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Dirichlet 경계조건하에서의 비선형 타원형 방정식

Nonlinear Elliptic Equations under Dirichlet boundary Condition

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ABSTRACT

이 논문에서는 *Dirichlet* 경계 조건을 갖는 비선형 타원형 방정식 $-\Delta u + g(u) = f(x)$ 의 해의 존재에 대한 연구를 하였다. 존재하는 해의 다중성을 증명하기 위하여 임계점 이론과 롤의 정리를 사용하였으며, 대응되는 범함수에 따라서 방정식의 해와 임계점이 동시에 나타난다는 정리를 이용하였다.

이 때 $g(u) = bu^{+} - au^{-}$ 으로 나타날 때 외력항 (방정식의 우변)이 상수로 주어지는 경우 적어도 두개의 해가 존재한다는 것을 증명하였다. 만약 우변(외력항)의 상수가 음수이거나 0인 경우 이 방정식의 해가 존재하지 않거나 자명한 해만 존재하기 때문에 상수는 양수인 것으로 가정하였다.

키워드 : *라플라스방정식, 임계점, 프레셋 미분가능, 롤의 정리*
Key Words : *Laplace equation, critical point, Frechet differentiable, Rolle's theorem*

1. Preliminaries

In applications to differential equations, critical point corresponds to weak solution of the equation.

$$-\Delta u = f(x);\ x \in \Omega$$
$$u = 0;\ x \in \partial\Omega, \quad \text{.......................(1.1)}$$

where Ω denotes a bounded domain in $\Re^n$ and the boundary of Ω is smooth. Suppose $f \in C(\overline{\Omega})$. A function u is a classical solution of (1.1) if $u \in C^2(\Omega) \cap C(\overline{\Omega})$. For such a solution, multiplying (1.1) by $\varphi \in C_0^\infty(\Omega)$ yields

$$\int_\Omega (\nabla u \cdot \nabla \varphi - f\varphi)\ dx = 0 \quad \text{.....................(1.2)}$$

after an integration by parts. Let $W_0^{1,2}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$, and the norm is

$$\|u\|_{W_0^{1,2}} \equiv \left(\int_\Omega |\nabla u|^2\ dx\right)^{1/2}$$

If $u \in W_0^{1,2}(\Omega)$ and satisfies (1.2) for all $\varphi \in C_0^\infty(\Omega)$, then u is said to be a weak solution of (1.1). By our above remarks, any classical solution of (1.1) is a weak solution. Under slightly stronger hypotheses on f (e.g. f is *Hölder* continuous) the converse is also true. Choosing $E \equiv W_0^{1,2}(\Omega)$, set

$$I(u) \equiv \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right)dx \quad \text{.....................(1.3)}$$

It is not difficult to verify that I is Frechet differentiable on E and

$$I'(u)\varphi = \int_\Omega (\nabla u \cdot \nabla \varphi - f\varphi)dx \quad \text{...............(1.4)}$$

for $\varphi \in E$. Thus u is a critical point of I if and only if u is a weak solution of (1.1).

Let Ω be a bounded domain in $\Re^n$ with smooth boundary $\partial\Omega$. We consider the equation

$$Lu = f(x) \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega \quad \text{........................(1.5)}$$

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with eigenvalues λ_i.

Recently, the research of the multiplicity of solutions of several operators in the elliptic partial differential equations has been done. Many authors try to find the solutions of several operators.

In [1], the authors investigated the multiplicity of solutions of the nonlinear suspension bridge equation

$$-K_1u_{xxtt}+u_{tt}+K_2u_{xxxx}+K_3u^+=1+k\cos x+\varepsilon h(x,t),$$

$$u(\pm\frac{\pi}{2},t)=u_{xx}(\pm\frac{\pi}{2},t)=0.$$

In [3], the authors investigated the multiplicity of solutions of the nonlinear wave equation

$$u_{tt}-u_{xx}+bu^+-au^-=f(x,t) \text{ in } (c,d)\times\Re,$$

$$u(c,t)=u(d,t)=0,$$

$$u(x,t+T)=u(x,t).$$

In these cases the boundary conditions are given by the periodic functions. It is natural to ask if the boundary condition is given by the general smooth function. In this paper, we investigate the existence of solutions of a nonlinear elliptic equation (the *Laplace* operator) with jumping nonlinearity when the boundary has smooth curve. In particular, we investigate the multiplicity of solutions of a nonlinear elliptic equation with constant load

$$-\Delta u+bu^+-au^-=s,\ s>0 \text{ in } \Omega, \quad \text{..........(1.7)}$$

when $a<\lambda_1<b<\lambda_2$ and $g(u)=bu^+-au^-$.

