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점프가 일어나는 비선형 빔방정식에 대한 연구

Jumping Problem in a Nonlinear Beam Equation

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Abstract

이 논문에서는 Dirichlet 경계 조건을 갖는 비선형 빔방정식 $u_{tt}+u_{xxxx}+g(u)=f(x,t)$의 해의 존재에 대한 연구를 하였다. 이 때 $g(u)=bu^{+}-au^{-}$으로 나타나고 우변의 외력항이 고유함수 $\{\phi_{00},\phi_{41}\}$로 확장된 함수로 나타날 때 $c_1\phi_{00}+c_2\phi_{41}$가 포함될 수 있는 원뿔형 공간을 만들고 사상을 정의하였고 이 사상의 역(逆)사상의 해의 존재여부에 따라서 빔방정식의 존재하는 해의 개수를 찾는데 이용하였다.

키워드 : *빔방정식, 해의 다중성, 고유값, 고유함수*
Key Words : *Beam equation, multiplicity of solution, eigenvalue, eigenfunction*

0. Introduction

In this paper we investigate the existence of solutions $u(x,t)$ for a beam operator $L=u_{tt}+u_{xxxx}$ under the *Dirichlet* boundary condition on the interval $(-\pi/2,\pi/2)$ and periodic condition on the variable t,

$$u_{tt}+u_{xxxx}+bu^{+}-au^{-}=f(x,t) \quad in \ (-\pi/2,\pi/2)\times\Re,$$

$$u(\pm\pi/2,t)=u_{xx}(\pm\pi/2,t)=0,$$

$$u(x,t)=u(-x,t)=u(x,-t)=u(x,t+\pi),$$

when the jumping nonlinearty crossing the first eigenvalue. The eigenvalues of L under the *Dirichlet* boundary condition and periodic condition on the variable t are given

$$\lambda_{mn}=(2n+1)^4-4m^2,(m,n=0,1,2,\cdots\cdots).$$

Let H be the Hilbert space defined by

$$H=\{u\in L^2(\Omega)\mid u \ is \ even \ in \ x \ and \ t\}.$$

Then the equation can be stated as

$$Lu+bu^{+}-au^{-}=f \quad in \quad H$$

Recently, the research of the multiplicity of solutions of several operators in the elliptic partial differential equations has been done. Many authors try to find the solutions of several operators.

In [3], the authors investigated the multiplicity of solutions of the nonlinear wave equation. In [4], the authors investigated the multiplicity of solutions of the nonlinear elliptic equation. McKenna[11] found the solutions of the equation

$$-\Delta u+bu^{+}-au^{-}=s\phi_1 \ in \ \Omega,$$

$$u=0 \quad on \ \partial\Omega.$$

has $\begin{cases} at\ least \ \ 2\ solutions, & if\ s>0 \\ exactly \ \ 1\ solutions, & if\ s=0 \\ no\ solution, & if\ s<0 \end{cases}$, if $a<\lambda_1<b<\lambda_2$,

and $\begin{cases} at\ least \ \ 4\ solutions, & if\ s>0 \\ no\ solution, & if\ s<0 \end{cases}$, if $a<\lambda_1,\ \lambda_2<b<\lambda_3$.

In this paper, we investigate the existence of

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solutions of a beam equation with jumping nonlinearity.

1. A variational reduction method

We consider the beam equation under *Dirichlet* boundary condition on the interval $(-\pi/2,\pi/2)$ and periodic condition on the variable t

$$u_{tt}+u_{xxxx}+bu^{+}-au^{-}=f(x,t) \quad in \ (-\pi/2,\pi/2)\times\Re, \quad \text{......(1.1)}$$

$$u(\pm\pi/2,t)=u_{xx}(\pm\pi/2,t)=0,$$

$$u(x,t)=u(-x,t)=u(x,-t)=u(x,t+\pi).$$

Here we suppose the relation between eigenvalues and the coefficients of the jumping nonlinearty a and b is

