J. KSIAM Vol.2, No.1, 1-14, 1998

MULTIPLICITY RESULTS OF DOUBLY-PERIODIC SOLUTIONS FOR SEMILINEAR HEAT EQUATIONS

Wan Se Kim

Department of Mathematics
College of Natural Sciences
Hanyang University
Seoul 133-791,KOREA

Abstract. Multiplicity results of doubly-periodic solution for semilinear heat equations having coercive nonlinearity are discussed. Our proofs rely on Mawhin's continuation theorem
keyword: Semilinear heat equation, Multiplicity result

1 Introduction

Let Z, Z^+ and R be the set of all integers, non-negative interges and real numbers, respectively and let $\Omega = [0, 2\pi] \times [0, 2\pi]$.

Let $p \in [1, \infty)$, by $L^p(\Omega)$ we denote be the space of all measurable real-valued functions $u : \Omega \to R$ for which $|u(t, x)|^p$ is Lebesgue lntegrable over Ω with usual norm $\|\cdot\|_{L^p}$ given by

$$\|u\|_{L^p} = [\iint_\Omega |u(t, x)|^p dtdx]^{1/p} \tag{1.1}$$

In particular, let $L^2(\Omega)$ be the space of measurable real-valued functions $u : \Omega \to R$ which are Lebesgue square integrable over Ω with usual inner product $(,)$ and usual norm $\|\cdot\|_{L^2}$ and let $L^\infty(\Omega)$ be the space of measurable real-valued functions $u : \Omega \to R$ which are essentially bounded with usual essential norm $\|\cdot\|_{L^\infty}$.

AMS Subject Classification: 35k05, 35k60

Let $C^k(\Omega)$ be the space of all continous functions $u : \Omega \to R$ such that the partial derivatives up to order k with respect to both variables are continuous on Ω, while $C(\Omega)$ is used for $C^0(\Omega)$ with the usual norm $\|\cdot\|_\infty$ and we write $C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega)$.

Let $H^1(\Omega)$ be the complection of the space $C^\infty(\Omega)$ with respect to the norm given by

$$\|u\|_1 = [\iint_\Omega (|u(t,x)|^2 + |u_t(t,x)|^2 + |u_x(t,x)|^2) dtdx]^{1/2} \tag{1.1}$$

and $H^{1.2}(\Omega)$ be the completion of the space $C^\infty(\Omega)$ with respect to the norm given by

$$\|u\|_{1.2} = [\iint_\Omega (|u(t,x)|^2 + |u_t(t,x)|^2 + |u_x(t,x)|^2 + |u_{xx}(t,x)|^2) dtdx]^{1/2} \tag{1.2}$$

Note that $H^1(\Omega)(H^{1.2}(\Omega)$ has distributional derivatives $u_t, u_x \in L^2(\Omega)(u_t, u_x, u_{xx} \in L^2(\Omega))$ and these derivatives can be obtained as limit in $L^2(\Omega)$ of the corresponding derivatives of a sequence of $C^\infty(\Omega)$ functions which tend to u in $H^1(\Omega)$
$(H^{1.2}(\Omega))$. Moreover if derivatives are interpreted in distributional sence, the morms in $H^1(\Omega)$ and $H^{1.2}(\Omega)$ are given by (1.1) and (1.2), respectively, and $H^{1.2}(\Omega) \subseteq C(\Omega)$, and the embedding of $H^{1.2}(\Omega)$ in $C(\Omega)$ is continuous. Let $H^{0.1}(\Omega)$ be the closure in $H^1(\Omega)$ of all functions in $C^\infty(\Omega)$.

The purpose of this work is to investigate multiplicity results for weak doubly-periodic solutions of the semilinear heat equations of the form

$$u_t - u_{xx} + g(t,x,u) = h(t,x) \quad \text{in} \quad \Omega \tag{1.3}$$

where $u = u(t,x)$, $h \in L^2(\Omega)$ and $g : \Omega \times R \to R$ is a continuous function .

