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PETROV-GALERKIN METHOD FOR NONLINEAR SYSTEM

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Abstract. Petrov-Galerkin method is investigated for solving nonlinear systems without monotonicity. A monotone iteration is provided for solving the resulting problem. The numerical results show the advantages of such method.

Key words. Petrov-Galerkin method, Nonlinear systems without monotonicity.

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1. Introduction

In studying some problems arising in electromagnetism, biology and some other topics, we have to consider systems of nonlinear equations and their numerical solutions, e. g., see [1–4]. Some authors developed numerical methods by constructing the sequences of supersolutions and subsolutions. The main merit of those methods is the monotonicity of the related sequences. But their accuracies are of second order usually. Recently the authors proposed a new method with the accuracy of fourth order, see [5]. However, the corresponding theoretical analysis is valid only when the nonlinear term is monotone. On the other hand, the Petrov-Galerkin method has been used as a powerful tool for numerical solutions of partial differential equations. A Petrov-Galerkin scheme was proposed for a system of nonlinear equations in [4]. But all results in [4] are valid only when the nonlinear term satisfies some conditions of monotonicity. We now develop a Petrov-Galerkin scheme, and construct an iteration by using the local extreme values for solving the resulting problem. This scheme possesses high accuracy and the iteration is monotonically convergent, even if the nonlinear term is not monotone. Thus it improves the previous results essentially.

2. Petrov-Galerkin Scheme

Let $I = \{x|\ 0 < x < 1\}$, $\bar{I}$ be the closure of I, and $u = (u_1, u_2, \cdots, u_m)^T$ be a vector function of x. The given function

$$f(x,u) \in [C^0(\bar{I} \times \mathbf{R}^m) \cap C^1(I \times \mathbf{R}^m)]^m$$

has the components $f_i(x,u)$, $1 \leq i \leq m$. Also let $a_i(x) \in C^1(\bar{I})$ and assume that for certain positive constants $\alpha_0 \leq \alpha_1$, $\alpha_0 \leq a_i(x) \leq \alpha_1$ for all $x \in I$ and $1 \leq i \leq m$. Furthermore let

$$l = \mathrm{diag}(l_1, \cdots, l_m)$$

with

$$l_i u_i(x) = -(a_i(x) u_i'(x))', \quad u_i'(x) = \frac{\mathrm{d}u_i}{\mathrm{d}x}(x), \qquad 1 \leq i \leq m.$$

Set $F_{i,j}(x,u) = \dfrac{\partial f_i}{\partial u_j}(x,u)$, $1 \leq i,j \leq m$. We consider the following coupled problem, i.e., finding $u(x) \in [C^0(\bar{I}) \cap C^2(I)]^m$ such that

$$\begin{cases} lu(x) + f(x,u(x)) = 0, & x \in I, \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

The solution of such a problem is a vector function $u(x) \in [C^2(I) \cap C^0(\bar{I})]^m$ satisfying (1). Without further mention, we assume that the inequalities involving vector are componentwise. If the vector functions $u_*(x)$, $u(x)$ and $u^*(x)$ satisfy that $u_*(x) \leq u(x) \leq u^*(x)$ for all $x \in \bar{I}$, then we say that $u \in \mathbf{K}(u_*, u^*)$. Furthermore, we define $[u]_i = (u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_m)^T$. Thus we can rewrite, e.g., $f_i(x,u) = f_i(x, u_i, [u]_i)$.

Let

$$a_i(u_i, v_i) = \int_0^1 a_i(x) u_i'(x) v_i'(x) \mathrm{d}x, \qquad 1 \leq i \leq m.$$

The weak formulation of (1) is to seek a solution $u(x) \in [H_0^1(I)]^m$ such that

$$a_i(u_i, v_i) + \int_0^1 f_i(x, u(x)) v_i(x) \mathrm{d}x = 0, \qquad \forall v_i(x) \in H_0^1(I), \quad 1 \leq i \leq m. \tag{2}$$