2. Constant Load

Let Ω be a bounded domain in $\Re^n$ with smooth boundary $\partial\Omega$. We consider the problem with constant load $s>0$

$$-\Delta u=-bu^++au^-+s, \text{ in } \Omega,$$

$$u=0 \text{ on } \partial\Omega, \text{ ....(2.1)}$$

where we assume $a<\lambda_1<b<\lambda_2$. Now we state the main result in this section.

Theorem 2.1 Assume that $a<\lambda_1<b<\lambda_2$ and $s>0$. Then the problem (2.1) has at least two solutions.

One of the solutions of the problem (2.1) is positive and the other solution will be found by the critical point theory. To prove this we need several lemmas and theorems.

Lemma 2.1 Let $a<\lambda_1<b<\lambda_2$. Then the problem

$$-\Delta u=-bu^++au^- \text{ in } \Omega \text{......................(2.2)}$$

has only the trivial solution.

Proof Rewrite the equation (2.2) as

$$(-\Delta+\lambda_1)u+(-\lambda_1+b)u^++(\lambda_1-a)u^-=0 \text{ in } W_0^{1,2}(\Omega).$$

Multiply to the both sides by ϕ_1 and integrate over Ω. Then since $((-\Delta+\lambda_1)u,\phi_1)=0$, we have

$$\int_\Omega\{(-\lambda_1+b)u^++(\lambda_1-a)u^-\}\phi_1=0 \text{ ....(2.3)}$$

But for all real valued function u for all $x\in\Omega$,

$$(-\lambda_1+b)u^++(\lambda_1-a)u^-\le 0.$$

Hence the left-hand side of the equation (2.3) is always less than or equal to 0. So the only possibility to hold (2.3) is $u\equiv 0$ ***Q.E.D.***

Lemma 2.2 Let $a<\lambda_1<b<\lambda_2$ and $s>0$. Then the unique solution u_1 of the problem

$$-\Delta u=-bu^++au^-+s \text{ in } L^2(\Omega) \text{ ....(2.4)}$$

is positive. And the boundary value problem (2.1) has a positive solution u_1.

Proof Let $a<\lambda_1<b<\lambda_2$ and $s>0$. Then the problem

$$-\Delta u+bu=\lambda u \text{ in } L^2(\Omega)$$

has eigenvalues which are positive. Since the inverse operator is positive, the solution of (2.4) is positive. The solution of the linear problem (2.4) is positive, hence u_1 is also a solution of the boundary value problem (2.1). ***Q.E.D.***

Now we investigate the existence of the other solution of the problem (2.1) under the condition $a<\lambda_1<b<\lambda_2$ and $s>0$ by critical point theory.

Let us define the functional corresponding to (2.1) in $W_0^{1,2}(\Omega)$

$$F_b(u,s)=\int_\Omega\left[\frac{1}{2}|\nabla u|^2-\frac{b}{2}|u^+|^2-\frac{a}{2}|u^-|^2-su\right]dx. \quad \cdots(2.5)$$

When b is fixed, we will write $F=F_b$ for simplicity. Then F is well defined. The solutions of (2.1) coincide with the critical points of $F(u,s)$.

Lemma 2.3 If b is fixed and $s\in\Re$. Then $F(u,s)=F_b(u,s)$ is continuous and *Frechet* differentiable in $W_0^{1,2}(\Omega)$.

Let V be the one-dimensional subspace of $L^2(\Omega)$ spanned by ϕ_1 whose eigenvalue is λ_1. Let W be the orthogonal complement of V in $W_0^{1,2}(\Omega)$. Let $P:W_0^{1,2}(\Omega)\to V$ be the orthogonal projection of $W_0^{1,2}(\Omega)$ onto V, and $I-P:W_0^{1,2}(\Omega)\to W$ be the orthogonal projection of $W_0^{1,2}(\Omega)$ onto W. Then every element $u\in W_0^{1,2}$ is expressed by $u=v+w$, where $v=Pu$ and $w=(I-P)u$. Then the problem (2.1) is equivalent to

$$-\Delta v=P[b(v+w)^+-a(v+w)^-+s],$$

$$-\Delta w=(I-P)[b(v+w)^+-a(v+w)^-+s].$$

We treat these equations as a system of two unknowns v and w.