A *weak solution* of (1.3) will be $u \in H^{1.2}(\Omega) \cap H^{0.1}(\Omega)$ which satisfies the equation (1.3) a.e. on Ω and the boundary conditions

$$v(t,0) - v(t,2\pi) = v_x(t,0) - v_x(t,2\pi), t \in [0,2\pi]$$

$$v(0,x) - v(2\pi,x) = v_t(0,x) - v_t(2\pi,x), x \in [0,2\pi].$$

Let us remark that a necessary condition for (1.3) to have meaning is that g is such that $g(.,.,u) \in L^2(\Omega)$ when $u \in L^2(\Omega)$

Besides, g is a continuous function on $\Omega \times R$, we assume the following.

(H_1). There exist $a, b > 0$ such that

$$|g(u)| \leq a|u| + b \qquad \text{on} \quad R.$$

In this note, we will treat multiplicity results for semilinear heat equations having coercive groth nonlinearity. We will prove the multiplicity using the range of nonlinear term. Our results are based on Ambrosetti-Prodi type multiplicity which has been initiated by Ambrosetti-Prodi[1]. For more historical background ,results and references, we refer to Hirano and Kim[5,6] and Kim[10,11]. Our proof is based on Mawhin's continuation theorem[12].

2 Preliminary results

Now consider the doubly periodic problem of equation

$$u_t - u_{xx} = h(t,x) \tag{2.1}$$

where $u = u(t,x)$.

Define a linear operator $L : DomL \subseteq L^2(\Omega) \to L^2(\Omega)$ by

$$(Lu)(t,x) = u_t(t,x) - u_{xx}(t,x)$$

where $DomL = H^{1.2}(\Omega) \cap H^{0.2}(\Omega)$. Then DomL is dense in $L^2(\Omega)$, $KerL = R$

$$ImL = \{h \in L^2(\Omega) : \int\int_{\Omega} h(t,x)\, dt\, dx = 0\},$$

ImL is closed and

$$[KerL]^{\perp} = ImL.$$

Moreover, $L^2(\Omega) = KerL \bigoplus ImL$. Consider a continuous projection

$$P : L^2(\Omega) \to L^2(\Omega) \quad \text{such that} \quad ImL = KerP.$$

Then $L^2(\Omega) = KerL \bigoplus KerP$. We consider another continuous projection $Q : L^2(\Omega) \to L^2(\Omega)$ defined by

$$(Qh)(t,x) = \frac{1}{|[\Omega]|} \int\int_{\Omega} h(t,x)\, dt\, dx.$$

Then we have $L^2(\Omega) = ImQ \bigoplus ImL$, $KerQ = ImL$, and $L^2(\Omega)/ImL$ is isomorphic to ImQ.

Since $dim[L^2(\Omega)/ImL] = dim[ImQ] = dim[KerL] = 1$, we have an isomorphism $J : ImQ \to KerL$ and L is a Fredholm mapping of index 0. Moreover, we have easily the following lemma.

Lemma 2.1. *$L : DomL \subseteq L^2(\Omega) \to L^2(\Omega)$ is a closed operator.*

If $h \in L^2(\Omega)$, then u is a weak solution of (2.1) if and only if $u \in DomL$, $Lu = h$. L is not bijective but the restriction

$$L|_{DomL \bigcap ImL} : ImL \cap DomL \to ImL$$

is bijective, so we can define a right inverse

$$K^R = [L|_{DomL \bigcap ImL}]^{-1} : ImL \to ImL \cap DomL$$

and we can represent K^R as a convolution product

$$(K^R h)(t,x) = (K * h)(t,x) = \iint_\Omega K(t-s, x-y)h(s,x)dsdy$$

where $K(t,x) : \frac{1}{4\pi^2} \sum_{\substack{(l,m)\in Z\times Z \\ (l,m)\neq(0,0)}} [\beta li + (m^2 - l^2)]^{-1} \exp[i(lt + mx)]$. We have the following lemmas.

Lemma 2.2. *$DomL \cap ImL = K^R[ImL] \subseteq H^{1.2}(\Omega) \cap C(\Omega) \cap ImL$ and*

Proof. cf. [4,7].

