State Transformations for Regenerative Sampling in Simulation Experiments

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〈Abstract〉

The randomness of the input variables in simulation experiments produce output responses which are also realizations of random variables. The random responses make necessary the use of statistical inferences to adequately describe the stochastic nature of the output. The analysis of the simulation output of non-terminating simulations is frequently complicated by the autocorrelation of the output data and the effect of the initial conditions that produces biased estimates.

The regenerative method has been developed to deal with some of the problems created by the random nature of the simulation experiments. It provides a simple solution to some tactical problems and can produce valid statistical results. However, not all processes can be modeled using the regenerative method. Other processes modeled as regenerative may not return to a given demarcating state frequently enough to allow for adequate statistical analysis.

This paper shows how the state transformation concept was successfully used in a queueing model and a job shop model. Although the first example can be analyzed using the regenerative method, it has the problem of too few recurrences under certain conditions. The second model has the problem of no recurrences. In both cases, the state transformation increase the frequency of the demarcating state. It was shown that time state transformations are regenerative and produce more cycles than the best typical discrete demarcating state in a given run length.

1. Introduction

Simulation is a technique often used for the study of large and/or complex systems. Simulation modeling and analysis is usually used when systems cannot be adequately studied analytically. The effects of some variables of the simulation model are generated randomly from statistical frequency distributions. The randomness of the input variables produce output responses which are also realizations of random variables. The random responses make necessary the use of statistical inferences－such as confidence intervals－to adequately describe the stochastic nature of the output. The analysis of the simulation output of non-terminating simulations is frequently complicated by the autocorrelation of the output data, as well as the effect of the initial conditions that produces biased estimates. Some tactical issues that the simulationist has to address are how to start the simulation, when to begin collecting data, how long to run the simulation and how to deal with the highly correlated output.

2. The Regenerative Method

The regenerative method has been developed to deal with some of the problems created by the random nature of the

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simulation experiments. It provides a simple solution to some tactical problems and can produce valid statistical results. However, not all processes can be modeled using the regenerative method. Other processes modeled as regenerative may not return to a given demarcating state frequently enough for adequate statistical analysis (Crane & Iglehart 1974).

2.1 Regenerative Processes in Discrete Time

A sequence $\{X_n, n \geq 1\}$ of random vectors in k dimensions is a regenerative process if there is an increasing sequence $1 \leq \alpha_1 < \alpha_2 < \ldots$ of random discrete times, called regeneration epochs, such that at each of these epochs the process starts afresh probabilistically according to the same probabilistic structure governing it at epoch α_1.

Let S_d represent the state of the system at time T_d. Suppose that the system starts at time t_d and let T_k denote the time at which the k^{th} return to state S_d occurs. We define the k^{th} epoch as the period between returns k-1 and k to state S_d (Fishman 1978). If the state behavior in each of the k epochs is independent from the others and each obeys the same probability law, then we say that the process is regenerative. Given a sequence of regeneration times, $\{T_1, T_2, \ldots, T_k\}$ with observed sums of a performance measure $\{Z_1, Z_2, \ldots, Z_k\}$ and number of observations $\{N_1, N_2, \ldots, N_k\}$, mathematically,

$$Z_i = \sum_{j=1}^{N_i} D_{ij} \qquad (1)$$

where D_{ij} is the observation for the performance measure of the j^{th} occurrence in the i^{th} cycle.

Further, the sequence $\{Z_1, Z_2, \ldots, Z_k\}$ of k, i.i.d. random variables is called a renewal process. Regenerative processes can thus be analyzed through the application of classical statistical analyses (Law and Kelton 1982).

2.2 Regenerative Processes in Continuous Time

A regenerative process $\{X(t), t \geq 0\}$ in m dimensions is a stochastic process which starts afresh probabilistically at each element of an increasing sequence $0 < \beta_1 < \beta_2$ of random epochs on the time axis $(0, \infty)$. Thus, between any two consecutive regeneration epochs β_j and β_{j+1}, the time spent in state X(t), $\{X(t), \beta_j \leq t < \beta_{j+1}\}$, is an i.i.d. replicate of the time between any other two consecutive regeneration epochs (Crane and Lemoine 1977).

