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(s, S) 재고정책하에서 단일제품의 확률적 Block Stacking 저장모형의 최적화*

Optimization of a Block Stacking Storage Model for a Single Product using (s, S) Inventory Policy*

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Abstract

Block stacking, which involves the storage of unit loads in stacks within storage rows, is typically used in traditional warehouses to achieve a high space utilization at a low investment cost.

In this paper, assuming that the demand size from a customer is an i.i.d. random variable, we develop a probabilistic block stacking storage model and its algorithm for a single product, which minimizes the time-average floor space requirement under an (s, S) inventory policy and the violation of the FIFO lot rotation rule only in a single partially-occupied row.

1. Introduction

Due to many factors including space cost and operating cost in warehousing, automated storage/retrieval(AS/R) systems have been dominating continuously over traditional warehouses. However the recent proportion of traditional warehouses is still higher than that of AS/R systems in the world. Block stacking, which involves the storage of unit loads in stacks within storage rows, is typically used in traditional warehouses to achieve a high space utilization at a low investment cost [4].

The design of a block stacking storage system is characterized by: the depth of the storage row, the number of storage rows required for a given product lot, and the height of the stack. For a single product, factors that may influence the optimum row depth include lot size, load dimensions, aisle widths, row clearance, allowable stacking heights, storage/retrieval times, and storage/retrieval distribution [4].

The research on the field of block stacking storage systems has been very inactive. Matson [3], one of the representative researcher in this field, developed various deterministic blocking stacking storage models for a single and multiple products in order to minimize expected floor space requirement assuming different inventory policies. However Matson did not suggest a deterministic model

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based on (s, S) inventory policy due to the mathematical complexity of deterministic approach. In this paper, we cevelop a basic probabilistic block stacking storage model with a slightly modified (s, S) inventory policy and suggest an algorithm which minimizing time-average floor space requirement. For convenience, we summarize the notations used in this paper in Table 4 in advance.

Table 4. Summary of notations used.

Notation	explanation
A_n	n-th renewal epoch defined as $\sum_{k=1}^{n} D_k$, $A_0=0$
A	width of aisle
c	clearance between rows
D_k	the demand in the k-th day
F(d)	Prob{$D_k \leq d$}
H	height of unit load
H_c	height of ceiling in a warehouse
I^0	{0,1,2,...}
I^+	{1,2,3,...}
I_k	inventory level at the start of day k
L	Length of unit load
N(a)	sup{n:$A_n \leq$a}
$N_L(t)$	number of rows required for storage at time t
M(a)	E[N(a)]
R(t)	floor space requirement at time t
s	safety stock level
S	maximum inventory on hand
S_{inv}	{s,(s+1),(s+2),...,S}
$R_L(x)$	floor space required for a row including half of the aisle floor space in front of a row
T_r	r-th regeneration epoch
U_j	a random variable representing the sojourn time of inventory level j
v	minimum number of rows required during the inventory cycle
W	width of unit load
x	row depth in unit load
y	maximum number of rows required during the inventory cycle
z	number of storage tiers or levels in unit load

2. A Probabilistic Block Stacking Storage Model

2.1 Limiting Probability of an (s, S) inventory policy

Space requirement for a blocking stacking storage system in a warehouse depends on various inventory policies. For two positive integer parameters s and S (s $\leq$S), consider an (s, S) inventory policy for replenishing inventory in a warehouse; A replenishment decision is made based on the current inventory observed at the start of each day. If the inventory position is at least s, then no action is taken; If the inventory is less than s, say j, than the amount (S-j) is ordered so that the amount on hand plus on order equals S. We assume that the additional stock which is ordered is delivered virtually immediately. It follows that the set of inventory on hand, S_{inv}, becomes {s,(s+1),⋯,S}. In addition, we assume that all of the demanded items requested during a day are immediately retrieved at the time of each request and begin to be delivered simultaneously from the same place at the same time since deliveries are restricted to a particular time band, for example, from 11 p.m. to 5 a.m. of the next day.