Lemma 2.3. *The operator $K^R : ImL \to C(\Omega)$ is compact. If $h \in ImL$, then $\|K^R h\|_\infty \leq C_2 \|h\|_{L^2}$ for some constant $C_2 > 0$ independent of h.*

Proof. cf. [7] .

Now we can extend K^R an operator from $L^1(\Omega)$ into $L^2(\Omega)$ by defiing $\bar{K}^R : L^1(\Omega) \to L^2(\Omega)$ by the formular

$$(\bar{K}^R h)(t,x) = (K * h)(t,x) = \iint_\Omega K(t-s, x-y)h(s,x)dsdy$$

for $h \in L^1(\Omega)$. Then, by Holder's theorem inequality and Fubini's theorem, we have

Lemma 2.4. *If $h \in ImL$, then$\|\bar{K}^R h\|_{L^2} \leq \|\bar{K}\|_{L^2} \|h\|_{L^1}$.*

Proof. cf.[7]

3 Multiplicty Results

To treat our problem, let us consider the following doubly-periodic boundary value problem for a family of homotopy equation

$$(3.1_\lambda) \qquad u_t - u_{xx} + \lambda g(t,x,u) = \lambda h(t,x), \lambda \in [0,1],$$

where $g : \Omega \times R \to R$ is a continuous function and $h \in ImL$. Let $L : DomL \subseteq L^2(\Omega) \to L^2(\Omega)$ be defined as befor and define a substitution operator $N_\lambda : L^2(\Omega) \to L^2(\Omega)$

$$(N_\lambda)(t,x) = \lambda g(t,x,u) - \lambda h(t,x)$$

for $u \in L^2(\Omega)$ and $(t,x) \in \Omega$. By Krasnosel'skii's results, N_λ is continuous and bounded. Let G be any open bounded subset of $L^2(\Omega)$, then $QN : \bar{G} \to L^2(\Omega)$ is bounded and $K^R(I-Q) : \bar{G} \to L^2(\Omega)$ is compact and continuous. Thus, N_λ is L-compact on $\bar{G}$. The coincidence degree $D_L(L+N_\lambda, G)$ is well-defined and constant in λ if $Lu + N_\lambda \neq 0$ for $\lambda \in [0,1]$ and $u \in DomL \cap \partial G$. It is easy to check that (u,λ) is a weak doubly-periodic solution of (3.1_λ) if and only if $u \in DomL$ and

$$Lu + N_\lambda u = 0. \tag{3.2_λ}$$

Here we assume the following;

$$g(t,x,u) \geq 0 \quad \text{on} \quad \Omega \times R, \tag{H_2}$$

$$\lim_{|u| \to +\infty} g(t,x,u) = +\infty \quad \text{uniformly} \quad \text{on} \quad \Omega. \tag{H_3}$$

Lemma 3.1. *If (H_1) and (H_2) are satisfied, then there exists $M > 0$ such that*

$$\|\tilde{u}\|_{L^2} \leq M$$

holds for each possible weak doubly-periodic solution $u = \bar{u} + \tilde{u}$, with $\bar{u} \in KerL$ and $\tilde{u} \in ImL$, of (3.1_λ) where $\lambda \in [0,1]$.

proof. Let (u,λ) be any weak doubly-periodic solution of (3.1_λ). Then (u,λ) is a solution of (3.2_λ) where $u = \bar{u} + \tilde{u}$ with $\bar{u} \in KerL$ and $\tilde{u} \in ImL$. By applying $\bar{K}^R$ on the both sides of equation (3.2_λ),we have, since

$$\bar{K}^R_{|ImL} = K^R,$$

$$\tilde{u} = -\lambda \bar{K}^R N_\lambda u = \lambda \bar{K}^R[-g(\cdot,\cdot,u) + h(\cdot,\cdot)].$$

Hence, by Lemma 2.4,

$$\|\tilde{u}\|_{L^2} \leq \|K\|_{L^2}[\|g(\cdot,\cdot,u)\|_{L^1} + \|h\|_{L^1}].$$

By taking the inner product with 1 on the both sides of (3.2_λ), since $1 \in kerL$, we have

$$\iint_\Omega g(t,x,u(t,x))dtdx = \iint_\Omega h(t,x)dtdx.$$

Hence, by (H_2), we have $\|g(\cdot,\cdot,u)\|_{L^1} \leq \|h\|_{L^1}$. Therefore, we have

$$\|\tilde{u}\|_{L^2} \leq 2\|K\|_{L^2}\|h\|_{L^1} \equiv M.$$

The proof is complete.

