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THE CONDITION NUMBER OF STIFFNESS MATRIX UNDER p-VERSION OF THE FEM

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Abstract. In this paper, we briefly study the condition number of stiffness matrix with h-version and analyze it with p-version of the finite element method.

1. Preliminaries

In this paper, we will consider the following elliptic partial differential equation of order 2.

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \subset R^2, \\
\Gamma u &= 0 \quad \text{on } \Gamma, \quad \text{where} \quad \Gamma u = u \quad \text{or} \quad \Gamma u = \frac{\partial u}{\partial n}
\end{aligned}
\tag{1}
$$

Let $\Omega \subset R^2$ and define Sobolev space

$$H^1(\Omega) = \{v \in L_2(\Omega) \,|\, \frac{\partial v}{\partial x_i} \in L_2(\Omega),\ i = 1, 2\}$$

Theorem 1.1. *Consider the following variational formulation.*

$$\textit{Find} \quad u \in H^1(\Omega) \quad \textit{such that} \quad a(u,v) = (f,v), \tag{2}$$

for any $v \in H^1(\Omega)$, where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$ and $(f,v) = \int_\Omega fv \, dx$. Then (1) and (2) are equivalent ,i.e., they have the same solution.

Proof. See Johnson [8]. □

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Consider a basis of $H^1(\Omega)$. In h-version of the finite element method, we can have the simplest usual basis $\{\varphi_i\}_{i=1}^M$, which is the basis for the finite dimensional space $V_F = \{v \in H^1(\Omega) : v|_K \in P_1(K), K \in T_h\}$. On the other hand, in p-version we get the basis $\{\varphi_i\}_{i=1}^M$ on the finite dimensional space $V_F = \{v \in H^1(\Omega) : v|_K \in P_p^{[T_p]}(\Omega), K \in T_p\}$, where $P_p^{[T_p]}(\Omega)$ consists of all functions in $H^1(\Omega)$ which are piecewise polynomials of degree at most p and T_p is a subdomain of Ω.

The notations $\{\varphi_i\}_{i=1}^M$ and V_F on the following statement are considered h or p-version case.

Now, we will derive the approximate solution $u_A = \sum_{i=1}^{M} a_i\varphi_i$ of the problem (1). In the finite dimensional space V_F, (2) becomes the problem:

$$\text{Find } u \in V_F \text{ such that } a(u,v) = (f,v) \quad \text{for any } v \in V_F. \tag{3}$$

But, if we choose $v = \varphi_i$ for $i = 1, \ldots, M$, then (3) becomes

$$\sum_{i=1}^{M} a_i\, a(\varphi_i, \varphi_j) = (f, \varphi_j), \quad j = 1, \ldots, M \tag{4}$$

which is a linear system of equations in M unknowns $a_1, \ldots, a_M$.

In the matrix form, the linear system (4) can be written as

$$A\mathbf{a} = \mathbf{b} \tag{5}$$

where $A = (a_{ij})$ is the $M \times M$ matrix with the element $a_{ij} = a(\varphi_i, \varphi_j)$, and $\mathbf{a} = (a_1, \ldots, a_M)$ and $\mathbf{b} = (b_1, \ldots, b_M)$ with $b_i = (f, \varphi_i)$.

2. The Order of the Condition Number

In this section we analyze the order of the condition number of the stiffness matrix for the h-version and p-version cases with the model problem given in the section 1.

2.1. *h*-version case. Let $\{\varphi_i\}_{i=1}^M$ be the usual basis for $V_h = \{v \in H_0^1(\Omega) : v|_K \in P_1(K), K \in T_h\}$,i.e., $\{\varphi_i\}_{i=1}^M$ are piecewise polynomials of degree 1. Let $a(v,u) = \int_\Omega \nabla v \cdot \nabla u\, dx$. And the stiffness matrix A is denoted by (a_{ij}), where $a_{ij} = a(\varphi_i, \varphi_j)$ for $i, j = 1, \ldots, M$. Then, we have following lemma.