To discretize (2), we introduce a set of mesh points $\{x_p\}_0^N$ such that

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1.$$

For each p, let $I_p = (x_{p-1}, x_p)$, $h_p = x_p - x_{p-1}$, and $h = \max\limits_{1 \leq p \leq N} h_p$. Suppose that there exists a positive constant β such that

$$\frac{\max_{1 \leq p \leq N} h_p}{\min_{1 \leq p \leq N} h_p} \leq \beta.$$

Let $S_h = \prod\limits_{i=1}^m S_{h,i}$ and $T_h = \prod\limits_{i=1}^m T_{h,i}$ be the finite-dimensional linear spaces of trial functions and test functions in $[H_0^1(I)]^m$ respectively. The corresponding approximate problem is to find $u_h(x) \in S_h$ such that

$$a_i(u_{h,i}, v_{h,i}) + \int_0^1 f_i(x, u_h(x)) v_{h,i}(x) \mathrm{d}x = 0, \qquad \forall v_h(x) \in T_h, \quad 1 \leq i \leq m. \tag{3}$$

Let $\{\varphi_p(x)\}_1^{N-1}$ and $\{\psi_p(x)\}_1^{N-1}$ be the bases of the spaces S_h and T_h respectively, where

$$\varphi_p(x) = (\varphi_{p,1}(x), \cdots, \varphi_{p,m}(x))^T, \quad \psi_p(x) = (\psi_{p,1}(x), \cdots, \psi_{p,m}(x))^T.$$

We assume that for all $1 \le i \le m$ and $1 \le p, q \le N-1$, the following conditions are fulfilled,

($\mathbf{H_1}$) $\mathrm{supp}\varphi_{p,i} \subset \overline{I_p \cup I_{p+1}}, \ \varphi_{p,i}(x_q) = \delta_{p,q}$, and for all $x \in I, \ \varphi_{p,i}(x) \ge 0, \ \sum_{p=1}^{N-1} \varphi_{p,i}(x) \le 1;$

($\mathbf{H_2}$) $\mathrm{supp}\psi_{p,i} \subset \overline{I_p \cup I_{p+1}}, \ \psi_{p,i}(x_q) = \delta_{p,q}$, and for all $x \notin \{x_p\}_0^N, \ l_i\psi_{p,i}(x) = 0.$

Following [4], we have from ($\mathbf{H_2}$) that

$$\psi_{p,i}(x) = \begin{cases} A_{i,p} \displaystyle\int_{x_{p-1}}^{x} \frac{1}{a_i(t)} \mathrm{d}t, & x \in I_p, \\ A_{i,p+1} \displaystyle\int_{x}^{x_{p+1}} \frac{1}{a_i(t)} \mathrm{d}t, & x \in I_{p+1}, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where

$$A_{i,p} = \left(\int_{x_{p-1}}^{x_p} \frac{1}{a_i(t)} \mathrm{d}t \right)^{-1}.$$

Clearly $u_h(x) \in S_h$ has the form

$$u_{h,i}(x) = \sum_{p=1}^{N-1} u_{h,i}(x_p)\varphi_{p,i}(x), \qquad x \in \overline{I}, \quad 1 \le i \le m.$$

Thus (3) is equivalent to the following integro-difference system

$$\begin{cases} \displaystyle\sum_{q=p-1}^{p+1} a_i(\varphi_{q,i}, \psi_{p,i}) u_{h,i}(x_q) + \int_0^1 f_i(x, u_h)\psi_{p,i}(x)\mathrm{d}x = 0, \\ u_h(0) = u_h(1) = 0, \qquad 1 \le i \le m, \quad 1 \le p \le N-1. \end{cases} \tag{5}$$

After integrating by parts, we deduce that

$$\begin{aligned} &a_i(\varphi_{p-1,i}, \psi_{p,i}) = -a_i(x_{p-1})\psi'_{p,i}(x_{p-1}+0) = -A_{i,p}, \\ &a_i(\varphi_{p,i}, \psi_{p,i}) = a_i(x_p)\psi'_{p,i}(x_p - 0) - a_i(x_p)\psi'_{p,i}(x_p+0) = A_{i,p} + A_{i,p+1}, \\ &a_i(\varphi_{p+1,i}, \psi_{p,i}) = a_i(x_{p+1})\psi'_{p,i}(x_{p+1} - 0) = -A_{i,p+1}. \end{aligned}$$

Let

$$l_{h,i}u_{h,i}(x_p) = -A_{i,p}u_{h,i}(x_{p-1}) + (A_{i,p} + A_{i,p+1})u_{h,i}(x_p) - A_{i,p+1}u_{h,i}(x_{p+1}).$$

Then (5) becomes

$$\begin{cases} l_{h,i}u_{h,i}(x_p) + \displaystyle\int_0^1 f_i(x, u_h)\psi_{p,i}(x)\mathrm{d}x = 0, \\ u_h(0) = u_h(1) = 0, \qquad\qquad 1 \le i \le m, \quad 1 \le p \le N-1. \end{cases} \tag{6}$$