Lemma 3.2. *If (H_1), (H_2) and (H_3) are satisfied, then there exists γ such that*

$$|\bar{u}| \leq \gamma$$

holds for each possible weak doubly-periodic solution $u = \bar{u} + \tilde{u}$, with $\bar{u} \in KerL$ and $\tilde{u} \in ImL$, of (3.1_λ) where $\lambda \in [0,1]$.

proof. Suppose there exist a sequence of weak doubly-periodic solutions $\{(u_n, \lambda_n)\}$ of (3.1_{λ_n}) with $\{|\bar{u}_n|\}$ is unbounded. Then (u_n, λ_n) is a solution of (3.2_{λ_n}) where $u_n = \bar{u}_n + \tilde{u}_n$ with $\bar{u}_n \in KerL$ and $\tilde{u}_n \in ImL$. We may choose a subsequence, say again $\{\bar{u}_n\}$ such that $|\bar{u}_n| \to +\infty$ as $n \to +\infty$. Now suppose that $\bar{u}_n \to +\infty$ as $n \to +\infty$. Let $M_0 > 2\pi M$ where M is given in Lemma 3.1 and let

$$\Omega_n = \{(t,x) | \tilde{u}_n(t,x) \leq -\frac{M_0}{4\pi^2}\}.$$

Then

$$\begin{aligned} 2\pi M &\geq \iint_\Omega |\tilde{u}_n(t,x)| dtdx \\ &\geq \iint_{\Omega_n} |\tilde{u}_n(t,x)| dtdx \\ &\geq [\frac{M_0}{4\pi^2}] |[\Omega_n]|. \end{aligned}$$

Therefore, $|[\Omega_n]| \leq 4\pi^2 \frac{2\pi M}{M_0}$ and hence $|[\Omega - \Omega_n]| = |\{(t,x) | u_n(t,x) > -\frac{M_0}{4\pi^2}\}| \geq 4\pi^2 [1 - \frac{2\pi M}{M_0}] > 0$.

Since $\lim_{|u| \to +\infty} g(t,x,u) = +\infty$ uniformly on Ω, there exists $C > 0$ such that

$$g(t,x,u) > \frac{1}{|[\Omega - \Omega_n]|} \iint_\Omega h(t,x) dtdx$$

for all n if $|u| \geq C$.
Since $\bar{u}_n \to +\infty$, there exists $N > 0$ such that

$$\bar{u}_n \geq \frac{M_o}{4\pi^2} + C \quad \text{if} \quad n \geq N.$$

Hence, for $(t,x) \in \Omega - \Omega_n$ and $n \geq N$, we have

$$u_n(t,x) = \bar{u}_n + \tilde{u}_n(t,x) \geq C.$$

Thus, for $n \geq N$, we have

$$\iint_{\Omega - \Omega_n} g(t,x,u_n(t,x)) dtdx > \iint_\Omega h(t,x) dtdx.$$

On the other hand, by taking the inner product with 1 on the both sides of (3.1_{λ_n}), we have

$$\iint_\Omega g(t,x,u_n(t,x))dtdx \le \iint_\Omega h(t,x)dtdx.$$

Therfore, for $n \ge N$, by (H_2),

$$\begin{aligned}\iint_\Omega h(t,x)dtdx &= \iint_\Omega g(t,x,u_n(t,x))dtdx \\ &\ge \iint_{\Omega-\Omega_n} g(t,x,u_n(t,x))dtdx \\ &> \iint_\Omega h(t,x)dtdx\end{aligned}$$

which is impossible.

Similary, we can treat the case where $\bar{u}_n \to -\infty$. The proof is complete.