$$\lambda_{20}=-15<a<\lambda_{00}=1<b<\lambda_{41}=17.$$

Let L be the beam operator $L=u_{tt}+u_{xxxx}$. Then the eigenvalue problem

$$Lu=\lambda u \quad in \ (-\pi/2,\pi/2)\times\Re,$$

$$u(\pm\pi/2,t)=u_{xx}(\pm\pi/2,t)=0,$$

$$u(x,t)=u(-x,t)=u(x,-t)=u(x,t+\pi)$$

has infinitely many eigenvalues λ_{mn} and corresponding eigenfunctions ϕ_{mn} $(m,n=0,1,2,3,\cdots\cdots)$ given by

$$\lambda_{mn}=(2n+1)^4-4m^2,(m,n=0,1,2,\cdots\cdots).$$

$$\phi_{mn}=\cos 2mt\cos(2n+1)x,(m,n=0,1,2,\cdots\cdots).$$

Then the set $\{\phi_{mn}\mid m,n=0,1,2,\cdots\cdots\}$ forms an orthogonal set in H.

Theorem 1 Let $f=c_1\phi_{00}+c_2\phi_{41}$ $(c_1,c_2\in\Re)$ in the equation (1.1). Then we have:

1) If $c_1<0$, then (1.1) has no solution.

2) If $c_1=0$ and $c_2\neq 0$, then (1.1) has no solution.

Proof Rewrite equation (1.1) as

$$(L-\lambda_{00})u+(b+\lambda_{00})u^{+}-(a+\lambda_{00})u^{-}=f(x).$$

Multiply ϕ_1 to the both sides and integrate over Ω, then we have

$$\int_\Omega[(b-\lambda_{00})u^{+}-(a-\lambda_{00})u^{-}]\phi_{00}=c_1\int_\Omega\phi_{00}^2. \quad \text{.......(1.2)}$$

By the self-adjointness of L and orthogonality of eigenfunctions, the first statement follows since the left hand side is nonnegative. If $c_1=0$, then $u\equiv 0$ is a solution for (1.1). But it does not satisfy (1.1) when $c_2\neq 0$. This proves the second statement. **Q.E.D.**

Let V be the two-dimensional subspace of $L^2(\Omega)$ spanned by $\{\phi_{00},\phi_{41}\}$ and W be the orthogonal complement of V in $L^2(\Omega)$. Let P be the orthogonal projection of $L^2(\Omega)$ onto V. Then every $u\in L^2(\Omega)$ can be written as $u=v+w$, where $v=Pu$ and $w=(I-P)u$. Then the equation (1.1) is equivalent to

$$Lv+P(b(v+w)^{+}-a(v+w)^{-})=c_1\phi_{00}+c_2\phi_{41} \quad \text{.........(1.3)}$$

$$Lw+(I-P)(b(v+w)^{+}-a(v+w)^{-})=0 \quad \text{...............(1.4)}$$

These are the system of equations with two unknowns v and w.

Lemma 1 For fixed $v\in V$, (1.4) has a unique solution $w=\theta(v)$. Furthermore, $\theta(v)$ is *Lipschitz* continuous in v.

Proof We use the contraction mapping theorem. Let $\delta=(a+b)/2$. Then (1.4) becomes

$$(L-\delta)w=(I-P)(b(v+w)^{+}-a(v+w)^{-}-\delta(v+w)) \quad \text{....(1.5)}$$

$$w=(L-\delta)^{-1}(I-P)g_v(w),$$

where $g_v(w)=b(v+w)^{+}-a(v+w)^{-}-\delta(v+w)$.

Since

$$|g_v(w_1)-g_v(w_2)|\leq|b-\delta|\cdot|w_1-w_2|,$$

$$\|g_v(w_1)-g_v(w_2)\|\leq|b-\delta|\cdot\|w_1-w_2\|,$$

where $\|\cdot\|$ denotes the norm in $L^2(\Omega)$. The operator $(L-\delta)^{-1}(I-P)$ is a self-adjoint compact linear map from $W=(I-P)L^2(\Omega)$ into itself. Its eigenvalues in W are $(\lambda_{mn}-\delta)^{-1}$, where $\lambda_{mn}\neq 1$. Therefore its L^2

norm is $\max\{1/|17-\delta|, 1/|-15-\delta|\}$. Since

$$\max\{|\delta-b|, |a-\delta|\} < \min\{|17-\delta|, |-15-\delta|\},$$

it follows that for fixed $v \in V$, the right hand side of (1.5) defines a *Lipschitz* mapping of W into itself with *Lipschitz* constant less than 1. Hence, by the contraction mapping principle, for each $v \in V$, there is a unique $w \in W$ which satisfies (1.4). And $\theta(v)$ is *Lipschitz* continuous in v.