Let D_k be i.i.d. nonnegative random variables with distribution function F(·), representing the demand in the k-th day where k$\in I^+$ = {1,2,3,⋯}. Define $A_n = \sum_{k=1}^{n} D_k$ and for a$\geq$0 N(a)=sup {n:$A_n \leq$a}. Then {N(a),a$\geq$0} is a renewal counting process with renewal epochs, {A_n,n$\in I^0$} where A_0=0 and I^0 is {0,1,2,⋯}. Note that we do not count a renewal at A_0, i.e., N(a)=0 for 0$\leq$a <A_1. Figure 1 shows a sample path of {N(a),a$\geq$0}.

Now determine the distribution of the number of items in inventory. Without loss of generality the inventory initially contains S items and let I_k be the inventory level at the start of day k, i.e., I_k=S-A_k for k$\in I^0$. Then {I_k,k$\in I^0$} is regenerative with state space, S_{inv}, since there exists the first regeneration epoch, T_1, such that T_1= N(S-s)+1 and the continuation of {I_k,k$\in I^0$} beyond T_1 is a probabilistic replica of the process beginning at time

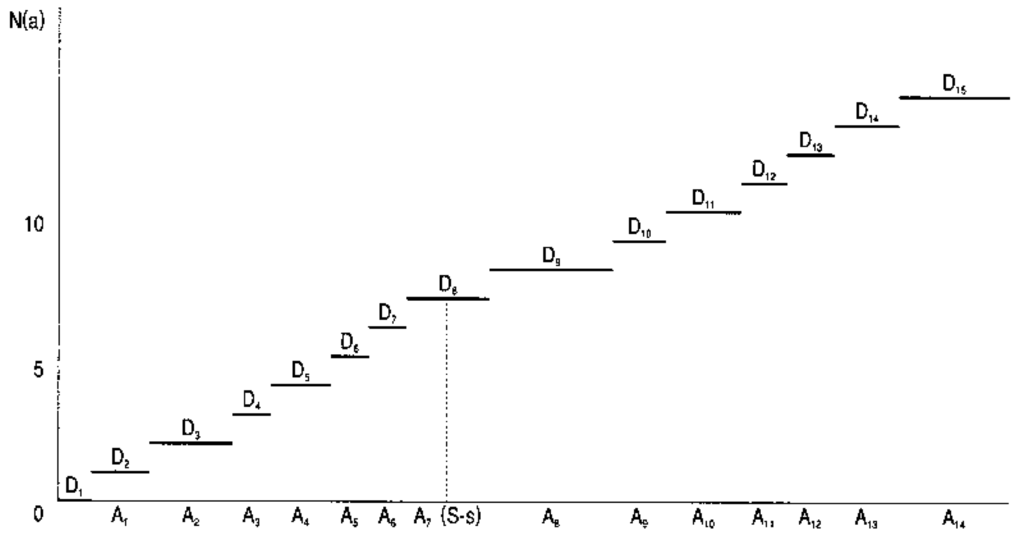

Figure 1. A sample path of $\{N(a), a \geq 0\}$.

zero as shown in Figure 2. Let the set of regeneration epochs be $\{T_r, r \in I^0\}$, where $T_{r+1} > T_r$ for $r \in I^+$ and $T_0=0$.