Let $\tau_j = \beta_{j+1} - \beta_j$, $j \geq 1$. Then the sequence $\{\tau_j, j \geq 1\}$ represents the time spans between consecutive epochs of regeneration and is a sequence of i.i.d. random variable.

Just as with regenerative processes in discrete time, regenerative processes in continuous time also have limiting or steady-state distributions. Let $f: \Re^k \rightarrow \Re$ (a function in k dimensions taking real values) and suppose the goal of the simulation is to estimate the value of $r \equiv E\{f(x)\}$. For $j \geq 1$, let

$$Z_j = \int_{\beta_j}^{\beta_{j+1}} f(x(t))dt$$

that is, Z_j is the integral of f(x(t)) over the j^{th} regeneration cycle. The following results are analogous to the discrete case. The sequence $\{(Z_j, \tau_j), j \geq 1\}$ consists of i.i.d. random vectors. If $E\{|f(x)|\} < \infty$, then, $r \equiv E(f(x)) = E(Z_1)/E(\tau_1)$. If we denote by $t_1, t_2, \ldots, t_m$, the time points in the interval β_j to β_{j+1} where the process has a change in state and we let $t_0 = \beta_j$ and $t_{m+1} = \beta_{j+1}$, then,

$$Z_j = \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} f(x(t))dt$$

2.3 Demarcating State Transformation as an Alternative

The regenerative method, as used in stochastic simulation, allows for data collection at the entry times to a single recurrent state of the process of interest. In order to implement the regenerative method a specific state to which the system returns has to be chosen. The system has to return to this state enough times during the simulation to yield good estimates. An example of such a state is empty and idle. Every time the system reaches the empty and idle state the system is known to restart afresh probabilistically. The main problem is that very few systems return to the empty and idle state frequently. Therefore, it is desirable to have other demarcating states that yield better statistical results.

The concept of state space transformations can also be applied to simulations in which the behavior of the existing recurrent states is already acceptable. Here, a transformed state can be used to produce more regeneration cycles in a fixed run length. This transformed demarcating state, that can be derived as a function of an actual demarcating state of the system, is a time lapse. The time lapse is the discrete or continuous time spent in a specific state.

Furthermore, few systems can be modeled as regeneration processes because they do not return to specific states. Therefore, the system states have to be approximated. Crane and Iglehart(1975) proposed four approximations to the regenerative method that allow the analyst to deal with this problem.

This paper shows how the state transformation concept can be successfully used. Although two of the models can be analyzed using the regenerative method, the problem is that it yields too few or zero return to the regeneration state under certain conditions. The state transformation used increases recurrence of the demarcating state. For all models it was shown that time state transformations are regenerative and produce more cycles than the best typical discrete demarcating state in a given run length. Therefore, it is an alternative procedure which can produce substantial savings in the simulation of stochastic processes.

3. Relevant Studies

In his theory of recurrent events, Feller (1950) studied stochastic processes, X_n, in discrete time in which a certain type of regenerative event is found. This event, R, is characterized by the property that if it is known that R happens at $t = t_0$, then further knowledge of X for all $t \leq t_0$ has no predictive value. Feller restated the analytic content of the convolution equations which served as a basis for the theory of recurrent events.

Smith (1955) expanded on Feller's work and developed a theory of regenerative processes applicable to continuous time stochastic processes. Smith (1958) presented a theoretical description of regenerative stochastic processes. Miller (1972) complemented Feller's work regarding non-lattice distributions. Crane and Iglehart (1974) introduced renewal theory in the analysis of simulation of stochastic systems in the steady-state. They presented regeneration as a step toward avoiding the problem of statistical dependence and the effects of initial conditions in simulations of stable stochastic systems. In their second paper Crane and Iglehart (1974) demonstrated that the choice of the starting state of regenerative simulations has no effect on the expected length of the confidence interval. In their third paper, Crane and Iglehart (1975) extended the regenerative method to discrete event simulations, characterized as a sample path realization of two-vector stochastic processes $\{T(t): t \geq 0\}$ and $\{Z(t): t \geq 0\}$ that change state at a finite number of event times $0 \leq t_1 < t_2 < \ldots$ generated in the course of the simulation. Crane and Iglehart (1975) presented four approximation techniques for obtaining confidence intervals when the simulation does not contain the required renewal process.