Lemma 2.1. *Let* $v = \sum_{i=1}^{M} \eta_i \varphi_i \in V_h$ *and* $h = \max h_K$ *where* h_K *is the longest side of* K. *Suppose there are positive constants* β_1, β_2 *such that* $h_K \geq \beta_1 h$ *and* $\dfrac{\rho_K}{h_K} \geq \beta_2$. *Then, there are constants* c *and* C *only depending on the constants* β_1, β_2 *such that*

$$ch^2|\eta|^2 \leq \|v\|_{L_2}^2 \leq Ch^2|\eta|^2$$

and

$$a(v,v) \equiv \int_\Omega |\nabla v|^2\, dx \leq Ch^{-2}\|v\|_{L_2}^2.$$

Proof. See Johnson [8]. □

Theorem 2.2. *Let* $\chi(A)$ *be the condition number of* A. *Then*

$$\chi(A) = O(h^{-2}).$$

Proof. It follows from Lemma 2.1 and see Johnson [8]. □

Whatever the finite element spaces are, the above result is not different(See [1]). Moreover, if we take the elliptic problem of order $2k$, then we will have $\chi(A) = O(h^{-2k})$(See [8]).

2.2. p-version case. In this case, we will consider $\Omega \subset R^2$ like h-version case. Let $\{\psi_i\}_{i=1}^N$ be the basis of $P_p^{[T_p]}(\Omega)$, where $P_p^{[T_p]}(\Omega)$ is the space of all piecewise polynomials on Ω of degree at most p.

Lemma 2.3. *Suppose* $f(x)$ *is a polynomial of degree* p. *Then,*

$$\int_{-1}^{1} f'(x)^2\, dx \leq \frac{(p+1)^4}{2} \int_{-1}^{1} f(x)^2\, dx.$$

Proof. See Bellman [4]. □

By Lemma 2.3, we can prove the following;

Lemma 2.4. *Let* k *be an integer and* I *be an interval in* R. *Then, for any* $\psi_p \in P_p(I)$ *where* $P_p(I)$ *is the set of all polynomials on* I *of degree at most* p,

$$\|\psi_p\|_{H^k(I)} \leq C(k)p^{2k}\|\psi_p\|_{L_2(I)},$$

where C *is independent of* p *and* ψ_p.

Proof. Using Lemma 2.3, we get

$$\int_I \left[\frac{d\psi_p}{dx}\right]^2 dx \le \frac{(p+1)^4}{2}\int_I {\psi_p}^2\, dx \le \frac{(2p)^4}{2}\int_I {\psi_p}^2\, dx \le 8p^4\int_I {\psi_p}^2\, dx,$$

since $p \ge 1$. Therefore,

$$\|\psi_p\|^2_{H^1(I)} = \int_I \left[{\psi_p}^2 + \left(\frac{d\psi_p}{dx}\right)^2\right] dx \le Cp^4\int_I {\psi_p}^2\, dx.$$

Now, the mathematical induction completes the proof. □

We will state the 2-dimensional version of Lemma 2.4

LEMMA 2.5. *Let k be an integer. If S is any triangle, then for each $\psi_p \in P_p(S)$,*

$$\|\psi_p\|_{H^k(S)} \le C(k)p^{2k}\|\psi_p\|_{L_2(S)},$$

where C is independent of p and ψ_p.

Proof. For each $x_1 \in I$, it follows from Lemma 2.4 that

$$\|\psi_p(x_1,\cdot)\|^2_{H^k(I)} \le Cp^{4k}\,\|\psi_p(x_1,\cdot)\|^2_{L_2(I)}. \tag{6}$$

Similarly for $x_2 \in I$,

$$\|\psi_p(\cdot,x_2)\|^2_{H^k(I)} \le Cp^{4k}\,\|\psi_p(\cdot,x_2)\|^2_{L_2(I)}. \tag{7}$$

If we now integrate (6) and (7) with respect to x_1 and x_2 respectively, we get

$$\|\psi_p\|_{H^k(Q)} \le Cp^{2k}\,\|\psi_p\|_{L_2(Q)}, \quad \text{where } Q = I^2.$$

With an affine mapping, we can infer that Q is a parallelogram. Then, $S = \bigcup_{i=1}^{m} Q_i$ and thus,

$$\|\psi_p\|_{H^k(S)} \le \sum_{i=1}^{m}\|\psi_p\|_{H^k(Q_i)} \le Cp^{2k}\sum_{i=1}^{m}\|\psi_p\|_{L_2(Q_i)} \le Cp^{2k}\,\|\psi_p\|_{L_2(S)}.$$

□

Now we have similar result of the Lemma 2.1 for the p-versions.