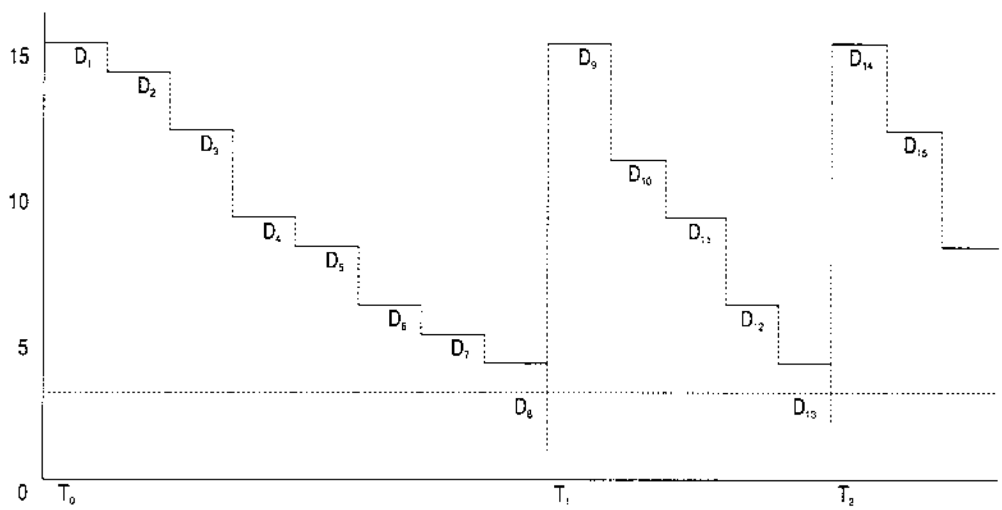

Figure 2. A sample path of $\{I_k, k \in I^0\}$ with S=15 and s=3.

Assume that $E(T_{r+1}-T_r) < \infty$ for $r \in I^0$, and that $P\{T_1 \leq t\}$ has a density on some interval, then there exists a limiting distribution of I_k according to the existence theorem[2].

Let p_j be $\lim_{k \to \infty} \text{Prob}\{I_k=j\}$ and define M(a) to be a renewal function, E[N(a)]. By renewal argument, the renewal equation can be obtained as

$$M(a) = F(a) + \sum_{j=0}^{a} M(a-j) f_j \quad \text{for } a \in I^0$$

where f_j=F(j)-F(j-1) for $a \in I^+$ and f_0=F(0). For the easy computation of M(a), replacing a with zero results in

$$M(0) = \frac{f_0}{(1 - f_0)} \text{ and}$$

$$M(a) = \frac{F(a) + \sum_{j=1}^{a} M(a-j)f_j}{(1 - f_0)} \quad \text{for} \quad a \in I^+ \qquad (1)$$

From (1), M(a) for $a \in I^+$ can be computed in turn since the right hand side of (1) involves only M(0), M(1), ... , M(a-1).

For $j \in S_{inv}$, let U_j be a random variable representing the sojourn time of inventory level j during $[T_0, T_1]$. Since the number of days before the inventory drops below a is N(S-a)+1 for $s \le a \le S$ during $[T_0, T_1]$, and U_j can be considered as the duration, the number of days before the inventory drops below j minus the number of days before the inventory drops below j+1, we have

$$U_j = \begin{cases} N(0)+1 & \text{for } j=S \\ N(S-j)-N(S-j-1) & \text{for } s \le j \le (S-1) \text{ and} \end{cases}$$

$$E(U_j) = \begin{cases} M(0)+1 & \text{for } j=S \\ M(S-j)-M(S-j-1) & \text{for } s \le j \le (S-1) \end{cases} \qquad (2)$$

It follows that

$$p_j = \lim_{k \to \infty} \text{Prob}\{I_k = j\} = \begin{cases} \frac{M(0)+1}{M(S-s)+1} & \text{if } j=S \\ \frac{M(S-j)-M(S-j-1)}{M(S-s)+1} & \text{if } s \le j \le (S-1) \\ 0 & \text{otherwise} \end{cases} \qquad (3)$$

where $\sum_{j \in S_{inv}} p_j = 1$. Table 1 summarizes (3).

2.2 Block Stacking Storage Requirement based on our inventory policy

During the storage and retrieval cycle of a product lot in a block stacking storage system, vacancies can occur in a storage row. To achieve first-in, first-out (FIFO) lot rotation, these vacant storage positions cannot be used for storage of other products or lots until all loads have been withdrawn from the row [4]. However, in order to save floor space, we assume FIFO lot rotation rule except for some safety stock in a single partially-occupied row.

For example, suppose that safety stock occupies three rows, two rows fully occupied and one row partially

Table 1. The limiting probability of $\{I_k, k \in I^0\}$.