Theorem 3.1. *Assume (H_1), (H_2), then the doubly-periodic boundary value problem on Ω for the equation (1.3) has at least two weak solutions if there exists a constant $r_0 \in R$ such that*

$$\iint_\Omega g(t,x,r_0+\tilde{u}(t,x))dtdx < \iint_\Omega h(t,x)dtdx \tag{3.3}$$

for every $\tilde{u} \in L^2(\Omega)$ having mean value zero on Ω, satisfying the doubly-periodic conditions and such that

$$\|\tilde{u}\|_{L^2} \le 2\|K\|_{L^2}\|h\|_{L^1}. \tag{3.4}$$

proof. To prove our multiplicity result, we construct two disjoint bounded open sets G_1 and G_2 on which the coincidence degree is well-defined and non-zero, respectively.

Since $\lim_{|u|\to\infty} g(t,x,u) = \infty$ uniformly on Ω, there exists $\delta > 0$ such that

$$g(t,x,u) > \iint_\Omega h(t,x)dtdx$$

for all $|u| > \delta$ and uniformly on Ω.

Let

$$G_1 = \{u \in L^2(\Omega) | r_0 < \bar{u} < \bar{r} + \bar{M}, \|\tilde{u}\|_{L^2} < \bar{M}\}$$

where $u = \bar{u} + \tilde{u}$ with $\bar{u} \in KerL, \tilde{u} \in ImL$ and $\bar{M}, \tilde{u}$ and $\bar{r}$ are constant such that $\bar{M} > M, \bar{r} > \max\{r, \delta\}$.

If $u \in \partial G_1$, then necessary $\bar{u} = r_0$ or $\bar{u} = \bar{r} + \bar{M}$ and if (u, λ) satisfies the equation (3.2_λ), then (u, λ) satisfies

$$\iint_\Omega g(t, x, u(t, x))dtdx = \iint_\Omega h(t, x)dtdx. \tag{3.5}$$

If $\bar{u} = r_0$, then, from (3.3), we have a contradiction. If $\bar{u} = \bar{r} + \bar{M}$, then $u = \bar{r} + \bar{M} + \tilde{u}$. let

$$\Omega_0 = \{(t, x) | |\tilde{u}(t, x)| \geq \bar{M}\}.$$

Then

$$\begin{aligned} 2\pi\bar{M} &\geq \iint_\Omega |\tilde{u}(t, x)|dtdx \\ &\geq \iint_{\Omega_0} |\tilde{u}(t, x)|dtdx \\ &\geq |[\Omega_0]|\bar{M}. \end{aligned}$$

Therefore $|[\Omega_0]| \leq 2\pi$ and hence $|[\Omega - \Omega_0]| = |[\{(t, x)||\tilde{u}(t, x)| < \bar{M}\}]| > 1$. Thus we have $|u| > \delta$ on $\Omega - \Omega_0$ and hence

$$\begin{aligned} \iint_\Omega g(t, x, u(t, x))dtdx &\geq \iint_{\Omega - \Omega_0} g(t, x, u(t, x))dtdx \\ &\geq |[\Omega - \Omega_0]| \iint_\Omega h(t, x)dtdx \\ &> \iint_\Omega h(t, x)dtdx \end{aligned}$$

which leads another contradiction. Therefore the coincidence degree $D_L(L - N, G_1)$ is well defined on Ω

Now, since, for $u \in KerL \cap \partial G_1$, we have $u = r_0$ or $u = \bar{r} + \bar{M}$, we conclude

$$(QN)(r_0) = \frac{1}{|[\Omega]|} \iint_\Omega [h(t, x) - g(t, x, r_0)]dtdx > 0,$$

$$(QN)(\bar{r} + \bar{M}) = \frac{1}{|[\Omega]|} \iint_\Omega [h(t, x) - g(t, x, \bar{r} + \bar{M})]dtdx < 0.$$

Hence, the coincidence degree exists and the corresponding value

$$D_L(L - N, G_1) = d_B[JQN, KerL \cap G_1, 0] = 1$$

where d_B is Brouwer degree (see, [12]). Therefore, the equation $(3, 2_1)$ has at least one solution in $DomL \cap Cl(G_1)$

Similary, we can prove that the equation $(3, 2_1)$ has at least one solution in $DomL \cap Cl(G_2)$ where

$$G_2 = \{u \in L^2(\Omega) | -(\bar{r} + \bar{M}) < \bar{u} < r_0, \|\tilde{u}\|_{L^2} < \bar{M}\}.$$

Since, by (3.3), $u \equiv r_0$ is not solution to (3.2_1) and $Cl(G_1) \cap Cl(G_2) = \{r_0\}$, the doubly-periodic boundary value problem to the equation (1.3) has at least two weak solutions.