THEOREM 2.6. *Let $v = \sum_{i=1}^{N} a_i\psi_i$, where $\{\psi_i\}$ is the basis. Then,*

$$a(v,v) \equiv \int_\Omega |\nabla v|^2\, dx \le Cp^4\,\|v\|^2.$$

Proof. Take $k = 1$ on Lemma 2.5. Then,

$$\|\psi_p\|_{H^1(S)} \le Cp^2 \|\psi_p\|_{L_2(S)}, \quad \text{where } S \text{ is a triangle.}$$

That is,

$$\|\psi_p\|_{H^1(S)}^2 = \int_S \left[{\psi_p}^2 + |\nabla\psi_p|^2\right] dx \le Cp^4 \int_S {\psi_p}^2 \, dx.$$

Thus,

$$\int_S |\nabla\psi_p|^2 \, dx \le Cp^4 \int_S {\psi_p}^2 \, dx.$$

Let $\{K_i\}$ be subdivision triangles of Ω. Then, with $C = \sup C_i$

$$\begin{aligned} a(v,v) &= \int_\Omega |\nabla v|^2 \, dx = \sum \int_{K_i} |\nabla(v|_{K_i})|^2 \, dx \\ &\le \sum C_i p^4 \int_{K_i} (v|_{K_i})^2 \, dx \le Cp^4 \int_\Omega v^2 dx \end{aligned}$$ □

The following theorem gives bounds of the finite element solution.

Theorem 2.7. *Let* $v = \sum_{i=1}^{N} a_i\psi_i$ *and* $\mathbf{a} = (a_1, \ldots, a_N)$. *Then,*

$$\|v\|^2 \le C_1|\mathbf{a}|^{\mathbf{2}}, \quad \text{where } C_1 \text{ is dependent of } \{\psi_i\}.$$

Proof. Let $M = \max_{1\le i\le N}\{\|\psi_i\|\}$. Then we have

$$\|v\|^2 \le \left(\sum_{i=1}^{N} |a_i|\|\psi_i\|\right)^2 \le \left(M\sum_{i=1}^{N} |a_i|\right)^2 = M^2\left(\sum_{i=1}^{N} |a_i|\right)^2.$$

By the Hölder's inequality, $\left(\sum_{i=1}^{N} |a_i|\right)^2 \le K \sum_{i=1}^{N} |a_i|^2$ with K is a constant. Thus,

$$\|v\|^2 \le M^2K\sum_{i=1}^{N} |a_i|^2.$$

If we denote $C_1 = M^2K$, then we have

$$\|v\|^2 \le C_1 \sum_{i=1}^{N} |a_i|^2 = C_1|\mathbf{a}|^{\mathbf{2}}.$$ □

THEOREM 2.8. *Let* $v = \sum_{i=1}^{N} a_i\psi_i$. *Then, there exists a constant* C_2 *such that*

$$C_2|\mathbf{a}|^{\mathbf{2}} \leq \|\mathbf{v}\|^{\mathbf{2}}.$$

Proof. Using that $\|a+b\|^2 + \|a-b\|^2 = 2\left(\|a\|^2 + \|b\|^2\right)$, we have

$$\left\|\sum_{i=1}^{N-1} a_i\psi_i + a_N\psi_N\right\|^2 + \left\|\sum_{i=1}^{N-1} a_i\psi_i - a_N\psi_N\right\|^2 = 2\left(\left\|\sum_{i=1}^{N-1} a_i\psi_i\right\|^2 + \|a_N\psi_N\|^2\right),$$

$$\begin{aligned}\left\|\sum_{i\neq N-1} a_i\psi_i + a_{N-1}\psi_{N-1}\right\|^2 + \left\|\sum_{i\neq N-1} a_i\psi_i - a_{N-1}\psi_{N-1}\right\|^2 \\ = 2\left(\left\|\sum_{i\neq N-1} a_i\psi_i\right\|^2 + \|a_{N-1}\psi_{N-1}\|^2\right),\end{aligned}$$

$$\vdots$$

$$\left\|\sum_{i\neq 1} a_i\psi_i + a_1\psi_1\right\|^2 + \left\|\sum_{i\neq 1} a_i\psi_i - a_1\psi_1\right\|^2 = 2\left(\left\|\sum_{i\neq 1} a_i\psi_i\right\|^2 + \|a_1\psi_1\|^2\right).$$