Inventory, j	Number of rows required, m	Limiting probability, p_j
S	$\lceil \frac{S}{xz} \rceil$	$\frac{\{M(0)+1\}}{\{M(S-s)+1\}}$
S-1	$\lceil \frac{(S-1)}{xz} \rceil$	$\frac{\{M(1)-M(0)\}}{\{M(S-s)+1\}}$
S-2	$\lceil \frac{(S-2)}{xz} \rceil$	$\frac{\{M(2)-M(1)\}}{\{M(S-s)+1\}}$
⋮	⋮	⋮
S-j	$\lceil \frac{(S-j)}{xz} \rceil$	$\frac{\{M(j)-M(j-1)\}}{\{M(S-s)+1\}}$
⋮	⋮	⋮
s+1	$\lceil \frac{(s+1)}{xz} \rceil$	$\frac{\{M(S-s-1)-M(S-s-2)\}}{\{M(S-s)+1\}}$
s	$\lceil \frac{s}{xz} \rceil$	$\frac{\{M(S-s)-M(S-s-1)\}}{\{M(S-s)+1\}}$

occupied. When a replenishment lot arrives, we assume that some loads of the replenishment lot are stacked in the partially-occupied row first of all and other remaining loads will be stored in the other available empty rows. Hence the safety stock in the partially occupied row do not follow the FIFO rule any longer.

Let $R_L(x)$ be the floor space requirement for a row with a row depth being x in unit load, which is the grayed area as shown in Figure 3. Assume that half of the aisle floor space in front of a row belongs to the floor space requirement for a row. Then $R_L(x) = (W+c)(0.5A+xL)$ where W is the width of unit load, c is the clearance between rows, A is the width of aisle, L is the length of unit load.

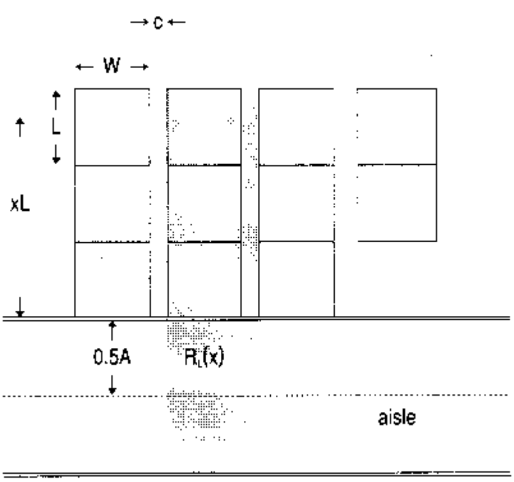

Figure 3. A blocking stacking storage with x=3 and y=4.

Let R(t) be the floor space requirement at time t and let $N_L(t)$ be the number rows required at time t. Since $R(t)=R_L(x)N_L(t)$ and $N_L(t)$ depends on inventory level, say j, $R(t)=R_L(x)\lceil \frac{j}{xz} \rceil$ where z is the number of storage tiers or levels and $\lceil \xi \rceil$ denotes the smallest integer greater than or equal to xi. Since each of $\{R(t), t\geq 0\}$ and $\{N_L(t), t\geq 0\}$ is clearly a regenerative process, the expectation of R(t) in the long run, E(R), will be

$$E(R) = \lim_{t\to\infty} E(R(t))$$
$$= R_L(x)\lim_{t\to\infty} E[N_L(t)]$$
$$= R_L(x)E(N_L) \quad (4)$$
$$= R_L(x)\sum_{j=s}^{S}\lceil \frac{j}{xz} \rceil p_j \quad (5)$$

2.3 Problem Definition of a Probabilistic Block Stacking Storage Model and Algorithm

Now the depth of the storage row can be obtained by minimizing the time-average floor space requirement as follows; for integers x and y,

$$\text{BSP: Minimize } E(R) = R_L(x)\sum_{j=s}^{S}\lceil \frac{j}{xz} \rceil p_j$$
$$\text{subject to} \quad xy \geq \lceil \frac{S}{z} \rceil$$
$$z \leq \lfloor \frac{H_c}{H} \rfloor$$

where H is the height of the unit load, H_c is the maximum stacking height., $\lfloor \xi \rfloor$ denotes the largest integer less than or equal to ξ.