Gunther and Wolff (1980) presented the almost regenerative method applicable to processes not having the regenerative property. Iglehart (1975) evaluated a variety of point and interval estimators which can be used in conjunction with the regenerative method.

Iglehart (1976) and Seila (1976) worked on the estimation of quantiles of the stationary distribution of regeneration processes. Seila (1982) used the batch quantile method in which a simulation run is divided in batches and the batches in cycles.

Fishman (1977) and Lavenberg and Sauer (1977) dealt with sequential stopping rules for regenerative simulations. Fishman used the Chow-Robbins sequential estimation procedure and the Shapiro-Wilk test for the normality test.

Heildelberger and Lewis (1981) exploited two aspects of the regenerative structure of the simulated process combining them in a graphical picture of the bias structure that can be obtained. Meketon and Heildelberger (1982) derived a new point estimate which they claim reduces the bias. Lavenberg, Moeller and Saucer (1979) investigated using multiple concomitant control variables to reduce the width of confidence intervals when estimating steady-state response variables via the regenerative method.

4. Methodology

4.1 Development of State Transformations

The use of transformations as a tool to help in the design and analysis of stochastic problems had been accepted in the fields of statistical analysis and experimental design for a long time (Box, et al. 1978). This paper is based on an extension of this concept to the regenerative method in simulation (Crane & Iglehart 1975).

1) Complete State-Space Discretization

Since the problem is that the input random variables are continuous and therefore the waiting times are also continuous, one alternative is to modify the generated interarrivals and service times in such a way that they are discrete distributions. Using this approximations the interarrival and service times could only take on values $\{0, \delta, 2\delta, 3\delta, ...\}$ for some $\delta > 0$, δ represents the size of the discrete interval. This will cause the waiting times to be restricted to take on values $\{0, \delta, 2\delta, 3\delta, ...\}$. This should result in enough regeneration points.

The choice of δ will affect the accuracy of the approximation and the estimates. The smaller the value for δ, the better the approximation will resemble the original process. However, if the values for δ are too small too few cycles will be generated.

2) Partial State-Space Discretization

The second approximation does not modify variables, but rather modifies certain values for the waiting times. Whenever the waiting time falls in a "trapping interval" $[S-\varepsilon, S+\varepsilon]$, the waiting time is set equal to S.

Mathematically this can be expressed as,

$$
\begin{aligned}
W'_i &= 0 && \text{if } W_i \le 0 \\
W'_i &= W_i && \text{if } W_i \le S-\varepsilon \text{ or } W_i \ge S+\varepsilon \\
\text{or,} \quad W'_i &= S && \text{if } W_i \ge S-\varepsilon \text{ and } W_i \le S+\varepsilon
\end{aligned} \tag{7}
$$

Therefore the only discretization necessary is in the neighborhood of S. Since $\varepsilon > 0$ it can be shown that the expected time to return to the trapping interval is finite for any S. As expected, the smaller the value for ε, the closer the approximation is to the actual process and the better the estimates will be. Moreover, if ε is too small too few cycles will be generated.

3) Stochastic Bounding

In this case we will approximate the process in such a way that the final result will be a confidence interval of the real process. In this case we define two modified waiting time estimates, W′ and W″

$$
\begin{aligned}
W'_i &= 0 && \text{if } W_i \ge 0, \\
W'_i &= W_i && \text{if } W_i \ge S+\varepsilon \text{ or } W_i \le S-\varepsilon \\
W'_i &= S-\varepsilon && \text{if } W_i \ge S-\varepsilon \text{ and } W_i \le S+\varepsilon
\end{aligned} \tag{8}
$$

And,

$$
\begin{aligned}
W''_i &= 0 && \text{if } W_i \ge 0, \\
W''_i &= W_i && \text{if } W_i \ge S+\varepsilon \text{ or } W_i \le S-\varepsilon \\
W''_i &= S+\varepsilon && \text{if } W_i \ge S-\varepsilon \text{ and } W_i \le S+\varepsilon
\end{aligned} \tag{9}
$$

It can be shown that $E(W') \le E(W) \le E(W'')$, therefore we can form a confidence interval for the real process by taking the limits of the two modified processes the following way.