Remark 3.1 If

$$\frac{1}{|[\Omega]|} \iint_\Omega h(t,x)dtdx < \inf_{\substack{(t,x)\in\Omega \\ u\in R}} g(t,x,u),$$

then the doubly-periodic boundary value problem for the equation (1.3) has no solution. Indeed, let

$$g(t_0, x_0, u_0) = \inf_{\substack{(t,x)\in\Omega \\ u\in R}} g(t,x,u)$$

Let F be any open bounded set in ImL such that

$$F \supseteq \{\tilde{u} \in ImL | \|\tilde{u}\|_{L^2} \bar{M}\}$$

and let, for any $\delta > 0$, $G = (u_0 - \delta, u_0 + \delta) \bigoplus F$. Suppose $u \in \partial G$ and (u, λ) satisfies the equation (3.2_λ), then (u, λ) satisfies $(3, 5)$. But $u = \bar{u} + \tilde{u}$ and

$$\iint_\Omega g(t, x, \bar{u} + \tilde{u})dtdx \geq |[\Omega]| \inf_{\substack{(t,x)\in\Omega \\ u\in R}} g(t,x,u) > \iint_\Omega h(t,x)dtdx$$

which contradicts to (3.5). Therefore the coincidence degree $D_L(L - N, G)$ is well-defined. But, for any $u \in KerL \cap G$,

$$(QN)(u) = \frac{1}{|[\Omega]|} \iint_\Omega [h(t,x) - g(t,x,u)]dtdx < 0$$

$$D_L(L - N, G) = d_B(JQN, KerL \bigcap G, 0) = 0.$$

Therefore the double-periodic boundary value problem to the equation (1.3) has no solution.

Next we consider multiplicity result for equation (1.3) when the nonlinear term $g(t,x,u)$ depends only on u, i.e.

$$u_t - u_{xx} + g(u) = h(t,x) \tag{3.6}$$

To treat our problem, let us consider the following doubly-periodic boundary value problom for a family of homotopy equations

$$u_t - u_{xx} + \lambda g(u) = \lambda h(t,x), \lambda \in [0.1] \tag{3.7_λ}$$

where $g : R \to R$ is continus and $h \in ImL$.

Let $L : DomL \subseteq L^2(\Omega) \to L^2(\Omega)$ be defined us before and define the substitution operator by

$$(N_\lambda)(t,x) = \lambda g(u(t,x)) - \lambda h(t,x)$$

for $u \in L^2(\Omega)$ and $(t,x) \in \Omega$. Then N_λ is L-compact on $\bar{G}$ for any open bounded subset of $L^2(\Omega)$, and u is a weak doubly-periodic solution to (3.7_λ) if and only if $u \in DomL$ and

$$Lu + N_\lambda u = 0. \tag{3.8_λ}$$

Here we assume the following.

(H_2') $\lim_{|u|\to+\infty} g(u) = +\infty$,

(H_3') there exists $0 < \alpha < 1$ such that

$$|g(u) - g(v)| \leq \frac{\alpha}{2\pi C_2}|u - v| \quad \text{for all} \quad u, v \in \mathbf{R},$$

where C_2 is a constant defined in Lemma 2.3.

Lemma 3.3. *If (H_1) is satisfied, then there exists $M > 0$ such that*

$$\|\tilde{u}\|_{L^2} \leq M$$

holds for each possible weak solution $u = \bar{u} + \tilde{u}$, with $\bar{u} \in KerL$ and $\tilde{u} \in ImL$, of (3.7_λ) where $\lambda \in [0.1]$.