If we sum right and left hands of the above equalities respectively, we get

$$N\left\|\sum_{i=1}^{N} a_i\psi_i\right\|^2 + \sum_{k=1}^{N}\left\|\sum_{i\neq k} a_i\psi_i - a_k\psi_k\right\|^2 = 2\sum_{k=1}^{N}\left(\left\|\sum_{i\neq k} a_i\psi_i\right\|^2 + \|a_k\psi_k\|^2\right).$$

So, we see that

$$2\sum_{k=1}^{N}\|a_k\psi_k\|^2 \leq N\left\|\sum_{i=1}^{N} a_i\psi_i\right\|^2 + \sum_{k=1}^{N}\left\|\sum_{i\neq k} a_i\psi_i - a_k\psi_k\right\|^2.$$

We denote A by $N\|\sum_{i=1}^{N} a_i\psi_i\|^2$ for efficiency. Thus,

$$\begin{aligned}
2\sum_{k=1}^{N}\|a_k\psi_k\|^2 &\le A+\sum_{k=1}^{N}\left\|\sum_{i\neq k}a_i\psi_i - a_k\psi_k\right\|^2\\
&\le A+\sum_{k=1}^{N}\left(\sum_{i\neq k}\|a_i\psi_i\|+\|a_k\psi_k\|\right)^2\\
&= A+N\left(\sum_{i=1}^{N}\|a_i\psi_i\|\right)^2 = A+N\left(\sum_{i=1}^{N}|a_i|\|\psi_i\|\right)^2\\
&\le A+NM^2\left(\sum_{i=1}^{N}|a_i|\right)^2 \le A+NM^2K\sum_{i=1}^{N}|a_i|^2.
\end{aligned}$$

Here $M = \max_{1\le i\le N}\|\psi_i\|$ and the last inequality follows by the Hölder's inequality. Therefore, we have

$$2\sum_{k=1}^{N}\|a_k\psi_k\|^2 \le N\left\|\sum_{i=1}^{N}a_i\psi_i\right\|^2 + NM^2K\sum_{i=1}^{N}|a_i|^2.$$

So,

$$2\sum_{k=1}^{N}|a_k|^2\|\psi_k\|^2 \le N\left\|\sum_{i=1}^{N}a_i\psi_i\right\|^2 + NM^2K\sum_{i=1}^{N}|a_i|^2. \tag{8}$$

Using (8) with $L = \min_{1\le i\le N}(\|\psi_i\|)$, we have

$$2L^2\sum_{i=1}^{N}|a_i|^2 \le 2\sum_{k=1}^{N}|a_k|^2\|\psi_k\|^2 \le N\left\|\sum_{i=1}^{N}a_i\psi_i\right\|^2 + NM^2K\sum_{i=1}^{N}|a_i|^2.$$

Let $C_2 = (2L^2 - NM^2K)/N$. Then we get the following inequality which completes the proof.

$$C_2\sum_{i=1}^{N}|a_i|^2 \le \left\|\sum_{i=1}^{N}a_i\psi_i\right\|^2 = \|v\|^2.$$

□

Now, we will prove the main theorem. This theorem is parallel to Theorem 2.2, that is, the following theorem is a version of Theorem 2.2 for p-version case.

THEOREM 2.9. *Let* $\{\psi_i\}$ *be the basis of* $P_p^{[T_p]}(\Omega)$. *Let* A *be the stiffness matrix, i.e.,* $A = (a_{ij})$, *where* $a_{ij} = a(\psi_i, \psi_j) = \int_\Omega \nabla\psi_i \cdot \nabla\psi_j \, dx$. *And let* $\chi(A)$ *be the condition number of* A. *Then,*

$$\chi(A) = O(p^4).$$

Proof. Let $v = \sum_{i=1}^{N} a_i\psi_i$. Then, $a(v,v) = \mathbf{a} \cdot A\mathbf{a}$. Using Theorem 2.6 and Theorem 2.7, we have

$$\frac{\mathbf{a} \cdot A\mathbf{a}}{|\mathbf{a}|^2} = \frac{a(v,v)}{|\mathbf{a}|^2} \le Cp^4 \frac{\|v\|^2}{|\mathbf{a}|^2} \le Cp^4 \frac{C_1|\mathbf{a}|^2}{|\mathbf{a}|^2} = CC_1p^4, \tag{9}$$