We now introduce a new concept of supersolutions and subsolutions for (6). A pair of vector functions $\overline{u}_h(x)$, $\underline{u}_h(x) \in S_h$ is an ordered pair of supersolution and subsolution of (6), if they satisfy that

(i) for all $x \in \overline{I}$, $\overline{u}_h(x) \ge \underline{u}_h(x)$; (7)

(ii) for all $v_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h) \cap S_h$, $1 \le i \le m$ and $1 \le p \le N-1$,

$$\begin{cases} l_{h,i}\overline{u}_{h,i}(x_p) + \displaystyle\int_0^1 f_i(x, \overline{u}_{h,i}, [v_h]_i)\psi_{p,i}(x)\mathrm{d}x \ge 0, \\ l_{h,i}\underline{u}_{h,i}(x_p) + \displaystyle\int_0^1 f_i(x, \underline{u}_{h,i}, [v_h]_i)\psi_{p,i}(x)\mathrm{d}x \le 0. \end{cases} \tag{8}$$

In order to present the result for the existence of solutions of (6), we use some terminologies. Let A be a matrix. If for any vector U, $AU \ge 0$ implies $U \ge 0$, then we say that A is a monotone matrix. A necessary and sufficient condition for the monotonicity of A is the existence of the inverse $A^{-1} \ge 0$.

Now, let $A_h^i = (A_{p,q}^i)$ and $B_h = (B_{p,q})$ be the two tridiagonal matrices with the following elements

$$\begin{aligned} &A_{p,p-1}^i = -A_{i,p}, \quad A_{p,p}^i = A_{i,p} + A_{i,p+1}, \quad A_{p,p+1}^i = -A_{i,p+1}, \\ &B_{p,p-1} = h, \quad B_{p,p} = 2h, \quad B_{p,p+1} = h, \qquad\qquad 1 \le i \le m, \quad 1 \le p \le N-1. \end{aligned}$$

Let M be any non-negative constant, and

$$h(M) = \begin{cases} \text{arbitrary positive constant}, & \text{if } \ M = 0, \\ \sqrt{\dfrac{\alpha_0}{M}}, & \text{if } \ M > 0. \end{cases} \tag{9}$$

Then the matrix $A_h^i + MB_h$ with $h \le h(M)$ is monotone.

Theorem 1. Suppose that $\{\overline{u}_h, \underline{u}_h\}$ is an ordered pair of supersolution and subsolution for (6), and

$$F_{i,i}(x, \xi_h) \le \overline{M}, \qquad\qquad x \in \overline{I}, \quad \xi_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h).$$

Let $M^* = \max(\overline{M}, 0)$ and $h \le h(M^*)$ where $h(M^*)$ is given in (9). Then problem (6) has at least one solution $u_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$.

3. A New Monotone Iteration

So far, we have shown that if (6) possesses an ordered pair of supersolution and subsolution, then it has at least one solution. Moreover the supersolutions and the subsolutions may serve as the upper bounds and the lower bounds for the exact numerical solutions. We now propose a monotone iteration which improves the bounds monotonically. Moreover, under certain additional condition, the sequences of the upper and the lower bounds converge to the unique solution. In particular, we do not require any monotonicity of the nonlinear term $f(x,u)$.

Let $\tilde{L}$,$\tilde{T}$ and $\tilde{Q}\in\mathbf{R}^m$ with the elements $\tilde{l}_i$, $\tilde{t}_i$ and $\tilde{q}_i$, respectively. We define

$$\begin{aligned}g_i(x_p;\tilde{L},\tilde{T},\tilde{Q}) = &\int_{x_p}^{x_{p+1}} f_i(x,\tilde{t}_1\varphi_{p,1}(x)+\tilde{q}_1\varphi_{p+1,1}(x),\cdots,\tilde{t}_m\varphi_{p,m}(x)+\tilde{q}_m\varphi_{p+1,m}(x))\psi_{p,i}(x)\mathrm{d}x\\ &+\int_{x_{p-1}}^{x_p} f_i(x,\tilde{l}_1\varphi_{p-1,1}(x)+\tilde{t}_1\varphi_{p,1}(x),\cdots,\tilde{l}_m\varphi_{p-1,m}(x)+\tilde{t}_m\varphi_{p,m}(x))\psi_{p,i}(x)\mathrm{d}x.\end{aligned}$$