Proof. Let (u, λ) be any weak solution of (3.7_λ) where $u = \bar{u} + \tilde{u}$ with $\bar{u} \in KerL$ and $\tilde{u} \in ImL$.

By taking the inner product with $\tilde{u}_t$ on the both sides of (3.7_λ), we have

$$(L\tilde{u}, \tilde{u}_t) + \lambda \iint_\Omega g(u)\tilde{u}_t dtdx = \lambda \iint_\Omega h(t,x)\tilde{u}_t dtdx.$$

Since $u \in DomL$, there exists a sequence $\{u_n\}$ in $C^\infty(\Omega)$, u_n satisfies boundary conditions, such that the distributional derivatives u_t, u_x, u_{xx} can be obtained as limit in $L^2(\Omega)$ of the corresponding derivatives of u_n which tend to u in $H^{1.2}(\Omega)$. Now the integration of these smooth functions, using boundary conditions, shows that for each $n \in Z^+$, $(Lu_n, u_{nt}) = \|u_{nt}\|_L^2$. Letting $n \to +\infty$, we have $(L\tilde{u}_t, \tilde{u}_t) = \|\tilde{u}_t\|^2_{L^2}$. Moreover, since, for each n, the periodicity of $\tilde{u}_n(t,x)$ in t implies $(g(u_n), \tilde{u}_{n_t}) = 0$, we have $(g(u), \tilde{u}_t) = 0$.

Hence, we have

$$\|\tilde{u}_t\|^2_{L^2} = \lambda(h, \tilde{u}_t)$$

and

$$\|\tilde{u}_t\|^2_{L^2} \leq \|h\|_{L^2}.$$

But since $\|\tilde{u}\|_{L^2} \leq \|\tilde{u}_t\|^2_{L^2}$ for all $\tilde{u} \in DomL \cap ImL$, we have

$$\|\tilde{u}\|_{L^2} \leq \|h\|_{L^2}.$$

The proof is complete.

Theorem 3.2. *Assume (H_1), (H_2'), and (H_3').Then the doubly-periodic boundary value problem on Ω for the equation* (3.6) *has at least two solutions if*

$$\inf_{\bar{u} \in R} \iint_\Omega g(\bar{u} + \tilde{u}(t,x))dtdx < \frac{1}{|[\Omega]|} \iint h(t,x)dtdx \tag{3.6}$$

for every $\tilde{u} \in L^2(\Omega)$ having mean value zero on Ω, satisfying the doubly-periodic conditions such that

$$\|\tilde{u}\|_{L^2} \leq \|h\|_{L^2}.$$

proof. It is easy to see that (3.7_1) is equivalent to

$$L\tilde{u} + (I - Q)g(\bar{u} + \tilde{u}) = h - \bar{h} \tag{3.8}$$

$$Qg(\bar{u} + \tilde{u}) = \bar{h} \tag{3.11}$$

where $u = \bar{u} + \tilde{u}$ with $\bar{u} \in KerL$ and $\tilde{u} \in ImL$, $\bar{h} = \frac{1}{|[\Omega]|} \iint_\Omega h(t,x)dtdx$ and Q is the continuous projection defined in section 2. For fixed $\bar{u} \in R$, consider the equation (3.10). Define an operator $N : L^2(\Omega) \to ImL$ by

$$(Nu)(t,x) = -(I - Q)g(\bar{u} + \tilde{u}(t,x)) + h(t,x).$$

Then N is continuous and maps bounded sets into bounded sets. Since the inclusion mapping $i : C(\Omega) \to L^2(\Omega)$ is continuous, the right inverse $K^R : ImL \to$

$L^2(\Omega)$ is compact. Hence $K^R N : L^2(\Omega) \to L^2(\Omega)$ is completely continuous and (3.10) is equivalent to

$$\tilde{u} = K^R N \tilde{u}.$$

By Lemma 3.3, all possible solutions to the family of equations

$$\tilde{u} = \lambda K^R N \tilde{u}, \quad \lambda \in [0,1]$$

are bounded in $L^2(\Omega)$ independently of $\lambda \in [0,1]$.