Let $L' \le E(W') \le U'$ be a confidence interval for the first approximation, and $L'' \le E(W'') \le U''$ be a confidence interval for the second approximation. We can combine both intervals to obtain, $L' \le E(W) \le U''$, a confidence interval for the actual process.

Since the actual process is being estimated, how the width of the trapping interval influences the estimates is of no concern. However, this approximation requires two simulation runs.

4) Approximate Regeneration Times

This case is a special case of the *partial state-space discretization*, where the observation that lies within the trapping interval is not modified to take the value S. As we can see there is no modification of the state space. However, the times $\alpha_i(S)$, at which the value of W_i returns to the trapping interval are not regeneration times for the process because they do not result in identically distributed blocks [Crane & Iglehart 1975]. Furthermore, the observations taken out of the regeneration cycles less correlated. This correlation decreases as the value for ε decreases.

This technique yields approximate confidence intervals because the observations are treated as i.i.d. random variables when they are not. However, if the analyst is able to keep the correlation low and a large sample size (number of cycles) is obtained, the method will give accurate estimates.

For implementation purposes the *classical regenerative method* and the *partial state-space discretization* approximation have been chosen. The major goal is to compare the classical method with one of the approximations describe above.

The *partial state-space discretization* was chosen because it appears to yield the best estimates while it also has some other convenient characteristics, namely, low simulation cost and ease of programming.

The method of *stochastic bounding* was eliminated because it takes twice the computing time to generate a single confidence interval and two different programs have to be written. The *complete state-space discretization* approximation requires a large modification of the original system. Also, the impact of the discretization is unknown to the analyst. These facts make it an undesirable system to implement.

The real difference between the two remaining methods is not specifically known if the conditions are favorable for the *approximate regeneration times*. Since it is sometimes difficult to know when the correlation is low a priori, and the method is approximate, the *partial state-space discretization* method was chosen for implementation.

5. Implementation of the Regenerative Method

5.1 Description of the System to be Analyzed

A Single Server-Single Queue system (M/M/1 queue) with traffic intensity factor (utilization of the server), ρ=0.8 has been chosen for the experiment. This experiment was chosen because the results can be compared with analytical solutions obtained from *queueing theory*. The process of interest is a sequence of customer waiting times $\{W_i: i \geq 0\}$. Let $\alpha_i(S)$ the i^{th} value of n such that W_i=S, the demarcating state. It is known that the times $\{\alpha_i(S): i \geq 0\}$ are regeneration times for the system. Since the distributions of the interarrival times are continuous the expected time between returns to S is infinite except for the case where S=0, where there is no line and the server is idle, thus there is no waiting time. Since this is a problematic state, because most systems do not return to it frequently, in most cases the process must be approximated.

1) Description of the Experiment

The performance measure of interest is the average queue waiting time. The comparison is based on four criteria, the effect of sample size on the estimators, accuracy of the estimator, half-length of the confidence interval generated, and the coverage of the confidence intervals.

In order to study the effect of the sample size on the estimators the study was performed with runs of 1,000, 10,000, 50,000, and 100,000 customers.

The accuracy of the estimator is determined by comparing the estimates generated by each method and comparing them with the analytical value. Statistical testing is done by running the simulation 100 times and recording the average of the estimators and the standard deviation. Then, confidence intervals are generated for the average waiting time.

In order to compare the half-lengths of the confidence intervals generated, the half-lengths of the corresponding intervals generated by each method are compared.

2) Implementation of the Classical Regenerative Method

The system under study is an M/M/1 Queue with i.i.d. interarrival times $\{A_1, A_2, \ldots, A_k\}$, and i.i.d. service times $\{S_1, S_2, \ldots, S_k\}$. The arrival times and service times are generated as exponentially with means of λ and μ, respectively. The discrete time stochastic process of delays in queue $\{D_i, i \geq 1\}$ can be defined as follows [Law & Kelton 1982].