Q.E.D.

This lemma says that the study of the multiplicity of solutions of (1.1) is reduced to the study of the multiplicity of solutions of an equivalent problem

$$Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) = c_1\phi_{00} + c_2\phi_{41} \quad \text{......(1.6)}$$

defined on the two-dimensional subspace V spanned by $\{\phi_{00}, \phi_{41}\}$.

If $v \geq 0$ or $v \leq 0$, then $\theta(v) \equiv 0$. For instance if we take $v \geq 0$ and $\theta(v) = 0$, the equation (1.5) reduces to

$$-\Delta 0 + (I-P)(bv^+ - av^-) = 0,$$

which holds since $v^+ = v$, $v^- = 0$ and $(I-P)v = 0$ for $v \in V$.

Since V is spanned by $\{\phi_{00}, \phi_{41}\}$, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_{00} + c_2\phi_{41} \mid c_1 \geq 0, \ |c_2| \leq kc_1\},$$

for some $k > 0$, so that $v \geq 0$ for all $v \in C_1$. And a cone C_3 defined by

$$C_3 = \{v = c_1\phi_{00} + c_2\phi_{41} \mid c_1 \leq 0, \ |c_2| \leq k|c_1|\},$$

for some $k > 0$, so that $v \leq 0$ for all $v \in C_3$. Thus even if we do not know $\theta(v)$ for all $v \in V$, we know $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$. And C_2 and C_4 are defined as follows;

$$C_2 = \{v = c_1\phi_{00} + c_2\phi_{41} \mid c_2 \geq 0, \ k|c_1| \leq c_2\}$$

$$C_4 = \{v = c_1\phi_{00} + c_2\phi_{41} \mid c_2 \geq 0, \ k|c_1| \leq |c_2|\}.$$

Now we define a map $\Phi : V \to V$ by

$$\Phi(v) = Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-), \quad v \in V.$$

Then Φ is continuous on V and we have the following lemma.

Lemma 2 For $v \in V$ and $c \geq 0$, $\Phi(cv) = c\Phi(v)$.

Proof Let $c \geq 0$. If v satisfies

$$L\theta(v) + (I-P)(b(v+\theta(v))^+ - a(v+\theta(v))^-) = 0,$$

then

$$Lc\theta(v) + (I-P)(b(cv+c\theta(v))^+ - a(cv+c\theta(v))^-) = 0,$$

and hence $\theta(cv) = c\theta(v)$. Therefore we have

$$\begin{aligned}\Phi(cv) &= L(cv) + P(b(cv+\theta(cv))^+ - a(cv+\theta(cv))^-) \\ &= L(cv) + P(b(cv+c\theta(v))^+ - a(cv+c\theta(v))^-) \\ &= cL(v) + cP(b(v+\theta(v))^+ - ca(v+\theta(v))^-) \\ &= c\Phi(v).\end{aligned}$$

Q.E.D.

2. Multiplicity results for $a < \lambda_{00} < b < \lambda_{41}$

Now we want to investigate the image of C_i under Φ. From now, we will use the notation

$$\zeta = \frac{2\lambda_{00}\lambda_{41} - a(\lambda_{00} + \lambda_{41})}{\lambda_{00} + \lambda_{41} - 2a}$$

for simplicity. To make it easy to generalize, we use $\lambda_{00}, \lambda_{41}$ instead of its real values (constants).

First we consider the image of C_1 under Φ. If $v = c_1\phi_{00} + c_2\phi_{41} \geq 0$, then we have

$$\begin{aligned}\Phi(v) &= Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) \quad (= Lv + P(b(v))) \\ &= -c_1\lambda_{00}\phi_{00} - c_2\lambda_{41}\phi_{41} + b(c_1\phi_{00} + c_2\phi_{41}) \\ &= c_1(b-\lambda_{00})\phi_{00} - c_2(\lambda_{41} - b)\phi_{41}.\end{aligned}$$

So the images of the rays $c_1\phi_{00} \pm kc_2\phi_{41}, (c_1 \geq 0)$ can be calculated and they are

$$c_1(b-\lambda_{00})\phi_{00} \pm kc_2(\lambda_{41} - b)\phi_{41}, (c_1 \geq 0)$$