Lemma 2.4 Let $a<\lambda_1<b<\lambda_2$ and $s>0$. Then we have

1) There exists a unique solution $w\in W$ of the equation

$$-\Delta w+(I-P)[-b(v+w)^++a(v+w)^--s]=0 \text{ in } W \quad \ldots\ldots(2.6)$$

If for fixed $s\in\Re$,we put $w=\theta(v,s)$, then θ is continuous on V. In particular, θ satisfies a uniform *Lipschitz* condition in v with respect to the L^2 norm (also the *Sobolev* norm $\|\cdot\|_{W_0^{1,2}}$).

2) If $\tilde{F}:V\to\Re$ is defined by $\tilde{F}(v,s)=F(v+\theta(v,s),s)$, then $\tilde{F}$ has a continuous *Frechet* derivative $D\tilde{F}$ with respect to v and

$$D\tilde{F}(v,s)(\tilde{v})=DF(v+\theta(v,s),s)(\tilde{v})=0$$

for all $\tilde{v}\in V$. If v_0 is a critical point of $\tilde{F}$, then $v_0+\theta(v_0,s)$ is a solution of the problem (2.1) and conversely every solution of (2.1) is of this form.

Proof Let $a<\lambda_1<b<\lambda_2$ and $s>0$. Let $\delta=\frac{b+a}{2}$ and $g(\zeta)=b\zeta^+-a\zeta^-$. If $g_1(\zeta)=g(\zeta)-\delta\zeta$, then equation (2.6) is equivalent to

$$w=(-\Delta-\delta)^{-1}(I-P)(g_1((v+w)^+)+s) \quad \ldots\ldots\ldots\ldots\ldots(2.7)$$

Since $(-\Delta-\delta)^{-1}(I-P)$ is compact self-adjoint linear map from $(I-P)W_0^{1,2}(\Omega)$ into itself, the eigenvalues are $\frac{1}{\lambda_n-\delta}$ for $n\ge 2$. Therefore its norm is $\frac{1}{\lambda_2-\delta}$. Since

$$|g_1(\zeta_2)-g_1(\zeta_1)|\le\max\{|b-\delta|,|\delta|\}\cdot|\zeta_2-\zeta_1|,$$

it follows that the right hand side of (2.7) defines, for fixed $v\in V$, a *Lipschitz* mapping of $(I-P)W_0^{1,2}(\Omega)$ into itself with Lipschitz constant $\gamma<1$, where

$$\gamma=\frac{|b|}{2}\frac{1}{\lambda_2-\frac{b}{2}}<1.$$

Therefore, by the contraction mapping principle, for given $v\in V$, there exists a unique $w\in(I-P)W_0^{1,2}(\Omega)$, which satisfies (2.7). Since the constant δ does not depend on v and s, it follows from standard arguments that if $\theta(v,s)$ denotes the unique $w\in(I-P)W_0^{1,2}(\Omega)$ which solves (2.7), then θ is continuous with respect to v. In fact, if $w_1=\theta(v_1,s)$ and $w_2=\theta(v_2,s)$, then we have

$$\|w_1-w_2\|=\|(-\Delta-\delta)^{-1}(I-P)(g_1(v_1+w_1)-g_1(v_2+w_2))\|$$

$$=\gamma\|(v_1+w_1)-(v_2+w_2)\|$$

$$\le \gamma(\|v_1 - v_2\| + \|w_1 - w_2\|).$$

Hence we have

$$\|w_1 - w_2\| \le C\|v_1 - v_2\|, \quad \text{where} \quad C = \frac{\gamma}{1-\gamma},$$

which shows that $\theta(v,s)$ satisfies a uniform *Lipschitz* condition in v with respect to the $\|\cdot\|_{W_0^{1,2}}$ -norm. With the above inequality we have