In terms of g_i, we can rewrite (6) as

$$\begin{cases} l_{h,i}u_{h,i}(x_p)+g_i(x_p;u_h(x_{p-1}),u_h(x_p),u_h(x_{p+1}))=0,\\ u_h(0)=u_h(1)=0, \qquad\qquad 1\le i\le m,\quad 1\le p\le N-1.\end{cases}\tag{10}$$

Let $\overline{u}_h^{(k)}(x)$ and $\underline{u}_h^{(k)}(x)$ be two vector functions such that $\underline{u}_h^{(k)}(x_p)\le\overline{u}_h^{(k)}(x_p)$, $1\le p\le N-1$. We consider the following iteration

$$\begin{cases} l_{h,i}\overline{u}_{h,i}^{(k+1)}(x_p)+MP_{h,i}\overline{u}_{h,i}^{(k+1)}(x_p)\\ \quad=\max\limits_{\substack{\underline{u}_h^{(k)}(x_q)\le v_h(x_q)\le\overline{u}_h^{(k)}(x_q)\\ q=p-1,p,p+1}}\left\{MP_{h,i}v_{h,i}(x_p)-g_i(x_p;v_h(x_{p-1}),v_h(x_p),v_h(x_{p+1}))\right\},\\ l_{h,i}\underline{u}_{h,i}^{(k+1)}(x_p)+MP_{h,i}\underline{u}_{h,i}^{(k+1)}(x_p)\\ \quad=\min\limits_{\substack{\underline{u}_h^{(k)}(x_q)\le v_h(x_q)\le\overline{u}_h^{(k)}(x_q)\\ q=p-1,p,p+1}}\left\{MP_{h,i}v_{h,i}(x_p)-g_i(x_p;v_h(x_{p-1}),v_h(x_p),v_h(x_{p+1}))\right\},\\ \overline{u}_h^{(k+1)}(x)=\underline{u}_h^{(k+1)}(x)=0, \qquad\qquad x=0,1,\end{cases}\tag{11}$$

where M denotes some nonnegative constant specified later. For each i, the system (11) represents two uncoupled systems of linear algebraic equations for the components of the vector $(\overline{u}_{h,i}^{(k+1)}(x_1),\cdots,\overline{u}_{h,i}^{(k+1)}(x_{N-1}))^T$ and $(\underline{u}_{h,i}^{(k+1)}(x_1),\cdots,\underline{u}_{h,i}^{(k+1)}(x_{N-1}))^T$. Since for each i, the right-hand side of the first formula of (11) is not less than the right-hand side of the second one, we can prove that $\underline{u}_h^{(k+1)}(x_p)\le\overline{u}_h^{(k+1)}(x_p)$, $1\le p\le N-1$, provided $h\le h(M)$. Thus for all $h\le h(M)$, the iteration (11) is well defined.

Theorem 2. Suppose that $\{\overline{u}_h,\underline{u}_h\}$ is an ordered pair of supersolution and subsolution for (6), and

$$F_{i,i}(x,\xi_h)\le\overline{M}, \qquad\qquad x\in I,\quad \xi_h\in\mathbf{K}(\underline{u}_h,\overline{u}_h).$$

Let $M^*=\max(\overline{M},0)$ and $h\le h(M^*)$. Then the two sequences $\left\{\overline{u}_{h,i}^{(k)}(x_p)\right\}$ and $\left\{\underline{u}_{h,i}^{(k)}(x_p)\right\}$ defined by iteration (11) with $M=M^*$ and the initial values $\overline{u}_{h,i}^{(0)}(x_p)=$

$\overline{u}_{h,i}(x_p)$ and $\underline{u}^{(0)}_{h,i}(x_p) = \underline{u}_{h,i}(x_p)$, converge monotonically to the limits $\overline{u}^*_{h,i}(x_p)$ and $\underline{u}^*_{h,i}(x_p)$, respectively. Let $\overline{u}^{(k)}_h(x)$, $\underline{u}^{(k)}_h(x)$, $\overline{u}^*_h(x)$ and $\underline{u}^*_h(x)$ denote the vector functions in S_h with the components

$$\overline{u}^{(k)}_{h,i}(x) = \sum_{p=1}^{N-1} \overline{u}^{(k)}_{h,i}(x_p)\varphi_{p,i}(x), \qquad \underline{u}^{(k)}_{h,i}(x) = \sum_{p=1}^{N-1} \underline{u}^{(k)}_{h,i}(x_p)\varphi_{p,i}(x),$$
$$\overline{u}^*_{h,i}(x) = \sum_{p=1}^{N-1} \overline{u}^*_{h,i}(x_p)\varphi_{p,i}(x), \qquad \underline{u}^*_{h,i}(x) = \sum_{p=1}^{N-1} \underline{u}^*_{h,i}(x_p)\varphi_{p,i}(x).$$