$D_1 = 0$, and

$D_{i+1} = MAX\{D_i + S_i - A_{i+1}, 0\}$ for $i=1,2,\ldots,k$ (11)

These recursive relationships were used in modeling the system and the programming language used was FORTRAN.

In order to use the algorithm described above we must realize that the waiting time for a customer will be zero if the system is empty and idle. Therefore when a customer's waiting time is zero we have a regeneration time.

3) Implementation of the Partial State-Space Discretization

In this case the variable selected as the basis for determining the regeneration points is the average waiting time. The approach used was to set a trapping interval of the form $\{S+\varepsilon\}$ where S represents the analytical expected waiting time.

In this case, with $\lambda=1$ and $\mu=10/8$ the expected waiting time can be shown to be equal to 3.2. Since the values for ε affect the outcome of the experiment, the effect that different values have were studied. Specifically the values $\varepsilon=\{3, 1, 0.5\}$ were studied.

4) Results

Classical Regenerative Method

Tables 1-2 show the results of the experiment. As it can be seen from ⟨Table 1⟩, the accuracy of the estimates increases as the sample size increases. This is evident from the increase in the number of cycles as the sample size is increased. We find that when 1000 customers are included the average waiting time generated by the classical method is much higher than the theoretical value of 3.2. When the number of customers is increased to 10,000, the average waiting time starts to converge to the analytical value of 3.2.

The half-length of the confidence interval decreases very fast as the number of customers increases. This results from the fact that as the sample size increases there is less variability on the estimates and smaller half-lengths. The half-length seem to level off at 50,000 observations.

From ⟨Table 2⟩ it can be seen that the coverage increases as the sample size increases. This is a proof that the estimates are better as the sample size increases. Once again the coverage is approaching the desired value of 90%. Note that the coverage for the case of 100,000 customers was found to be 100%. However the half-length of the interval is very small (6% of the mean).

Partial State-Space Discretization

In this case there is another variable to take into consideration, the size of the trapping interval. From ⟨Table 1⟩ it can be seen that with a constant number of customers, the smaller the interval, the closer our estimates are to the analytical value. This follows from the fact that the smaller the interval, the closer the approximation is to reality.

It can be seen that with a number of customers of 50,000 and small (1 or 0.5) the estimate starts to converge to the analytical value of 3.2. Actually, the difference between the two values of ε is very small. At 50,000 customers the values for the average waiting time are the same for both $\varepsilon=1$ and 0.5. Nevertheless, the half-lengths are slightly less as ap-

⟨Table 1⟩ 90% C.I. on Average Waiting Time based on 100 runs

Number of Customers	Classical Method	Partial State-Space Discretization		
		$\varepsilon=3$	$\varepsilon=1$	$\varepsilon=0.5$
1,000	3.889 ±1.177	3.253 ± 1.746	2.106 ± 1.347	2.399 ± 1.253
10,000	3.202 ± 0.589	3.528 ± 0.558	3.013 ± 0.596	2.967 ± 0.595
50,000	3.208 ± 0.270	3.630 ± 0.296	3.186 ± 0.264	3.186 ± 0.246
100,000	3.205 ± 0.193	3.621 0.176	3.221 ± 0.193	3.183 ± 0.192

〈Table 2〉 Percent Coverage of C.I. based on 100 runs

Number of Customers	Classical Method	Partial State-Space Discretization		
		ε=3	ε=1	ε=0.5
1,000	70	73	44	44
10,000	86	76	77	75
50,000	90	100	89	87
100,000	100	100	100	86

proaches zero. However, the half-lengths of the intervals are not systematically reduced as the value of ε decreases. This is evident from the fact that the smaller the intervals, fewer cycles are obtained and thus the less precise the estimates are. Therefore, the estimate is closer to the real value, but with a slightly higher variability.

Also, as the number of customers increases and ε approaches zero this effect decreases. In fact, the half-length for 100,000 customers and ε=1 is actually less than the one for the same number of customers and ε=0.5. Thus, by carefully choosing the value for ε and the number of customers, good and consistent estimates can be obtained.