$$\begin{aligned}\|w_1 - w_2\|_{W_0^{1,2}} &= \|(-\Delta-\delta)^{-1}(I-P)(g_1(v_1+w_1) - g_1(v_2+w_2))\| \\ &\le C_1\|(I-P)(g_1(v_1+w_1) - g_1(v_2+w_2))\| \\ &\le C_1 \frac{b}{2}\|(v_1+w_1)-(v_2+w_2)\| \\ &\le C_1 \frac{b}{2}(\|v_1 - v_2\| + \|w_1 - w_2\|) \\ &\le C_1 \frac{b}{2}(1+C)\|v_1 - v_2\|\end{aligned}$$

for some $C_1 > 0$. Hence we have

$$\|w_1 - w_2\|_{W_0^{1,2}} \le C_2\|v_1 - v_2\|_{W_0^{1,2}} \quad \text{.................(2.8)}$$

for some $C_2 > 0$. This shows that $\theta(v,s)$ satisfies a uniform *Lipschitz* condition in v with respect to the $\|\cdot\|_{W_0^{1,2}}$ -norm.

Let $v \in V$ and $z = \theta(v,s)$. If $\tilde{w} \in W$, then from (2.6) we see that

$$\int_\Omega [\nabla w \cdot \nabla \tilde{w} - (I-P)(b(v+w)^+ - a(v+w)^- - s) \cdot \tilde{w}]dx = 0 \quad \text{...(2.9)}$$

Since $\int_\Omega \nabla w \cdot \nabla \tilde{w} = 0$, we have

$$DF(v+\theta(v,s),s)(\tilde{w}) = 0 \quad \text{.....................(2.10)}$$

for $\tilde{w} \in W$. From Lemma 2.3, $\tilde{F}(v,s)$ has a continuous *Frechet* derivative $D\tilde{F}$ with respect to v,

$$D\tilde{F}(v,s)(\tilde{v}) = DF(v+\theta(v,s),s)(\tilde{v}) \quad \text{..........(2.11)}$$

for all $\tilde{v} \in V$. Suppose that there exists $v_0 \in V$ such that $D\tilde{F}(v_0,s) = 0$ for some fixed $s > 0$. Then it follows from (2.11) that

$$DF(v_0 + \theta(v_0,s),s)(v) = 0$$

for all $v \in V$. Since (2.10) holds for all $\tilde{w} \in W$ and $W_0^{1,2}(\Omega)$ is the direct sum of V and W, it follows that

$$DF(v_0 + \theta(v_0,s),s) = 0 \text{ in } W_0^{1,2}.$$

Therefore $u = v_0 + \theta(v_0,s)$ is a solution of (2.1). Conversely, above procedure shows that if u is a solution of (2.1) and $v = Pu$, then $D\tilde{F}(v,s) = 0$ in V. ***Q.E.D.***

Let $a < \lambda_1 < b < \lambda_2$ and $s > 0$. From Lemma 2.2, we see that (2.1) has a positive solution $u_1(x)$ which is of the form $u_1(x) = v_1 + \theta(v_1,s)$.

Lemma 2.5 Let $a < \lambda_1 < b < \lambda_2$ and $s > 0$. Then there exists a small neighborhood B of v_1 in V such that $v = v_1$ is a strict local point of maximum of $\tilde{F}$.

Proof Let $s > 0$. Then equation (2.1) has a positive solution $u_1(x)$ which is of the form $u_1(x) = v_1 + \theta(v_1,s) > 0$, $\theta(v_1,s) \in W$. Since $I + \theta$, where I is an identity map on V, is continuous on V, there exists a small open neighborhood B of v_1 in V such that if $v \in B$, then $v + \theta(v,s) > 0$. Therefore if $w = \theta(v,s)$, $w_1 = \theta(v_1,s)$ and $v + w = (v_1 + w_1) + (\tilde{v} + \tilde{w})$, then we have