Then for all $x \in \overline{I}$ and $k \geq 1$,

$$\underline{u}_h(x) \leq \underline{u}^{(k)}_h(x) \leq \underline{u}^{(k+1)}_h(x) \leq \underline{u}^*_h(x) \leq \overline{u}^*_h(x) \leq \overline{u}^{(k+1)}_h(x) \leq \overline{u}^{(k)}_h(x) \leq \overline{u}_h(x). \tag{12}$$

In addition, for any possible solution $u_h(x)$ of problem (6) in $\mathbf{K}(\underline{u}_h, \overline{u}_h)$, we have $u_h \in \mathbf{K}(\underline{u}^*_h, \overline{u}^*_h)$.

Theorem 3. Assume that all the hypotheses in Theorem 2 hold, and that $\overline{u}^*_h(x)$ and $\underline{u}^*_h(x)$ are the limits obtained from the corresponding monotone sequences. If $\overline{u}^*_h(x) = \underline{u}^*_h(x)$ for all $x \in \overline{I}$, then $u_h(x) := \overline{u}^*_h(x) = \underline{u}^*_h(x)$ is the unique solution of problem (6) in $\mathbf{K}(\underline{u}_h, \overline{u}_h)$.

Let A^i_h and B_h denote the tridiagonal matrices as before, and λ_i be the least eigenvalue of the symmetric matrix $B^{-1}_h A^i_h$. Now, we provide a condition in terms of λ_i, ensuring that the iteration (11) converges to the unique solution of (6).

Theorem 4. Assume that $\{\overline{u}_h, \underline{u}_h\}$ is an ordered pair of supersolution and subsolution for (6), and

$$|F_{i,j}(x, \xi_h)| \leq \overline{M}, \qquad x \in \overline{I},\ \xi_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h), \quad i, j = 1, 2, \cdots, m.$$

Let $h \leq h(\overline{M})$, $h(\overline{M})$ being as in (9), and let G be the matrix with the elements

$$G_{i,j} = \delta_{i,j}(\lambda_i - \overline{M}) - 2\overline{M}(1 - \delta_{i,j}). \tag{13}$$

If the matrix G is positive definite, then the iteration (11) with $M = \overline{M}$ and the initial values $\overline{u}_h(x)$ and $\underline{u}_h(x)$, yields the sequences $\left\{\overline{u}^{(k)}_h(x)\right\}$ and $\left\{\underline{u}^{(k)}_h(x)\right\}$ as the upper bounds and the lower bounds converging monotonically to the unique solution $u_h(x)$ of problem (6) in $\mathbf{K}(\underline{u}_h, \overline{u}_h)$.

Next, we give another condition ensuring the convergence of iteration (11). For this purpose, we introduce the following discrete norm,

$$|z_h|^2_1 = \max_{1\leq i\leq m} \sum_{p=1}^{N} \frac{(z_{h,i}(x_p) - z_{h,i}(x_{p-1}))^2}{h_p}.$$

Theorem 5. Assume that $\{\overline{u}_h, \underline{u}_h\}$ is an ordered pair of supersolution and subsolution for (6), and

$$|F_{i,j}(x, \xi_h)| \leq \overline{M}, \qquad x \in \overline{I},\ \xi_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h).$$

Let $h \le h(\overline{M})$. If $4\beta\overline{M}(2m-1) < \alpha_0$, then all results of Theorem 4 hold.

We now estimate the errors between $\overline{u}_h^{(k)}(x_p)$ and $\overline{u}_h^*(x_p)$, and the errors between $\underline{u}_h^{(k)}(x_p)$ and $\underline{u}_h^*(x_p)$.

Theorem 6. If the hypotheses in Theorem 5 hold, then

$$|\overline{u}_h^{(k)} - \overline{u}_h^*|_1^2 + |\underline{u}_h^{(k)} - \underline{u}_h^*|_1^2 \le \gamma^k \left(|\overline{u}_h^{(0)} - \overline{u}_h^*|_1^2 + |\underline{u}_h^{(0)} - \underline{u}_h^*|_1^2\right)$$

where

$$\gamma = \frac{4\beta\overline{M}(m-1)}{\alpha_0 - 4\beta\overline{M}m} < 1.$$

Theorem 6 shows the geometric convergence of the iteration (11). rate.