Thus Φ maps C_1 onto the cone

$$R_1 = \{d_1\phi_{00} + d_2\phi_{41} \mid d_1 \geq 0, \ |d_2| \leq k\frac{\lambda_{41} - b}{b - \lambda_{00}}d_1\}.$$

Next we consider the image of C_2 under Φ. If

$$v = -c_1\phi_{00} + c_2\phi_{41} \le 0,\ (c_1 \ge 0,\ |c_2| \le kc_1),$$

then we have

$$\Phi(v) = Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-)\ \ (= Lv + P(-a(v)))$$
$$= c_1\lambda_{00}\phi_{00} - c_2\lambda_{41}\phi_{41} - a(c_1\phi_{00} - c_2\phi_{41})$$
$$= c_1(\lambda_{00} - a)\phi_{00} - c_2(\lambda_{41} - a)\phi_{41}\ .$$

So the images of the rays $-c_1\phi_{00} \pm kc_2\phi_{41}, (c_1 \ge 0)$ can be calculated and they are

$$c_1(\lambda_{00} - a)\phi_{00} \mp kc_2(\lambda_{41} - a)\phi_{41}, (c_1 \ge 0).$$

Thus Φ maps C_3 onto the cone

$$R_3 = \{d_1\phi_{00} + d_2\phi_{41}\ |\ d_1 \ge 0,\ \ |d_2| \le k\frac{\lambda_{41} - a}{\lambda_{00} - a}d_1\}.$$

Lemma 3 For every $v = c_1\phi_{00} + c_2\phi_{41} \ge 0$, there exists a constant $d > 0$ such that $(\Phi(v), \phi_{00}) \ge d|c_2|$.

Proof Let $g(u) = bu^+ - au^-$ and

$$v = c_1\phi_{00} + c_2\phi_{41} + \theta(c_1\phi_{00} + c_2\phi_{41}).$$

Then

$$\Phi(v) = L(c_1\phi_{00} + c_2\phi_{41}) + P(g(c_1\phi_{00} + c_2\phi_{41} + \theta(c_1\phi_{00} + c_2\phi_{41}))).$$

Hence if $u = c_1\phi_{00} + c_2\phi_{41} + \theta(c_1\phi_{00} + c_2\phi_{41})$, then

$$(\Phi(v), \phi_1) = ((L + \lambda_{00})(c_1\phi_{00} + c_2\phi_{41}), \phi_{00}) + (g(u) - \lambda_{00}u, \phi_{00}).$$

Since L is self-adjoint, $((L + \lambda_{00}), \phi_{00}) = 0$. And

$$g(u) - \lambda_{00}u = bu^+ - au^- - \lambda_{00}u^+ + \lambda_{00}u^-$$
$$= (b - \lambda_{00})u^+ + (\lambda_{00} - a)u^- \ge \gamma|u|,$$

where $\gamma = \min\{b - \lambda_{00}, \lambda_{00} - a\} > 0$.

Hence $(\Phi(v), \phi_{00}) \ge \gamma\int|u|\phi_{00}$. Thus there exists $d > 0$ such that $\gamma\phi_{00} \ge d|\phi_{20}|$ and therefore

$$\gamma\int|u|\phi_{00} \ge d\int|u|\cdot|\phi_{41}| \ge d\left|\int u\phi_{41}\right| = d|c_2|.$$ **Q.E.D.**

This lemma says that the image of Φ is contained in the right-half plane, *i.e.* $\Phi(C_2)$ and $\Phi(C_4)$ are the cones in the right-half plane.

Here we have three cases, $R_1 \subset R_3$, $R_3 \subset R_1$ $R_1 = R_3$. The first case holds if and only if the nonlinearity $bu^+ - au^-$ satisfies $b > \zeta$. The second case holds if and only if the nonlinearity $bu^+ - au^-$ satisfies $b < \zeta$. The last case holds if and only if the nonlinearity $bu^+ - au^-$ satisfies $b = \zeta$.

Consider the restrictions $\Phi|_{C_i}$,$(1 \le i \le 4)$ of Φ to the cones C_i. Let $\Phi_i = \Phi|_{C_i}$, *i.e.* $\Phi_i : C_i \to V$.

