . Then  $(F * G)_z$  belongs to  $\mathcal{F}(n; p)$ .*



- (ii) Let  $F \in \mathcal{F}(n; p)$  and  $G \in \mathcal{F}(n; p')$ , where  $1 \leq p < \infty$ , and  $1/p + 1/p' = 1$ . Then  $(F * G)_z$  belongs to  $\mathcal{F}(n; \infty)$ .
- (iii) Let  $F \in \mathcal{F}(n; 1)$  and  $G \in \mathcal{F}(n; \infty)$ . Then  $(F * G)_z$  belongs to  $\mathcal{F}(n; \infty)$ .

*Proof.* (i) It suffices to show that  $H_z(\cdot)$  given by the formula (4.2) belongs to  $L_p(\mathbb{R}^n)$  for every  $z \in \mathbb{C}_+^\sim$ . With the assistance of the Minkowski's integral inequality [12; p.3], we have

$$\begin{aligned}
 & \int_{\mathbb{R}^n} |H_z(\vec{w})|^p d\vec{w} \\
 & \leq \left| \frac{z}{2\pi} \right|^{\frac{np}{2}} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left| f\left(\frac{\vec{w} + \vec{u}}{\sqrt{2}}\right) \cdot g\left(\frac{\vec{w} - \vec{u}}{\sqrt{2}}\right) \right| d\vec{u} \right\}^p d\vec{w} \\
 & = \left| \frac{z}{\pi} \right|^{\frac{np}{2}} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} |f(\vec{v}) \cdot g(\sqrt{2}\vec{w} - \vec{v})| d\vec{v} \right\}^p d\vec{w} \\
 & = \left| \frac{z}{\pi} \right|^{\frac{np}{2}} \left(\frac{1}{2}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} |f(\vec{v})| |g(\vec{w} - \vec{v})| d\vec{v} \right\}^p d\vec{w} \\
 & \leq \left| \frac{z}{\pi} \right|^{\frac{np}{2}} \left(\frac{1}{2}\right)^{\frac{n}{2}} \left[ \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} |g(\vec{w} - \vec{v})|^p d\vec{v} \right\}^{1/p} |f(\vec{v})| d\vec{v} \right]^p \\
 & = \left| \frac{z}{\pi} \right|^{\frac{np}{2}} \left(\frac{1}{2}\right)^{\frac{n}{2}} \|f\|_1^p \|g\|_p^p,
 \end{aligned}$$

where the fourth inequality follows from the Minkowski's integral inequality. Therefore we have

$$\|H_z(\cdot)\|_p \leq \left| \frac{z}{\pi} \right|^{\frac{n}{2}} \left(\frac{1}{2}\right)^{\frac{n}{2p}} \|f\|_1 \|g\|_p,$$

and so we have the desired result.

(ii) Using the Hölder's inequality, we have, for every  $\vec{w} \in \mathbb{R}^n$ ,

$$\begin{aligned}
 & |H_z(\vec{w})| \\
 & \leq \left| \frac{z}{2\pi} \right|^{\frac{n}{2}} \int_{\mathbb{R}^n} \left| f\left(\frac{\vec{w} + \vec{u}}{\sqrt{2}}\right) \cdot g\left(\frac{\vec{w} - \vec{u}}{\sqrt{2}}\right) \right| d\vec{u} \\
 & \leq \left| \frac{z}{2\pi} \right|^{\frac{n}{2}} \left\{ \int_{\mathbb{R}^n} \left| f\left(\frac{\vec{w} + \vec{u}}{\sqrt{2}}\right) \right|^p d\vec{u} \right\}^{1/p} \left\{ \int_{\mathbb{R}^n} \left| g\left(\frac{\vec{w} - \vec{u}}{\sqrt{2}}\right) \right|^{p'} d\vec{u} \right\}^{1/p'}
 \end{aligned}$$

$$= \left| \frac{z}{\pi} \right|^{\frac{n}{2}} \|f\|_p \|g\|_{p'}.$$

Thus  $H_z(\cdot)$  belongs to  $L_\infty(\mathbb{R}^n)$ . Next using a standard argument we can show that  $H_z(\cdot)$  belongs to  $C_o(\mathbb{R}^n)$ . Hence  $(F * G)_z$  belongs to  $\mathcal{F}(n; \infty)$ .

(iii) We can easily prove this statement.  $\square$

In our next theorem, we show that the Fourier-Feynman transform of the convolution product is the product of Fourier-Feynman transforms for each functional.