4. The Convergence of the Petrov-Galerkin Method

We now deal with the convergence of Petrov-Galerkin scheme (6). Let $u(x)$ be the solution of (1) and $u_h(x) \in S_h$ be the solution of (6). We introduce the local Green's function as follows,

$$G_p(x,s) = \text{diag}(G_{p,1}(x,s), G_{p,2}(x,s), \cdots, G_{p,m}(x,s))$$

where

$$\begin{cases} l_i G_{p,i}(x,s) = \delta(x,s), & (x,s) \in I_p \times \overline{I}_p,\ 1 \le p \le N-1,\ 1 \le i \le m, \\ G_{p,i}(x_{p-1},s) = G_{p.i}(x_p,s) = 0, & s \in \overline{I}_p,\ \ 1 \le p \le N-1,\ 1 \le i \le m. \end{cases}$$

By [4],

$$G_{p,i}(x,s) = \begin{cases} \dfrac{1}{A_{i,p}} g_{p,i,1}(s) g_{p,i,2}(x), & x \le s, \\ \dfrac{1}{A_{i,p}} g_{p,i,1}(x) g_{p,i,2}(s), & x > s \end{cases}$$

where $A_{i,p}$ are the same as before, and

$$g_{p,i,1}(x) = A_{i,p} \int_x^{x_p} \frac{1}{a_i(t)} \mathrm{d}t, \qquad g_{p,i,2}(x) = A_{i,p} \int_{x_{p-1}}^{x} \frac{1}{a_i(t)} \mathrm{d}t.$$

Using these local Green's functions, we can prove that $u(x)$ satisfy

$$\begin{cases} l_{h,i} u_i(x_p) + \displaystyle\int_0^1 f_i(x,u(x)) \psi_{p,i}(x) = 0, \\ u(0) = u(1) = 0, \qquad 1 \le i \le m,\ \ 1 \le p \le N-1. \end{cases} \tag{14}$$

Define the map $T_h = \text{diag}(T_{h,1}, \cdots, T_{h,m}) :\ [H_0^1(I)]^m \to S_h$ as

$$T_{h,i} u_i(x) = \sum_{p=1}^{N-1} u_i(x_p) \varphi_{p,i}(x).$$

Clearly, if

$$\max_i \max_{x\in\overline{I}} |T_{h,i}u_i(x) - u_i(x)| \le C_1 h^\alpha, \tag{15}$$

then

$$\max_i \left| \int_0^1 f_i(x,u)\psi_{p,i}(x)\mathrm{d}x - \int_0^1 f_i(x,T_h u)\psi_{p,i}(x)\mathrm{d}x \right| \le C_2 h^{\alpha+1}$$

where C_1, C_2 are some positive constants independent of h.

Theorem 7. Let $u(x)$ and $u_h(x)$ be the solution of (1) and (6) in $\mathbf{K}(u_*, u^*)$ respectively, and $T_h u \in \mathbf{K}(u_*, u^*)$. Moreover,

$$|F_{i,j}(x,\xi)| \le \overline{M}, \qquad x \in I,\ \xi \in \mathbf{K}(u_*, u^*),\ i,j = 1,2,\cdots,m.$$

If $4\beta\overline{M}m < \alpha_0$ and the map T_h has the approximation property as in (15), then

$$\|u - u_h\|_{L^\infty(\overline{I})} \le C^* h^\alpha$$

where C^* is a positive constant independent of h.

5. Numerical Results

We consider the following system

$$\begin{cases} -u_1''(x) + f_1(x,u_1,u_2) = 0, & 0 < x < 1, \\ -u_2''(x) + f_2(x,u_1,u_2) = 0, & 0 < x < 1, \\ u_1(x) = u_2(x) = 0, \quad x = 0,1 \end{cases} \tag{16}$$

where

$$f_1(x,u_1,u_2) = -p_1(x)\cos(q_1(x)u_2(x)),$$
$$f_2(x,u_1,u_2) = -p_2(x)\cos(q_2(x)u_1(x)).$$