$$\begin{aligned}\tilde{F}(v,s) &= F(v+w,s) \\ &= \int_\Omega [\frac{1}{2}|\nabla(v+w)|^2 - \frac{b}{2}|(v+w)^+|^2 - \frac{a}{2}|(v+w)^-|^2 - s|v+w|] \, dx \\ &= \int_\Omega [\frac{1}{2}|\nabla(v_1+w_1) + \nabla(\tilde{v}+\tilde{w})|^2 - \frac{b}{2}|(v_1+w_1)+(\tilde{v}+\tilde{w})|^2 \\ &\qquad - s\{(v_1+w_1)+(\tilde{v}+\tilde{w})\}] \, dx \\ &= \int_\Omega \frac{1}{2}|\nabla(v_1+w_1)|^2 - \frac{b}{2}|(v_1+w_1)|^2 - s(v_1+w_1)] \, dx \\ &\quad + \int_\Omega [\nabla(v_1+w_1)\cdot\nabla(\tilde{v}+\tilde{w}) - b(v_1+w_1)\cdot(\tilde{v}+\tilde{w}) \\ &\quad - s(\tilde{v}+\tilde{w})] \, dx + \int_\Omega [\frac{1}{2}|\nabla(\tilde{v}+\tilde{w})|^2 - \frac{b}{2}|\tilde{v}+\tilde{w}|^2] \, dx\end{aligned}$$

Here

$$\int_\Omega [\frac{1}{2}|\nabla(v_1+w_1)|^2 - \frac{b}{2}|v_1+w_1|^2 - s|v_1+w_1|]\ dx$$
$$= F(v_1+w_1,s)$$
$$= \widetilde{F}(v_1,s)$$

and

$$\int_\Omega \nabla(v_1+w_1)\cdot\nabla(\widetilde{v}+\widetilde{w}) - b(v_1+w_1)\cdot(\widetilde{v}+\widetilde{w}) - s(\widetilde{v}+\widetilde{w})\ dx$$

$$= \int_\Omega [\nabla(v_1+w_1) - b(v_1+w_1) - s]\cdot(\widetilde{v}+\widetilde{w})\ dx = 0,$$

since v_1+w_1 is a positive solution of (2.1). Since $\widetilde{v}+\widetilde{w}$ can be expressed by

$$\widetilde{v}+\widetilde{w} = e_1\phi_1 + e_2\phi_2 + e_3\phi_3 + \cdots\cdots,$$

we have

$$\widetilde{F}(v,s) - \widetilde{F}_1(v_1,s) = \int_\Omega [\frac{1}{2}|\nabla(\widetilde{v}+\widetilde{w})|^2 - \frac{b}{2}|\widetilde{v}+\widetilde{w}|^2]dx$$

$$= \frac{1}{2}\{(\lambda_1-b)e_1^2 + (\lambda_2-b)e_2^2 + \cdots\cdots\} < 0,$$

since $a<\lambda_1<b<\lambda_2$. Therefore $v=v_1$ is a strict local point of maximum of $\widetilde{F}$. This completes the proof. ***Q.E.D.***

We now define the functional on $W_0^{1,2}$

$$F^*(u) = F(u,0) = \int_\Omega [\frac{1}{2}|\nabla u|^2 - \frac{b}{2}|u^+|^2 - \frac{a}{2}|u^-|^2]\ dx$$

corresponding to the equation

$$-\Delta u + bu^+ - au^- = 0$$

Then the critical points of $F^*(u)$ coincide with solutions of the equation

$$-\Delta u + bu^+ - au^- = 0 \text{ in } W_0^{1,2}. \qquad ....(2.12)$$

If $a<\lambda_1<b<\lambda_2$, then (2.12) has only the trivial solution, and hence $F^*(u)$ has only the critical point $u=0$. Given $v\in V$, let $\theta^*(v)=\theta(v,0)\in W$ be the unique solution of the equation

$$-\Delta w + (I-P)[-b(v+w)^+ + a(v+w)^-] = 0 \text{ in } W.$$

Let us define the reduced functional $\widetilde{F}^*(v)$ on V, by $F^*(v+\theta^*(v))$. We note that we can obtain the same result as Lemma 2.5 when we replace $\theta(v,s)$ and $\widetilde{F}(v+\theta(v,s))$ by $\theta^*(v)$ and $\widetilde{F}^*(v)$, respectively. We also note that $\widetilde{F}^*(v)$ has only the critical point $v=0$.

Lemma 2.6 For $d>0$ and $v\in V$, $\widetilde{F}^*(dv) = d^2\widetilde{F}^*(v)$.