On the other hand, by Theorem 2.8

$$\frac{\mathbf{a} \cdot A\mathbf{a}}{|\mathbf{a}|^2} = \frac{a(v,v)}{|\mathbf{a}|^2} \ge \alpha \frac{\|v\|^2}{|\mathbf{a}|^2} \ge \alpha \frac{C_2|\mathbf{a}|^2}{|\mathbf{a}|^2} = \alpha C_2. \tag{10}$$

But we know that

$$\lambda_{\max} = \max_{\mathbf{a}\neq\mathbf{0}} \frac{\mathbf{a} \cdot A\mathbf{a}}{|\mathbf{a}|^2} \quad \text{and} \quad \lambda_{\min} = \min_{\mathbf{a}\neq\mathbf{0}} \frac{\mathbf{a} \cdot A\mathbf{a}}{|\mathbf{a}|^2}.$$

Therefore, we get $\lambda_{\max} \le CC_1p^4$ and $\lambda_{\min} \ge \alpha C_2$ by (9) and (10). Hence

$$\chi(A) = \frac{\lambda_{\max}}{\lambda_{\min}} \le \frac{CC_1}{\alpha C_2} p^4,$$

where α, C_1, C_2 and C are constants. This completes the proof. □

In the case $(-\Delta u + u = f)$, the result and the proof are quite similar to those of Theorem 2.9. Roughly speaking, elliptic problem of order 4 will have the order p^8 and p-version has order twice h-version in the condition number.

3. Numerical Experiment

We consider the following problem:

$$\begin{cases} u''(x) = -x/\sqrt{1-x^2} & \text{for } x \in \Omega = (-1,1), \\ u(-1) = -\frac{\pi}{4}, \qquad u(1) = \frac{\pi}{4}. \end{cases}$$

TABLE 1. Eigenvalues and Condition Numbers

p	$\lambda_{\max}$	$\lambda_{\min}$	Condition Number
1	2.00000001	2.00000001	1.00000000
2	2.00000001	0.66666668	2.99999996
3	2.00000001	0.17777778	11.24999968
4	2.00000001	0.04571429	43.74999848
5	2.00000001	0.01160998	172.26562088

In this case, the bilinear product is $a(u,v) = \int_{-1}^{1} u'(x)v'(x)\,dx$. By the variational formulation, we seek a solution $u \in H^1(\Omega)$ which satisfies

$$a(u,v) = \int_{-1}^{1} u'(x)v'(x)\,dx = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} v(x)\,dx$$

for all $v \in H^1(\Omega)$. To get the approximate solution of the finite element method, we ought to take some basis functions. We choose as basis functions $\psi_0(x) = 1 - x$, $\psi_1(x) = x$ and

$$\psi_{k+1}(x) = \int_{-1}^{x} P_k(t)\,dt \qquad \text{for } k \geq 1,$$

where $P_k(t)$ is the monic Legendre polynomial of degree k. In this paper we analyze the condition number of stiffness matrix and so we will find it in place of the approximate solution.

The procedure is programmed using Turbo C, including the Gaussian quadrature (See [5]) with $n = 5$ for numerical integration and the Jacobi's method for finding eigenvalues of the stiffness matrix.

The result is shown in Table 1 with maximum degree of basis functions $p = 1, 2, 3, 4, 5$. The maximum, minimum eigenvalues and the condition numbers of stiffness matrices are given in Table 1. The result has coincidence with Theorem 2.9. As the degree p grows, the condition number is larger rapidly.

Since we use the Gaussian quadrature with $n = 5$, the approximation of integration is exact if the integrand has degree at most 9 except round-off error. Thus the approximate values of integrations are quite exact in this case because the integrands are polynomials with degree from 0 to 8. Also to deal with a non-singular stiffness matrix we take $p \times p$ matrix because of $\psi_0 = 1$ while one gets generally $(p+1) \times (p+1)$ matrix as the stiffness matrix.

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