**THEOREM 4.4.** *Let  $F$  and  $G$  belong to  $\mathcal{F}(n; 1)$  and  $\mathcal{F}(n; p)$ , respectively, where  $1 \leq p \leq 2$ . Then the following formula (4.4) holds for all non-zero real number  $q$  and  $s$ -a.e.  $y \in B$  :*

$$(4.4) \quad (\mathcal{T}_q^{(p)}(F * G)_q)(y) = (\mathcal{T}_q^{(1)}(F))\left(\frac{y}{\sqrt{2}}\right) \cdot (\mathcal{T}_q^{(p)}(G))\left(\frac{y}{\sqrt{2}}\right).$$

*Proof.* With the assistance of Theorems 3.4, 3.5 and 4.3, we can verify that all of the Fourier-Feynman transforms in both sides of the formula (4.4) exist. Finally, using the formula (4.3), we conclude that the formula (4.4) is established for all non-zero real number  $q$  and  $s$ -a.e.  $y \in B$ .  $\square$

**REMARKS.**

- (a) Let  $C_o[0, T]$  be the Banach space of real-valued continuous functions  $x$  on the closed interval  $[0, T]$  which vanish at 0 with the uniform norm. Let  $(C_o[0, T], \mathcal{B}(C_o[0, T]), m_w)$  be the classical Wiener space, where  $m_w$  is the Wiener measure on the Borel  $\sigma$ -algebra  $\mathcal{B}(C_o[0, T])$ .

Put

$$H_o = \left\{ f \in C_o[0, T] : f = \int_0^t v(s) ds, v \in L_2[0, T], t \in [0, T] \right\}$$

and define an inner product  $\langle \cdot, \cdot \rangle$  on  $H_o$  as follows:

$$\langle f, g \rangle = \int_0^T (Df)(s) (Dg)(s) ds, \quad f, g \in H_o,$$

where  $Df = \frac{df}{ds}$ , the derivative of  $f$ .

Then  $H_o$  is a real separable Hilbert space, and the space  $(C_o[0, T], H_o, m_w)$  is a typical example of an abstract Wiener space ( see [11]).

Let  $\{h_j : j = 1, 2, 3, \dots\}$  be a complete orthonormal set in  $H_o$ . Then  $\{Dh_j : j = 1, 2, 3, \dots\}$  is also a complete orthonormal set in  $L_2[0, T]$ , and

$$(h_j, x)^\sim = \int_0^T (Dh_j)(s) \tilde{d}x(s), \quad \text{for } s - a.e. x \in C_o[0, T],$$

where  $\int_0^T (Dh_j)(s) \tilde{d}x(s)$  is the Paley-Wiener-Zygmund stochastic integral of  $Dh_j$  ( see [9]).

In this case, most results in [5] are corollaries of our results in Sections 3 and 4. For instance, put  $Dh_j = \alpha_j, j = 1, \dots, n$ . Then

$$(h_j, x)^\sim = \int_0^T \alpha_j(s) \tilde{d}x(s), \quad j = 1, \dots, n,$$

and every element  $F$  of  $\mathcal{F}(n; p)$  ( $1 \leq p \leq 2$ ) is expressed by

$$\begin{aligned} F(x) &= f((h_1, x)^\sim, \dots, (h_n, x)^\sim) \\ &= f\left(\int_0^T \alpha_1(s) \tilde{d}x(s), \dots, \int_0^T \alpha_n(s) \tilde{d}x(s)\right), \quad x \in C_o[0, T], \end{aligned}$$

where  $f$  belongs to  $L_p(\mathbb{R}^n)$ .

Using Theorems 3.4 and 3.5, we obtain

$$\begin{aligned} (\mathcal{T}_q^{(p)}(F))(x) &\approx (G_{-iq}f)((h_1, x)^\sim, \dots, (h_n, x)^\sim) \\ &= (G_{-iq}f)\left(\int_0^T \alpha_1(s) \tilde{d}x(s), \dots, \int_0^T \alpha_n(s) \tilde{d}x(s)\right) \\ &= \left(\frac{-iq}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{\frac{iq}{2} \sum_{j=1}^n \left(u_j - \int_0^T \alpha_j(s) \tilde{d}x(s)\right)^2\right\} d\vec{u}. \end{aligned}$$

This is the same result as in [5; Theorems 2.1 and 2.2].

- (b) Throughout this paper, for convenience' sake we assumed that  $\{h_1, \dots, h_n\}$  was an orthonormal set of elements in  $H$ . However, all of our results hold provided that  $\{h_1, \dots, h_n\}$  is a linearly independent set of elements in  $H$ .

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