Proof If $v\in V$ satisfies

$$-\Delta w + (I-P)(-b(v+\theta^*(v))^+ + a(v+\theta^*(v))^-) = 0 \text{ in } W,$$

then for $d>0$,

$$-\Delta(dw) + (I-P)(-b(dv+d\theta^*(v))^+ + a(dv+d\theta^*(v))^-) = 0$$

in W. Therefore $\theta^*(dv) = d\theta^*(v)$ for $d>0$. From the definition of $F^*(u)$ we see that $F^*(du) = d^2F^*(u)$ for $u\in H$ and $d>0$. Hence, for $v\in V$ and $d>0$,

$$\widetilde{F}^*(du) = F^*(dv+\theta^*(dv)) = d^2F^*(v+\theta^*(v)) = d^2\widetilde{F}^*(u).$$ ***Q.E.D.***

Until now, we used the notation F, F^* and $\widetilde{F}^*$ which denote F_b (defined in (2.5)), F_b^* and $\widetilde{F}_b^*$, respectively. In the following lemma we use the latter notation.

Lemma 2.7 Let $a<\lambda_1<b<\lambda_2$. Then there exists v_1 and v_2 in V such that $\widetilde{F}_b^*(v_1)<0$ and $\widetilde{F}_b^*(v_2)>0$.

Proof First, we choose $v_1\in V$ such that $v_1+\theta(v_1,0)>0$. In this case $w=\theta(v_1,0)=0$. Hence $v_1+w=d_1\phi_1$, and we have

$$\widetilde{F}_b^*(v_1) = \int_\Omega [\frac{1}{2}|\nabla(v_1+w)|^2 - \frac{b}{2}|(v_1+w)^+|^2]\ dx$$

$$= \int_\Omega [\frac{1}{2}|\nabla(v_1+w)|^2 - \frac{b}{2}|v_1+w|^2]\ dx$$

$$= \int_\Omega [\frac{1}{2}\nabla|v_1+w|\cdot|v_1+w| - \frac{b}{2}|v_1+w|\cdot|v_1+w|]\ dx$$

$$= \frac{1}{2}(\lambda_1-b)d_1^2 < 0,$$

since $a<\lambda_1<b<\lambda_2$.

Next we choose $v_2\in V$ such that $v_2+\theta(v_2,0)<0$. In this case $w=\theta(v_2,0)=0$. Hence $v_2+w=e_1\phi_1$, and we have

$$\widetilde{F}_b^*(v_2) = \int_\Omega [\frac{1}{2}|\nabla(v_2+w)|^2 - \frac{a}{2}|(v_2+w)^-|^2]\ dx$$

$$= \int_\Omega [\frac{1}{2}|\nabla(v_2+w)|^2 - \frac{a}{2}|v_2+w|^2]\ dx$$

$$= \int_\Omega [\frac{1}{2}\nabla|v_2+w|\cdot|v_2+w| - \frac{a}{2}|v_2+w|\cdot|v_2+w|]\ dx$$

$$= \frac{1}{2}[(\lambda_1 - a)e_1^2] > 0,$$

since $a<\lambda_1<b<\lambda_2$. ***Q.E.D.***

Lemma 2.8 Let $a<\lambda_1<b<\lambda_2$ and $s>0$. Then $\widetilde{F}_b(v,s)$ is neither bounded above nor bounded below on V.

Proof From Lemma 2.7, $\widetilde{F}_b^*(v)$ has negative (or positive) value. Suppose that $\widetilde{F}_b^*(v)$ has negative value and that $\widetilde{F}_b(v,s)$ is bounded below. Let v_0 denote a fixed point in V with $\|v_0\|=1$. Let

$$w_n = nv_0 + \theta(nv_0,s) \text{ and } w_n^* = v_0 + \frac{\theta(nv_0,s)}{n} = v_0 + \widetilde{w}_n^*.$$

Since θ is *Lipschitzian*, the sequence $\{w_n^*\}_1^\infty$ is bounded in $L^2(\Omega)$. We have $DF(w_n,s)(y)=0$ for all n and arbitrary $y\in W$. Dividing this by n gives

$$\int_\Omega [\nabla w_n^*\cdot\nabla y - bw_n^{*+}y - aw_n^{*-}y - \frac{s}{n}y]\ dx = 0 \quad .....(2.13)$$