Since the number of practical alternatives is small, enumeration over x is the natural method for solving this problem. Our algorithm can be described as follows;

ALG[BSP]:

Step 1. $z \leftarrow \lfloor \frac{H_c}{H} \rfloor$
$x_{MAX} \leftarrow \lceil \frac{S}{z} \rceil$
$R_{MIN} \leftarrow$ big value
$V_{MIN} \leftarrow$ big value

Step 2. For x from 1 to x_{MAX}
Begin
Compute E(R) and V(R)
If $E(R)=R_{MIN}$ and $V(R) < V_{MIN}$, then $V_{MIN} \leftarrow V(R)$ and $x^* \leftarrow x$
If $E(R) < R_{MIN}$, then $R_{MIN} \leftarrow E(R)$
$x^* \leftarrow x$
$V_{MIN} \leftarrow V(R)$
End

2.4 Example

Consider the daily demands forming a sequence of Bernoulli random variables with $f_0=1-p$ and $f_1=p$, i.e., $f_k=p^k(1-p)^{(1-k)}$ for k=0 or 1, $0\le p\le 1$. From (1), M(a) can be recursively obtained as,

$$M(a)=\frac{(a+1-p)}{p},\ M(0)=\frac{(1-p)}{p}$$

Since $M(S-s)+1=\frac{(S-s+1)}{p}$, and $E(U_j)=\frac{1}{p}$ from (2), p_j can be computed from (3) as

$$p_j=\frac{1}{(S-s+1)}\quad \text{for } j\in S_{inv}.$$

Consider (4). Let v and y be the minimum and maximum number of rows required during the inventory cycle respectively. Since the maximum number of items per row is xz, v and y will be $\ulcorner \frac{s}{xz} \urcorner$ and $\ulcorner \frac{S}{xz} \urcorner$ respectively. For a given x value, the expected sojourn time of row number m can be obtained as $\frac{(vxz-s+1)}{p}$ for m=v, $\frac{xz}{p}$ for m=(v+1),..., (y-1), and $\frac{\{S-xz(y-1)\}}{p}$ for m=y from (2). Thus as shown in Table 2, we have

$$\text{Prob}\{N_L=m\}=\begin{cases}\dfrac{vxz-s+1}{(S-s+1)} & \text{for } m=v\\ \dfrac{xz}{(S-s+1)} & \text{for } m=(v+1),\ \dots,\ (y-1)\\ \dfrac{\{S-xz(y-1)\}}{(S-s+1)} & \text{for } m=y\end{cases}$$

$$\begin{aligned}E(R) &= R_L(x)E(R_L)\\ &= R_L(x)\left[\frac{y\{S-xz(y-1)\}}{(S-s+1)}+\frac{xz}{(S-s+1)}\sum_{j=v+1}^{y-1} j+\frac{v(vxz-s+1)}{(S-s+1)}\right]\\ &= R_L(x)\left[\frac{\{2yS-xz(y-v)(y+v-1)-2v(s-1)\}}{2(S-s+1)}\right]\end{aligned}$$

$$\text{BSPEX: Minimize } E(R)=R_L(x)\left[\frac{\{2yS-xz(y-v)(y+v-1)-2v(s-1)\}}{2(S-s+1)}\right]$$

$$\text{subject to}\quad xy\ge \ulcorner \frac{S}{z} \urcorner$$

$$z\le \llcorner \frac{H_c}{H} \lrcorner$$

Suppose that S=30 unit loads, s=5 unit loads, L=50′, W=40′, c=3′, A=144′, and z=3. We enumerate over x from 1 to $x_{MAX}=9$ as shown in Table 3 which summarized p_j given each value of x and other statistics. Given x, we compute $\text{Prob}\{N_L=m\}$ for $v\le m\le y$. For example, given x=3, since $1\le m\le 4$, we need one row when $5\le j\le 9$, two rows when $10\le j\le 18$, three rows when $19\le j\le 27$, and four rows when $28\le j\le 30$. Since $p_j=\frac{1}{26}$ for $j\in\{5,6,7,\cdots,30\}$, $\text{Prob}\{N_L=m\}$ will be 0.1923, 0.3462, 0.3462, and 0.1154 respectively for m=1,2,3,4.