Thus, by Leray-schauder's theory, (3.10) has at least one solution $\tilde{u}$ for each $\bar{u} \in \mathbf{R}$. Such a solution is unique. Indead, if $\tilde{u}_1$ and $\tilde{u}_2$ are two different solutions with $\bar{u}$, then

$$L(\tilde{u}_1 - \tilde{u}_2) + (I - Q)[g(\bar{u} + \tilde{u}_1) - g(\bar{u} + \tilde{u}_2)] = 0.$$

Applying K^R on the both sides of the above equation, we have, by Lemma 2.3 and (H_3'),

$$\|\tilde{u}_1 - \tilde{u}_2\|_\infty \le \alpha \|\tilde{u}_1 - \tilde{u}_2\|_\infty$$

which is impossible since $0 < \alpha < 1$. Thus $\tilde{u}_1 = \tilde{u}_2$.

Denote this unique solution of (3.10) by $V(\bar{u})$, by Lemma 2.2,
then $V : R \to C(\Omega) \cap ImL$ is a continuous function.

If $\bar{u}, \bar{u}_0 \in R$, then

$$L[V(\bar{u}) - V(\bar{u}_0)] + (I - Q)[g(\bar{u} + V(\bar{u})) - g(\bar{u}_0 + V(\bar{u}_0))] = 0.$$

By Lemma 2.3 and (H_3'), we have

$$\|V(\bar{u}) - V(\bar{u}_0)\|_\infty \le \frac{\alpha}{1-\alpha} |\bar{u} - \bar{u}_0|.$$

Thus V is continuous.

By Lemma 3.3, $\|V(\bar{u})\|_{L^2} \le M$ for all $\bar{u} \in R$. Let

$$\Omega_0 = \{(t,x) | |V(\bar{u})(t,x)| \ge \frac{1+M}{2\pi}\},$$

then

$$M^2 \ge \iint_\Omega |V(\bar{u})(t,x)|^2 dtdx \ge [\frac{1+M}{2\pi}]^2 |[\Omega_0]|.$$

Thus

$$|[\Omega_0]| \le 4\pi^2 [\frac{1+M}{2\pi}]^2.$$

Let $\Omega_1 = \Omega - \Omega_0 = \{(t,x) : |V(\bar{u})(t,x)| \le \frac{1+M}{2\pi}\}$, $|[\Omega_1]| \ge 4\pi^2 [1 - \frac{M}{1+M}]^2 > 0$.

Thus

$$\begin{aligned}\iint_\Omega g(\bar u+V(\bar u)(t,x))dtdx &\geq \iint_\Omega [g(\bar u+V(\bar u)(t,x))-\beta]dtdx+4\pi^2\beta\\ &\geq \iint_{\Omega_1}[g(\bar u+V(\bar u)(t,x))-\beta]dtdx+4\pi^2\beta\end{aligned}$$

where $\beta=\min_{u\in R} g(u)$.
Therefore, by (H_2'),

$$\iint_\Omega g(\bar u+V(\bar u)(t,x))dtdx \to +\infty \quad \text{as} \quad |\bar u|\to+\infty.$$

Define $G:R\to R$ by

$$G(\bar u)=Qg(\bar u+V(\bar u))=\frac{1}{4\pi^2}\iint_\Omega g(\bar u+V(\bar u)(t,x))dtdx,$$

G is continuous by the continuity of V and, by (H_2'), $G(\bar u)\to+\infty$ as $|\bar u|\to+\infty$. Equation (3.5_λ) is then reduced to the scalar equation in $\bar u$;

$$G(\bar u)=Qg(\bar u+V(\bar u))=\bar h. \tag{3.12}$$

Let $h_1=\inf_{u\in R} G(\bar u)$, then $ImG=[h_1,+\infty[$.

If $G(u_0)=h_1$,then from (3.9), we may easly prove (3.6) has one solution in $]-\infty,\bar u_0[$ and one in $]\bar u_0,+\infty[$ by intermediate value theorem.
This completes the proof.

Remark 3.2 We may see easly that if $\bar h<h_1$, clearly (3.6) has no solution.

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