Setting $y=w_n$, we know that $\{w_n^*\}_1^\infty$ is bounded in $L^2(\Omega)$. Hence $\{\widetilde{w}_n^*\}_1^\infty$ is bounded in $L^2(\Omega)$. So we may assume that it converges weakly to an element $\widetilde{w}^*\in W$. If we put $w^*=\widetilde{w}^*+v_0$ and let $n\to\infty$ in (2.13), then we obtain

$$\int_\Omega [\nabla w^*\cdot\nabla y - bw^{*+}y - aw^{*-}y]\ dx = 0 \quad ....(2.14)$$

for arbitrary $y\in W$. Hence $\widetilde{w}^* = \theta(v_0,s)$. If we set $y=\widetilde{w}_n$ and dividing by n in (2.13) , then we have

$$\int_\Omega [|\nabla\widetilde{w}_n^*|^2 - b|w_n^{*+}|\widetilde{w}_n^* - a|w_n^{*-}|\widetilde{w}_n^* - \frac{s}{n}\widetilde{w}_n^*]\ dx = 0 \quad .....(2.15)$$

Letting $n\to\infty$ in (2.15), we obtain

$$\lim_{n\to\infty}\int_\Omega |\nabla\widetilde{w}_n^*|^2 dx = \lim_{n\to\infty}\int_\Omega [b|w_n^{*+}|\widetilde{w}_n^* + a|w_n^{*-}|\widetilde{w}_n^* + \frac{s}{n}\widetilde{w}_n^*]\ dx$$

$$= \int_\Omega [b|w^{*+}|\widetilde{w}^* + a|w^{*-}|\widetilde{w}^*]\ dx$$

$$= \int_\Omega \nabla w^*\cdot\nabla\widetilde{w}^*\ dx$$

$$= \int_\Omega |\nabla\widetilde{w}_n^*|^2\ dx,$$

where we have used (2.14). Hence

$$\lim_{n\to\infty}\int_\Omega |\nabla\widetilde{w}_n^*|^2\ dx = \lim_{n\to\infty}\int_\Omega |\nabla w_n^*|^2\ dx.$$

The assumption that $\widetilde{F}(v,s)$ is bounded below implies the existence of a constant M such that

$$\widetilde{F}_b(nv_0,s)/n^2 \ge M/n^2.$$

Letting $n\to\infty$, our previous reasoning shows that

$$\widetilde{F}_b^*(v_0) = F_b(v_0,0) = \lim_{n\to\infty}\widetilde{F}_b(nv_0,s)/n^2 \ge 0.$$

Since v_0 was an arbitrary member of V with $\|v_0\|=1$ and

$$F_b(kv,0) = k^2\widetilde{F}_b(v,s),$$

this contradicts the assumption $\widetilde{F}_b(v,s)$ is negative for some value of $v\in V$. Hence $\widetilde{F}_b(v,s)$ cannot be bounded below. The proof that $\widetilde{F}_b(v,s)$ cannot be bounded above when $\widetilde{F}_b^*(v)$ has positive value is essentially same. ***Q.E.D.***

Proof of the Theorem 2.1 Let $a<\lambda_1<b<\lambda_2$ and $s>0$. By Lemma 2.2 and Lemma 2.4, (2.1) has a positive solution $u_1(x) = v_1 + \theta(v_1,s)$. By Lemma 2.5, there exists a small open neighborhood B of v_1 in V such that $v=v_1$ is a strict local point of maximum of $\widetilde{F}_b(v,s)$. Since $\widetilde{F}_b(v,s)$ is not bounded below, there exists a point $v_2\in V$ with $v_1\ne v_2$ and $\widetilde{F}_b(v_1,s)=\widetilde{F}_b(v_2,s)$. The *Rolle's* theorem and the fact that $\widetilde{F}_b(v,s)$ has a *Frechet* derivative imply that there exists a strict local point of maximum $\widetilde{F}_b$. Thus $\widetilde{F}_b$ has at least two critical points. Therefore (2.1) has at least two solutions. ***Q.E.D.***

3. Conclusion

The multiplicity of solutions of a semilinear elliptic equation under *Dirichlet* boundary condition depends on the source term of the equation

$$-\Delta u + bu^{+} - au^{-} = f(x).$$

Usually the existence of solutions of the equation can be determined by degree theory or critical point theory. In this paper we use the critical point theory to prove the existence of the solutions of the equation. Here we use the fact that the critical points coincide with the weak solutions of a partial differential equations. And we use *Rolle's* theorem to find multiplicity of the *Laplace* equation.

We treat the source term as a constant load. In this case, the *Laplace* equation has at least two solutions when the relation between eigenvalues and the coefficients a and b is $a<\lambda_1<b<\lambda_2$.

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