First we consider Φ_1. It maps C_1 onto R_1. Let l_1 be the segment defined by

$$l_1 = \{\phi_{00} + d_2\phi_{41}|\ |d_2| \le k\frac{\lambda_{41} - b}{b - \lambda_{00}}\}$$

Then the inverse image $\Phi_1^{-1}(l_1)$ is the segment

$$L_1 = \Phi_1^{-1}(l_1) = \{\frac{1}{b - \lambda_{00}}(\phi_{00} + c_2\phi_{41})|\ |c_2| \le k\}$$

By Lemma 2 ,$\Phi_1 : C_1 \to R_1$ is bijective.

Next we consider Φ_3. It maps C_3 onto R_3. Let l_3 be the segment defined by

$$l_3 = \{\phi_{00} + d_2\phi_{41}\ |\ |d_2| \le k\frac{a - \lambda_{41}}{a - \lambda_{00}}\}.$$

Then the inverse image $\Phi_3^{-1}(l_3)$ is the segment

$$L_3 = \Phi_3^{-1}(l_3) = \{\frac{1}{a - \lambda_{00}}(\phi_{00} + c_2\phi_{41})\ |\ |c_2| \le k\}.$$

By Lemma 2 ,$\Phi_3 : C_3 \to R_3$ is bijective.

2.1 The nonlinearity $bu^+ - au^-$ satisfies $b > \frac{2\lambda_{00}\lambda_{41} - a(\lambda_{00} + \lambda_{41})}{\lambda_{00} + \lambda_{41} - 2a}$

The relation $R_1 \subset R_3$ holds if and only if the nonlinearity $bu^+ - au^-$ satisfies $b > \zeta$. We investigate the images of the cones C_2 and C_4 under Φ, where

$$C_2 = \{v = c_1\phi_{00} + c_2\phi_{41}\ |\ c_2 \ge 0,\ \ k|c_1| \le c_2\}$$
$$C_4 = \{v = c_1\phi_{00} + c_2\phi_{41}\ |\ c_2 \ge 0,\ \ k|c_1| \le |c_2|\}.$$

The image of C_2 under Φ is a cone containing

$$R_2 = \{d_1\phi_{00} + d_2\phi_{41}\ |\ d_1 \ge 0,\ k\frac{\lambda_{41} - b}{b - \lambda_{00}}d_1 \le d_2 \le k\frac{\lambda_{41} - a}{\lambda_{00} - a}d_1\},$$

and the image of C_4 under Φ is a cone containing

$$R_4=\{d_1\phi_{00}+d_2\phi_{41}\mid d_1\ge 0,\ -k\frac{\lambda_{41}-a}{\lambda_{00}-a}d_1\le d_2\le -k\frac{\lambda_{41}-b}{b-\lambda_{00}}d_1\}.$$

Consider the restriction Φ_2 and Φ_4, and define the segment l_2 and l_4 as follows;

$$l_2=\{\phi_{00}+d_2\phi_{41}\mid k\frac{\lambda_{41}-b}{b-\lambda_{00}}\le d_2\le k\frac{\lambda_{41}-a}{\lambda_{00}-a}\}$$

$$l_4=\{\phi_{00}+d_2\phi_{20}\mid -k\frac{\lambda_{20}-a}{\lambda_{00}-a}\le d_2\le -k\frac{\lambda_{20}-b}{b-\lambda_{00}}\}.$$

We want to prove Φ_2 and Φ_4 are surjective.

Lemma 4 For $i=2,4$, let γ be a simple path in R_i with end points on ∂R_i (starting from the origin) where each ray in R_i intersects only one point of γ. Then the inverse image $\Phi_i^{-1}(\gamma)$ of γ is also a simple path in C_i with end points on ∂C_i, where any ray in C_i (starting from the origin) intersects only one point of this path.

Proof Since γ is closed and Φ is continuous in V, $\Phi_i^{-1}(V)$ is closed. Suppose that there is a ray (starting from the origin) in C_i, which intersects two points of $\Phi_i^{-1}(\gamma)$, say p and $\alpha p\ (\alpha>1)$. Then $\Phi(\alpha p)=\alpha\Phi(p)$, which implies $\Phi(p)\in\gamma$ and $\Phi(\alpha p)\in\gamma$. This contradicts to the fact that each ray (starting from the origin) in C_i intersects only one point of γ.