The functions $p_i(x), q_i(x) \in C^0(I)$ and $|p_i(x)| \le \alpha$ for $x \in I$. We solve (16) by the Petrov-Galerkin scheme (6). For simplicity, let the mesh be uniform with the spacing $h = h_p$, $1 \le p \le N$. We take the standard piecewise linear functions space as the trial space $S_h = \mathrm{span}\{\varphi_p(x)\}_1^{N-1}$, that is,

$$\varphi_{p,i}(x) = \begin{cases} \dfrac{x - x_{p-1}}{h}, & x \in I_p, \\ \dfrac{x_{p+1} - x}{h}, & x \in I_{p+1}, \\ 0, & \text{otherwise}. \end{cases} \qquad 1 \le i \le 2,\ 1 \le p \le N-1.$$

Clearly, the assumption ($\mathbf{H_1}$) is satisfied. If we can take $\varphi_{p,i}(x) = \psi_{p,i}(x)$, then the Petrov-Galerkin scheme (10) is reduced to

$$\begin{cases} -\frac{1}{h}u_{h,1}(x_{p-1}) + \frac{2}{h}u_{h,1}(x_p) - \frac{1}{h}u_{h,1}(x_{p+1}) + g_1(x_p; u_h(x_{p-1}), u_h(x_p), u_h(x_{p+1})) = 0, \\ -\frac{1}{h}u_{h,2}(x_{p-1}) + \frac{2}{h}u_{h,2}(x_p) - \frac{1}{h}u_{h,2}(x_{p+1}) + g_2(x_p; u_h(x_{p-1}), u_h(x_p), u_h(x_{p+1})) = 0, \\ u_h(0) = u_h(1) = 0, \qquad 1 \le p \le N-1 \end{cases} \tag{17}$$

where

$$\begin{aligned} & g_1(x_p; u_h(x_{p-1}), u_h(x_p), u_h(x_{p+1})) \\ &\quad = -\int_{x_p}^{x_{p+1}} \tfrac{x_{p+1}-x}{h} p_1(x) \cos\left(q_1(x_p)u_{h,2}(x_p)\tfrac{x_{p+1}-x}{h} + q_1(x_{p+1})u_{h,2}(x_{p+1})\tfrac{x-x_p}{h}\right) \mathrm{d}x \\ &\qquad - \int_{x_{p-1}}^{x_p} \tfrac{x-x_{p-1}}{h} p_1(x) \cos\left(q_1(x_{p-1})u_{h,2}(x_{p-1})\tfrac{x_p-x}{h} + q_1(x_p)u_{h,2}(x_p)\tfrac{x-x_{p-1}}{h}\right) \mathrm{d}x, \\ & g_2(x_p; u_h(x_{p-1}), u_h(x_p), u_h(x_{p+1})) \\ &\quad = -\int_{x_p}^{x_{p+1}} \tfrac{x_{p+1}-x}{h} p_2(x) \cos\left(q_2(x_p)u_{h,1}(x_p)\tfrac{x_{p+1}-x}{h} + q_2(x_{p+1})u_{h,1}(x_{p+1})\tfrac{x-x_p}{h}\right) \mathrm{d}x \\ &\qquad - \int_{x_{p-1}}^{x_p} \tfrac{x-x_{p-1}}{h} p_2(x) \cos\left(q_2(x_{p-1})u_{h,1}(x_{p-1})\tfrac{x_p-x}{h} + q_2(x_p)u_{h,1}(x_p)\tfrac{x-x_{p-1}}{h}\right) \mathrm{d}x. \end{aligned}$$

Since the functions $p_i(x)$ and $q_i(x)$ may oscillate arbitrarily, the monotonicity of the functions f is destroyed usually. Now let $\overline{u}_h(x)$, $\underline{u}_h(x) \in S_h$ such that

$$\overline{u}_{h,i}(x_p) = -\underline{u}_{h,i}(x_p) = \alpha x_p(1-x_p), \qquad 1 \le p \le N-1, \quad i = 1, 2. \tag{18}$$

It can be verified that $\{\overline{u}_h, \underline{u}_h\}$ is an ordered pair of supersolution and subsolution.