〈Table 2〉 shows that the coverage of the confidence intervals is not only dependent on the value of ε, but is also highly dependent on the number of customers. The size of the interval is more significant when the simulation is run for fewer customers than for larger numbers. There is much more variation for smaller sample sizes than for larger sample sizes.

〈Table 2〉 also shows many coverage values reached 100%. Obviously this is because the confidence interval is too wide. However, there are also coverage values around 90%.

Based on the previous analyses, the best approximation will be to set the values of to ε=0.5 and to run the simulation for at least 50,000 customers. This will produce a good confidence interval without significant bias. The coverage for this confidence interval is 87% which is very close to the desired 90%.

Despite the previous analysis, the *regenerative method* only calls for one run because that run provides the sample size necessary for adequate statistical analyses. The first confidence interval generated in the experiment was 3.2378 +/- .30555. As can be seen, the average waiting time is very close to the analytical value. The width of the interval is found to be small, therefore we can conclude that the interval is appropriate.

5) Comparison of both Methods

The classical method is now compared with the *partial state-space discretization* with ε=0.5. It has been already stated that this size for the trapping interval yields the best results. It can be seen from 〈Table 1〉 that the average waiting time estimator is slightly closer to the analytical value for the classical method than for the other method. This means that the approximation yields slightly biased estimates for the mean response. As the table shows this bias is greatly reduced as approaches zero.

In terms of the variability of the estimator we can see that the approximation provides a smaller half-length than the classical method. This difference decreases as the number of customers increases. This supports the previous claim that the approximation converges to the classical method as the sample size increases. Nonetheless, a smaller half-length for the estimates for the approximation is preferable because it produces more precise estimates.

In terms of coverage, it can be seen from 〈Table 2〉 that the coverages are higher for the classical method. This is due to the fact that the estimators are less biased with a wider confidence interval. When 50,000 customers are considered, the difference in coverage is only 3%. However, when 100,000 customers are included the difference is much larger, a 100% coverage resulted for the classical estimate.

All of the above results are obtained in spite of the significant difference between the number of cycles generated by the two methods.

5.2 Description of Another System: Job Shop Model

A job shop model will be used to represent the case of no recurrences to the empty and idle state. The job shop is comprised of six different groups of machines. Each group consists of a number of identical machines as summarized in ⟨Table 3⟩.

⟨Table 3⟩ Data of Machine Group

Machine Group	Type of Machine	# of Machine
1	Casting units	14
2	Lathes	5
3	Planners	4
4	Drill Presses	8
5	Shapers	16
6	Polishing machines	4

Jobs arrive to the jobshop according to a Poisson process with a mean interarrival time of 9.6 minutes. The job mix consists of three job types, with each type having a different machine visitation sequence and different-operation times on each machine. All operation times are exponentially distributed. The data for the three job types are summarized in ⟨Table 4⟩.

1) Description of the Experiment

A plot of the cumulative average time in system and planners utilization is used in determining when the system reaches steady-state. The simulation is executed for 70,000 time units (minutes). See Figures 1 and 2. We can presume that the system reaches the steady-state at around 10,000 simulation time units. In this study the variable selected as the basis for determining the regeneration points is the average time in the system. The analytical solution for the average time in the system is not attainable, so this pilot run is used to estimate the average time in the system. Using the OUTPUT Processor of the ARENA package we estimate the average time in the system as 405 time units, which will be used as S value, with 70,000 time units.