Regarding a point $p\in V$ as a radius vector in the plane V. Define the argument $\arg p$ to be the angle from the positive axis ϕ_{00} to p.

We claim that $\Phi_i^{-1}(\gamma)$ meets all the rays (starting from the origin) in C_i. If not, $\Phi_i^{-1}(V)$ is disconnected in C_i. Since $\Phi_i^{-1}(V)$ is closed and meets at most one point of any ray in C_i, there are two points p_1 and p_2 in C_i such that $\Phi_i^{-1}(\gamma)$ does not contain a point $p\in C_i$ with $\arg p_1<\arg p<\arg p_2$. Let l be the segment with end points p_1 and p_2 then $\Phi_i(l)$ is a path in R_i, where $\Phi_i(p_1)$ and $\Phi_i(p_2)$ belong to γ. Choose a point $q\in\Phi_i(l)$ such that $\arg q$ is between $\arg\Phi_i(p_1)$ and $\arg\Phi_i(p_2)$. Then there exists a point q' of γ such that $q'=\mu q$ for some $\mu>0$. Hence $\Phi_i^{-1}(q)$ and $\Phi_i^{-1}(q')$ are on the same ray (starting from the origin) in C_i and $\arg p_1<\arg\Phi_i^{-1}(q')<\arg p_2$, which is a contradiction. This completes the proof. **Q.E.D.**

Theorem 2 For $1\le i\le 4$, the restriction Φ_i maps C_i onto R_i. Then Φ maps V onto R_3. In particular, Φ_1 and Φ_3 are bijective.

Theorem 3 Suppose $b>\zeta$. Let $f=c_1\phi_{00}+c_2\phi_{41}\in V$ $(c_1,c_2\in\Re)$. Then we have:

1) If f belongs to interior of R_1, then (1.1) has exactly two solutions, one of which is positive and the other is negative.
2) If f belongs to boundary of R_1, then (1.1) has a positive and a negative solution.
3) If f belongs to boundary of R_3, then (1.1) has a negative solution.
4) If f belongs to interior of R_2 or interior of R_4, then (1.1) has a negative solution and at least one sign changing solution.
5) If f does not belong to R_3, then (1.1) has no solution.

2.2 The nonlinearity bu^+-au^- satisfies $b<\frac{2\lambda_{00}\lambda_{41}-a(\lambda_{00}+\lambda_{41})}{\lambda_{00}+\lambda_{41}-2a}$

The relation $R_3\subset R_1$ holds if and only if the nonlinearity $bu^+ - au^-$ satisfies $b < \zeta$. We investigate the images of the cones C_2 and C_4 under Φ, where

$$C_2 = \{v = c_1\phi_{00} + c_2\phi_{41} \mid c_2 \ge 0, \quad k|c_1| \le c_2\}$$

$$C_4 = \{v = c_1\phi_{00} + c_2\phi_{41} \mid c_2 \ge 0, \quad k|c_1| \le |c_2|\}.$$

The image of C_2 under Φ is a cone containing

$$R_2' = \{d_1\phi_{00} + d_2\phi_{41} \mid d_1 \ge 0, \ k\frac{\lambda_{41} - a}{\lambda_{00} - a} d_1 \le d_2 \le k\frac{\lambda_{41} - b}{b - \lambda_{00}} d_1\},$$

and the image of C_4 under Φ is a cone containing

$$R_4' = \{d_1\phi_{00} + d_2\phi_{41} \mid d_1 \ge 0, \ -k\frac{\lambda_{41} - b}{b - \lambda_{00}} d_1 \le d_2 \le -k\frac{\lambda_{41} - a}{\lambda_{00} - a} d_1\}$$

Consider the restriction Φ_2 and Φ_4, and define the segment l_2' and l_4' as follows;

$$l_2' = \{\phi_{00} + d_2\phi_{41} \mid k\frac{\lambda_{41} - a}{\lambda_{00} - a} \le d_2 \le k\frac{\lambda_{41} - b}{b - \lambda_{00}}\}$$

$$l_4' = \{\phi_{00} + d_2\phi_{41} \mid -k\frac{\lambda_{41} - b}{b - \lambda_{00}} \le d_2 \le -k\frac{\lambda_{41} - a}{\lambda_{00} - a}\}.$$

We want to prove Φ_2 and Φ_4 are surjective

Lemma 5 For $i = 2,4$, let γ' be a simple path in R_i' with end points on $\partial R_i'$, where each ray in R_i' (starting from the origin) intersects only one point of γ'. Then the inverse image $\Phi_i^{-1}(\gamma')$ of γ' is also a simple path in C_i with end points on ∂C_i, where any ray in C_i (starting from the origin) intersects only one point of this path.