We first take α=1 and $p_i(x)$=$\frac{1}{3}$, $q_i(x)$=1 for $i = 1, 2$. Then

$$\frac{\partial f_1}{\partial u_1} = 0, \quad |\frac{\partial f_1}{\partial u_2}| < \frac{1}{12}, \quad |\frac{\partial f_2}{\partial u_1}| < \frac{1}{12}, \quad \frac{\partial f_2}{\partial u_2} = 0, \qquad x \in I, \ \xi_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h).$$

We can take M=0 in (11), and so it is reduced to

$$\begin{cases} -\overline{u}_{h,i}^{(k+1)}(x_{p-1}) + 2\overline{u}_{h,i}^{(k+1)}(x_p) - \overline{u}_{h,i}^{(k+1)}(x_{p+1}) \\ \qquad = h \max\limits_{\substack{\underline{u}_h^{(k)}(x_q) \le v_h(x_q) \le \overline{u}_h^{(k)}(x_q) \\ q=p-1,p,p+1}} \left\{-g_i(x_p; v_h(x_{p-1}), v_h(x_p), v_h(x_{p+1}))\right\}, \\ -\underline{u}_{h,i}^{(k+1)}(x_{p-1}) + 2\underline{u}_{h,i}^{(k+1)}(x_p) - \underline{u}_{h,i}^{(k+1)}(x_{p+1}) \\ \qquad = h \min\limits_{\substack{\underline{u}_h^{(k)}(x_q) \le v_h(x_q) \le \overline{u}_h^{(k)}(x_q) \\ q=p-1,p,p+1}} \left\{-g_i(x_p; v_h(x_{p-1}), v_h(x_p), v_h(x_{p+1}))\right\}, \\ \overline{u}_h^{(k+1)}(x) = \underline{u}_h^{(k+1)}(x) = 0, \qquad x = 0, 1 \end{cases} \tag{19}$$

with the initial values $\overline{u}_{h,i}^{(0)}(x_p) = \overline{u}_{h,i}(x_p)$ and $\underline{u}_{h,i}^{(0)}(x_p) = \underline{u}_{h,i}(x_p)$. We take $h = \frac{1}{20}$ and use (19) to solve (17). The numerical results show that the sequence $\left\{\overline{u}_h^{(k)}(x)\right\}$ is nonincreasing, and the sequence $\left\{\underline{u}_h^{(k)}(x)\right\}$ is nondecreasing. It agrees with the monotonicity described in Theorem 2. Furthermore by Theorem 5, both of them converge to the unique solution of problem (17) in $\mathbf{K}(\underline{u}_h, \overline{u}_h)$. In actual calculation, if

$$\max_i \max_p |\overline{u}_{h,i}^{(k+1)}(x_p) - \overline{u}_{h,i}^{(k)}(x_p)| < 10^{-5}, \tag{20}$$

then we take $\overline{u}_h^{(k+1)}(x)$ as the approximate solution of (16). The numerical results are listed in Table 1. Similarly, if (20) holds for $\underline{u}_{h,i}^{(k+1)}(x_p)$ and $\underline{u}_{h,i}^{(k)}(x_p)$, then we also take $\underline{u}_h^{(k+1)}(x)$ as the approximate solution of (16). The corresponding results are given in Table 2. Since the results are symmetric with respect to the central point, we only list the half results. Table 1 and Table 2 support the theoretical analysis in Theorem 5.

Next, we take $p_i(x)$ and α as before, and set $q_i(x)$=2, $i = 1, 2$. In this case, we get the same results as in the first example, for instance, the monotonicity of the sequences as described in Theorem 2. In addition, we find that the sequences $\left\{\overline{u}_h^{(k)}(x)\right\}$ and $\left\{\underline{u}_h^{(k)}(x)\right\}$ have the same limit and so it is the unique solution of the resulting problem in $\mathbf{K}(\underline{u}_h, \overline{u}_h)$. Whereas the condition of Theorem 5 is now destroyed. Thus the condition in Theorem 5 is only a sufficient condition.

The proof of Theorems 1-7 can be found in [6].

Table 1

	$N = 10$		$N = 30$	
x_p	$u_1(x_p)$	$u_2(x_p)$	$u_1(x_p)$	$u_2(x_p)$
0.1	0.014992	0.014992	0.014992	0.014992
0.2	0.026652	0.026652	0.026652	0.026652
0.3	0.034979	0.034979	0.034979	0.034979
0.4	0.039975	0.039975	0.039975	0.039975
0.5	0.041641	0.041641	0.041640	0.041640

Table 2

	$N = 10$		$N = 30$	
x_p	$u_1(x_p)$	$u_2(x_p)$	$u_1(x_p)$	$u_2(x_p)$
0.1	0.014992	0.014992	0.014992	0.014992
0.2	0.026652	0.026652	0.026652	0.026652
0.3	0.034979	0.034979	0.034979	0.034979
0.4	0.039975	0.039975	0.039975	0.039975
0.5	0.041641	0.041641	0.041640	0.041640

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