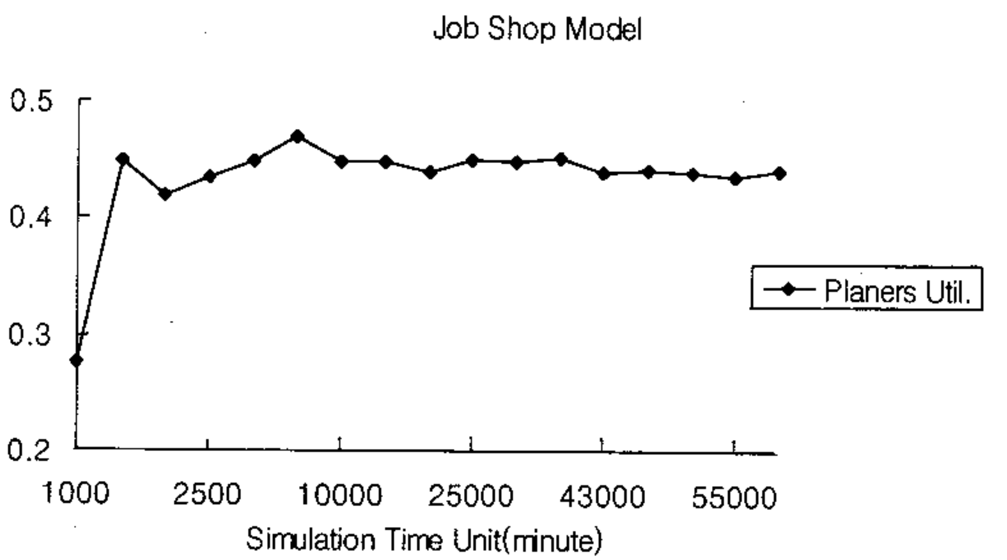
The approach used to set a trapping interval [S+ ε] where S represents the estimated time in the system. Since the value of ε affects the outcome of the experiment, the effect of the different ε values was studied. Specifically we set ε value as ε={5,10,15}. A FORTRAN program was written to handle the *Partial State-Space Discretization* approximation method and it was linked to the ARENA model. To determine the

⟨Table 4⟩ Data of Three Job Types

Job Type	% of Total Jobs	Machine Visitation Sequence	Mean Operation time (Minute)	
	Seq. No. Machine Type			
1	24	1	Casting unit(1)	125
		2	Planer(3)	35
		3	Lathe(2)	20
		4	Polishing Unit(6)	60
2	44	1	Shaper(5)	105
		2	Drill Press(4)	90
		3	Lathe(2)	65
3	32	1	Casting Unit(1)	235
		2	Shaper(5)	250
		3	Drill Press(4)	50
		4	Planer(3)	30
		5	Polishing Unit(6)	25

This model was taken from [Pegden,1995] with minor modifications and is written in ARENA.

〈Figure 1〉 Average Time in the System



〈Figure 2〉 Planers Utilatization

effect of the simulation run length in the accuracy of the confidence interval, the simulation run length as varied as 10,000, 50,000, 100,000 units. Finally, 100 multiple runs were executed for each ε value in conjunction with different simulation time lengths. A confidence interval is generated based on the result of 100 runs for each value of ε and each run length.

2) Results

Tables 5-12 show the results of this experiment. In a single run, the *partial state-space discretization method* keeps decreasing the half-length of confidence intervals as ε value and time length increases. In an identical time length simulation, the *partial state-space dicretization method* yields low variability as ε value varies. Comparison of single run results shows that in an identical simulation time length the regenerative method yields better estimates.

Even if the system is not regenerative, with approximation techniques the system can be implemented by regenerative method. Specifically, a job shop model was studied which does not have an analytical solution for the average time in the system. Thus, the regenerative method can be expanded to

〈Table 5〉 Results of a Single Run Using Batch Means

Without Truncation			10,000 Time Units Truncation		
Average Time in			Average Time in System		
405			406		
Lower Limit	Upper Limit	Half Width	Lower Limit	Upper Limit	Half Width
397	412	7.5	395	416	10.5

〈Table 6〉 Results fo a Single Run Using the Regenerative Method and 10,000 Run Length

10,000 Time Units	Avg. Time	Lower Limit	Upper Limit	Half Width	# of Cycles
ε=5	395.426	364.534	426.317	30.892	12
ε=10	395.452	366.356	424.549	29.356	19
ε=15	396.487	368.272	424.665	28.196	41

〈Table 7〉 Results of a Single Run Using the Regenerative Method and 50,000 Run Length

50,000 Time Units	Avg. Time	Lower Limit	Upper Limit	Half Width	# of Cycles
ε=5	395.457	384.412	406.502	11.045	65
ε=10	395.966	384.392	407.539	11.547	130
ε=15	395.968	385.246	406.689	10.721	206