Theorem 4 For $i = 2,4$, the restriction Φ_i maps C_i onto R_i'. And Φ_1 and Φ_3 are bijective. Therefore Φ maps V onto R_1.

This theorem implies the following results.

Theorem 5 Suppose $b < \zeta$. Let $f = c_1\phi_{00} + c_2\phi_{41} \in V$ $(c_1, c_2 \in \Re)$. Then we have:

1) If f belongs to interior of R_3, then (1.1) has exactly two solutions, one of which is positive and the other is negative.

2) If f belongs to boundary of R_3, then (1.1) has a positive and a negative solution.

3) If f belongs to boundary of R_1, then (1.1) has a negative solution.

4) If f belongs to interior of R_2 or interior of R_4, then (1.1) has a negative solution and at least one sign changing solution.

5) If f does not belong to R_1, then (1.1) has no solution.

2.3 The nonlinearity $bu^+ - au^-$ satisfies $b = \frac{2\lambda_{00}\lambda_{41} - a(\lambda_{00} + \lambda_{41})}{\lambda_{00} + \lambda_{41} - 2a}$

The relation $R_3 = R_1$ holds if and only if the nonlinearity $bu^+ - au^-$ satisfies $b = \zeta$. Consider the map $\Phi : V \to V$ defined by

$$\Phi(v) = Lv + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), \quad v \in V$$

where $b = \zeta$. Now we want to investigate the images of the cones C_2 and C_4 under Φ. For fixed v, we define a map $\Phi_v : (\lambda_1, \lambda_2) \to V$ as follows

$$\Phi_v(b) = Lv + P(b(v + w)^+ - a(v + w)^-), \quad b \in (\lambda_1, \lambda_2),$$

where $v \in V$ and a is fixed.

Lemma 6 If a is fixed, then Φ_v is continuous at $b_0 = \zeta$.

Proof Let $\delta = (a + b_0)/2$. Then (1.3) becomes

$$(L - \delta)w = (I - P)(b(v + w)^+ - a(v + w)^- - \delta(v + w))$$

$$w = (L - \delta)^{-1}(I - P)(b(v + w)^+ - a(v + w)^- - \delta(v + w)).$$

Let $g(b, w) = b(v + w)^+ - a(v + w)^- - \delta(v + w)$. The

$$w = (L - \delta)^{-1}(I - P)g(b, w).$$

By Lemma 1, this equation has a unique solution $w = \theta_b(v)$ for fixed b. Let $w_0 = \theta_{b_0}(v)$. Then we get

$$w - w_0 = (L - \delta)^{-1}(I - P)(g(b, w) - g(b_0, w_0))$$

$$= (L-\delta)^{-1}(I-P)[(g(b,w)-g(b,w_0)+g(b,w_0)-g(b_0,w_0)]$$
$$= (L-\delta)^{-1}(I-P)[(g(b,w)-g(b,w_0)] + (L-\delta)^{-1}(I-P)[g(b,w_0)-g(b_0,w_0)].$$

Since

$$\|g(b,w)-g(b,w_0)\| \le \max\{|b-\delta|,|\delta-a|\}\|w-w_0\|,$$

and

$$\gamma = \frac{1}{|\lambda_{41}-a|}\max\{|b-\delta|,|\delta-a|\} < 1,$$

then we have

$$\|w-w_0\| \le \gamma\|w-w_0\| + \frac{1}{|\lambda_{41}-a|}\|v+w_0\|\cdot|b-b_0|$$

$$\|w-w_0\| \le \frac{1}{|\lambda_{41}-a|\cdot(1-\gamma)}\|v+w_0\|\cdot|b-b_0|,$$

which shows that $w=\theta_b(v)$ is continuous at b_0. Thus $\Phi_v(b)$ is continuous at b_0. Therefore Φ_v is continuous at b_0. **Q.E.D.**

First, we investigate the image of the cone C_2 under Φ. Let

$$p_1 = \phi_{00} + k\frac{\lambda_{41}-b}{b-\lambda_{00}}\phi_{41} \quad \text{and} \quad p_2 = \phi_{00} + k\frac{\lambda_{41}-a}{\lambda_{00}-a}\phi_{41}.$$

Fix a. Define $\theta = |\arg p_1 - \arg p_2|$.