Table 2. The limiting probability of $\{N_L(t),\ t\ge 0\}$ in the Example.

Inventory range	Number of rows required, m	Limiting probability, $\text{Prob}\{N_L=m\}$
$xz(y-1) < j\le S$	$y=\ulcorner \frac{S}{xz} \urcorner$	$\frac{\{S-xz(y-1)\}}{(S-s+1)}$
$xz(y-2) < j\le xz(y-1)$	(y-1)	$\frac{xz}{(S-s+1)}$
$xz(y-3) < j\le xz(y-2)$	(y-2)	$\frac{xz}{(S-s+1)}$
⋮	⋮	⋮
$xzv < j\le xz(v+1)$	(v+1)	$\frac{xz}{(S-s+1)}$
$(s-1) < j\le xzv$	$v=\ulcorner \frac{s}{xz} \urcorner$	$\frac{(vxz-s+1)}{(S-s+1)}$

Table 3. p_i for Example and other statistics given x=1,2,...,9,10.

Number of rows \ x	1	2	3	4	5*	6	7	8	9	10
1	-	0.0769	0.1923	0.3077	0.4231	0.5385	0.6538	0.7692	0.8846	1.0000
2	0.0769	0.2308	0.3462	0.4615	0.5769	0.4615	0.3462	0.2308	0.1154	
3	0.1154	0.2308	0.3462	0.2308	-	-	-	-	-	
4	0.1154	0.2308	0.1154	-	-	-	-	-	-	
5	0.1154	0.2308	-	-	-	-	-	-	-	
6	0.1154	-	-	-	-	-	-	-	-	
7	0.1154	-	-	-	-	-	-	-	-	
8	0.1154	-	-	-	-	-	-	-	-	
9	0.1154	-	-	-	-	-	-	-	-	
10	0.1154	-	-	-	-	-	-	-	-	
$R_L(x)$ (ft^2)	5,246	7,396	9,546	11,696	13,846	15,996	18,146	20,296	22,446	24,596
$E(N_R)$	6.15	3.31	2.38	1.92	1.58	1.46	1.35	1.23	1.12	1.00
$E(R)$ (ft^2)	32,283	24,464	22,764	22,492	21,834	23,379	24,427	24,980	25,036	24,596
$V(R)$ (10^7 ft^2)	17.3	8.7	7.8	7.3	4.7	6.4	7.5	7.3	5.1	0.0

From (4), $E(N_L)$=2.38 rows. The optimal row depth is x^*= 5, which gives the minimum floor space requirement in the long run as E(R)=21,834 ft^2.

3. Conclusion

We developed a probabilistic block stacking storage model for a single product by assuming that demand size is a random variable under an (s, S) inventory policy and the violation of the FIFO lot rotation rule only in a single partially-occupied row. In addition, we suggested a methodology for determining an optimal row depth by minimizing the time-average floor space requirement during an inventory cycle. In addition, our model could be a starting basis for developing more elaborated probabilistic models.

Our probabilistic model is realistic in one sense that it is based on probabilistic demand assumption and an (s, S) inventory policy. It is, however, unrealistic in the other sense that the additional stock which is ordered is delivered virtually immediately. If we relax this assumption, p_i could be solved using a semi-Markov chain and this could be one of further researches. For multiple products case, other decision variables must be considered including the optimum unique row depths, the assignments of products to depths, and aggregate space requirements, etc. The related interaction of factors above is mathematically so complicated that this could be another one of further researches in the future.

Reference

[1] DeMars, N. A., J. O. Matson, and J. A. White. "Optimizing Storage System Selection", in Proceed-

ings of the 4th International Conference on Automation in Warehousing, Tokyo, Japan, October 1982
[2] Heyman, D. P., and J. S. Matthew, *Stochastic Models in Operations Research*, Volume I, McGraw-Hill Book Inc., 1982
[3] Matson, J. O., *The Analysis of Selected Unit Load Storage Systems*, Unpublished Doctoral thesis, Georgia Institute of Technology, September 1982
[4] Tompkins, J. A. and J. A. White, *Facilities Planning*, John Wiley & Sons Inc, 1984

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