〈Table 8〉 Results of a Single Run Using the Regenerative Method and 100,000 Run Length

50,000 Time Units	Avg. Time	Lower Limit	Upper Limit	Half Width	# of Cycles
ε=5	405.348	394.769	413.188	10.579	126
ε=10	405.335	395.836	414.834	9.449	254
ε=15	405.723	396.259	413.188	8.465	420

〈Table 9〉 Results of Multiple (100) Runs Using Batch Means

10,000 Time Length			50,000 Time Length			100,000 Time Length		
Average Time in System			Average Time in System			Average Time in System		
Lower Limit	Upper Limit	Half Width	Lower Limit	Upper Limit	Half Width	Lower Limit	Upper Limit	Half Width
393	403	4.52	408	412	2.28	410	414	2.03

〈Table 10〉 Results of Multiple (100) Runs Using the Regenerative Method and 10,000 Runs Length

10,000 Time Units	Avg. Time	Lower Limit	Upper Limit	Half Width
$\epsilon=5$	397.597	393.293	401.899	4.303
$\epsilon=10$	397.631	393.167	402.095	4.465
$\epsilon=15$	397.827	393.332	402.323	4.496

〈Table 11〉 Results of Multiple (100) Runs Using the Regenerative Method and 50,000 Runs Length

50,000 Time Units	Avg. Time	Lower Limit	Upper Limit	Half Width
$\epsilon=5$	410.204	407.998	412.409	2.206
$\epsilon=10$	410.196	407.991	412.402	2.205
$\epsilon=15$	410.216	407.998	412.434	2.201

〈Table 12〉 Results of Multiple (100) Runs Using the Regenerative Method and 100,000 Runs Length

100,000 Time Units	Avg. Time	Lower Limit	Upper Limit	Half Width
$\epsilon=5$	412.151	410.147	414.154	2.003
$\epsilon=10$	412.117	410.113	414.121	2.004
$\epsilon=15$	412.114	410.112	414.115	2.002

those models which do not have analytical solutions. A pilot run was suggested to replace the analytical solution with the average of that pilot run. The estimate from the pilot run can be used as S, the demarcating state. Also if the analyst knows the process well, his knowledge may help to determine the demarcating state.

6. Conclusions and Recommendations

The regenerative method is a statistical approach which has been used successfully in dealing with some of the most important tactical issues of stochastic simulation. Not every system possesses regeneration points, so this regenerative method of collecting data cannot always be used. Furthermore, even when there are regeneration points, the one chosen to demarcate the epochs may not recur frequently resulting in a substantial amount of computer time needed to produce the desired number of cycles.

It has been shown that the regenerative method is a viable approach for the analysis of simulation output. Furthermore, a method was described to implement the regenerative method in systems that are not regenerative. This method, partial state-space discretization, yields slightly biased estimates with a lower variability that the classical method. Since the difference in the half-length of the confidence intervals generated by both methods is so small, one should implement the classical method whenever possible. In many cases, however, the approximation is easier to implement than the classical method and can provide reliable results.

In most cases the analyst does not have access to the analytical solution of the process to be studied. In those cases, a demarcating state must be found that yields good estimates. Sometimes a trial run can be done and the average for that run can be used as the demarcating state. In other cases, knowledge of the process may help in determining the demarcating state.

The contribution of this research is twofold. First, it is expected that through the use of state time transformations the regenerative method can be extended to other non-regenerative models. Second, computer time requirements on regenerative models may be reduced through the use of state time transformations with increased recurrence.

There are several areas were additional research should prove fruitful. One, is the investigation of the extension of time transformed states to other simulation models. The ideas generated by this research should be tested in many other models including models with regeneration structure. Further, other transformations should also be developed and tested for regenerative properties. Second, the use of the independence property in regenerative cycles should be taken advantage of in the estimation of confidence intervals. Sequential rules to determine the optimum number of epochs within a replication have been developed. But, the availability of increased recurrence transformation states, should provide additional reasons for further work in these sequential rules which may result in additional computer time savings.

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