Since $0 \le \theta \le \pi/2$,

$$\tan\theta = \left|\frac{(a+b)(\lambda_{00}+\lambda_{41})-2k\lambda_{00}\lambda_{41}-2ab}{(\lambda_{00}-a)(\lambda_{00}-b)-k^2(\lambda_{41}-a)(\lambda_{41}-b)}\right|.$$

When b converges to ζ, $\tan\theta$ converges to 0. Since $0 \le \theta \le \pi/2$, θ converges to 0. Note that Φ_2 maps C_2 onto R_2, and Φ_2 maps C_2 onto R_2' when $b<\zeta$. So if b converges to ζ, the angle between R_2 and R_2' converges to 0. Since Φ_2 is continuous at $b=\zeta$, Φ_2 maps C_2 onto the ray

$$R_2'' = \{d_1\phi_{00}+d_2\phi_{41} \mid d_1 \ge 0, \quad d_2 = k\frac{\lambda_{41}-b}{b-\lambda_{00}}d_1\}.$$

Second we investigate the image of the cone C_4 under Φ. Let

$$q_1 = \phi_{00} - k\frac{\lambda_{41}-b}{b-\lambda_{00}}\phi_{41} \quad \text{and} \quad q_2 = \phi_{00} - k\frac{\lambda_{41}-a}{\lambda_{00}-a}\phi_{41}.$$

Fix a. Define $\theta' = |\arg q_1 - \arg q_2|$. Since $0 \le \theta' \le \pi/2$,

$$\tan\theta' = \left|\frac{(a+b)(\lambda_{00}+\lambda_{41})-2k\lambda_{00}\lambda_{41}-2ab}{(\lambda_{00}-a)(\lambda_{00}-b)-k^2(\lambda_{41}-a)(\lambda_{41}-b)}\right|.$$

When b converges to ζ, $\tan\theta'$ converges to 0. θ' converges to 0. Since $0 \le \theta \le \pi/2$, Note that Φ_4 maps C_4 onto R_4, and Φ_4 maps C_4 onto R_4' when $b<\zeta$. So if b converges to ζ, the angle between R_4 and R_4' converges to 0. Since Φ_4 is continuous at $b=\zeta$, Φ_4 maps C_4 onto the ray

$$R_4'' = \{d_1\phi_{00}+d_2\phi_{41} \mid d_1 \ge 0, \quad d_2 = k\frac{\lambda_{41}-a}{\lambda_{00}-a}d_1\}.$$

Theorem 6 For $i=2,4$, the restriction Φ_i maps C_i onto R_i''. And Φ_1 and Φ_3 are bijective. Therefore, Φ maps V onto R, where $R=R_1=R_3$.

Theorem 7 Suppose $b=\zeta$. Let $f=c_1\phi_{00}+c_2\phi_{41} \in V$ $(c_1,c_2 \in \Re)$. Then we have

1) If f belongs to interior of R, then (1.1) has exactly two solutions, one of which is positive and the other is negative.
2) If f belongs to boundary of R, then (1.1) has a positive solution and a negative solution, and infinitely many sign changing solutions.
3) If f does not belong to R, then (1.1) has no solution.

3. Conclusion

We investigate the existence of solutions of the nonlinear beam equation under the *Dirichlet* boundary condition with jumping nonlinearity. The nonlinearty term is given by $bu^+ - au^-$ and the forcing term is given by $c_1\phi_{00}+c_2\phi_{41}$. We divide into three cases, which are $R_1 \subset R_3$, $R_3 \subset R_1$ and

$R_1 = R_3$. The eqation has two solutions, a negative solution, a negative solution or one sign changing solution according to where the function f belongs to.

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