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THE DISCRETE SLOAN ITERATE FOR CAUCHY SINGULAR INTEGRAL EQUATIONS

SEKI KIM

DEPARTMENT OF MATHEMATICS,
SUNGKYUNKWAN UNIVERSITY, SUWON 440-746, KOREA
skkim@math.skku.ac.kr

Abstract. The superconvergence of the Sloan iterate obtained from a Galerkin method for the approximate solution of the singular integral equation based on the use of two sets of orthogonal polynomials is investigated. The discrete Sloan iterate using Gaussian quadrature to evaluate the integrals in the equation becomes the Nyström approximation obtained by the same rules. Consequently, it is impossible to expect the faster convergence of the Sloan iterate than the discrete Galerkin approximation in practice.

Key words. discrete Sloan iterate, Cauchy singular integral equation, Nyström approximation

AMS subject classification. 65R20, 45E05

1. Introduction.

$$\left(\begin{array}{c} \begin{bmatrix} x & Y \\ z & w \end{bmatrix} \\ a \\ b \end{array} \right)$$

Singular integral equations (*SIEs*) differ from Fredholm integral equations (*FIEs*) because they contain a Cauchy principal value integral. Because of this difference, the number of applicable numerical methods for *SIEs* is less than that for *FIEs.* Among numerical methods used for *FIEs,* the Sloan iterate is well-known to solve *FIEs* of the second kind with its superconvergence. It is taken as a better approximation than the Galerkin method as its general case [**?**, **?**]. Recently the Sloan iterate of projection method for solving *SIEs* has been considered without practical examples in a number of papers [**?**, **?**, **?**]. They suggest the Sloan iterate as a very useful method for accelerating the convergence of some projection methods for *SIEs.* However, its superconvergence was not shown in actual calculations. In the special case of *SIEs*, the airfoil equation which is the *SIEs* of the first kind, the Sloan iterate does not converge at the rate no faster than the original approximation [**?**]. Here we will extend this result to the *SIEs* with variable coefficients. This shows we may not obtain the superconvergence of the Sloan iterate for *SIEs.*

The paper is organized as follows: after recalling some preliminary results, in section 3 we consider the quadrature methods used in *SIEs* including the Hunter's method to evaluate numerically Cauchy principle value integrals, and derive some facts which are important in proving the main result. In section 4 we review the two sets of orthonormal polynomial-based Galerkin method for solving the *SIEs* with the order of convergence. Then we investigate the functional equation satisfied by the discrete Galerkin approximation, that is essential to obtain the discrete Sloan iterate. In the last section, we establish that the convergence rate of the Sloan iterate is exactly same to the rate of the Nyström approximation by examining the effects of numerical integration errors on the Galerkin solution.

2. Preliminaries. Consider the second kind singular integral equation on $(-1,1)$

$$\tilde{a}\,(x)\,\tilde{\mu}\,(x)+\frac{1}{\pi}\int_{-1}^{1}\frac{\eta(x,t)}{t-x}\tilde{\mu}(t)dt=\tilde{f}(x)\qquad -1<x<1, \tag{1}$$

with $\tilde{a}(x)$ and $\eta(x,t)$, real valued functions. We assume here that the kernel η is a Hölder continuous function on $[-1,1]\times[-1,1]$, and moreover a continuously differentiable function of the variable x. We now consider the function of a single variable $\eta\,(x,x)$, which is also Hölder continuous. Since it may have zeros at the points λ_i, $i=1,...,N_\lambda$, each respectively of multiplicity ζ_i, we can define the polynomial

$$\tilde{b}\,(x)=\prod_{i=1}^{N_\lambda}(x-\lambda_i)^{\zeta_i}\qquad\text{with}\qquad N=\sum_{i=1}^{N_\lambda}\varsigma_i \tag{2}$$

and consider the function

$$v\,(x)=\frac{\eta\,(x,x)}{\tilde{b}\,(x)} \tag{3}$$

which is of one sign, here taken to be positive, bounded near the endpoints ±1. Since η is Hölder continuous, $v(x)$ must be integrable in $[-1,1]$. After subtracting the singularity, the *SIE* then becomes

$$\tilde{a}\,(x)\,\tilde{\mu}\,(x)+\frac{\tilde{b}\,(x)}{\pi}\int_{-1}^{1}\frac{v\,(t)\,\tilde{\mu}\,(t)}{t-x}dt+\int_{-1}^{1}\tilde{k}\,(x,t)\,\tilde{\mu}\,(t)\,dt=\tilde{f}(x),\qquad -1<x<1. \tag{4}$$

where the coefficients $\tilde{a}$ and $\tilde{b}$ do not have common zeros [?]. Since $v\,(x)$ is one-signed, we can then normalize them introducing the function

$$r^2\,(x)\equiv\frac{\tilde{a}^2\,(x)}{v^2(x)}+\tilde{b}^2\,(x)>0,\quad\forall x\in[-1,1]\,,$$

after appropriate simplifying and finally by changing the dependent variable

$$\mu\,(x)=v\,(x)\,\tilde{\mu}\,(x)\,,$$

the *SIE* (**??**) then takes the standard form

$$a(x)\mu(x) + \frac{b(x)}{\pi}\int_{-1}^{1}\frac{\mu(t)}{t-x}dt + \int_{-1}^{1}k(x,t)\mu(t)\,dt = f(x), \qquad -1<x<1, \tag{5}$$

where we recall that $b(x)$ is a polynomial of degree N in (**??**).

We denote the space of Hölder continuous functions on $[-1,1]$ by H, but we seek the solution in the space denoted by H^*, see [**?**], of Hölder continuous functions on $(-1,1)$, such that near the endpoints their behavior is given by

$$\mu(x) = \frac{\sigma_{-1}(x)}{(1+x)^{\tilde{\xi}_{-1}}},\ \mu(x) = \frac{\sigma_1(x)}{(1-x)^{\tilde{\xi}_1}},\ 0 \le \tilde{\xi}_{-1}, \tilde{\xi}_1 < 1, \tag{6}$$

with $\sigma_{-1}(x)$ and $\sigma_1(x)$ also in H. Also the real valued function f is assumed to lie in H^*.

Now it is convenient to introduce L_Z and $L_{Z^{-1}}$ Hilbert spaces of real-valued square-integrable functions with weights Z and $1/Z$ respectively for Galerkin methods used in this study. The inner products on L_Z and $L_{Z^{-1}}$ are denoted by $<,>_Z$and $<,>_{Z^{-1}}$where

$$<f,g>_Z = \int_{-1}^{1} Z(t)f(t)g(t)dt \quad \text{ and } <f,g>_{Z^{-1}} = \int_{-1}^{1} Z^{-1}(t)f(t)g(t)dt,$$

and the induced norms by $||\ ||_Z$ and $||\ ||_{Z^{-1}}$. The symbol $||\ ||$ without subscript will be used for operator norms. In operator notation, equation (**??**) can be written as

$$\hat{S}\mu + \hat{K}\mu = f, \tag{7}$$

where the so-called dominant operator is

$$\hat{S}\mu(x) \equiv a(x)\mu(x) + \frac{b(x)}{\pi}\int_{-1}^{1}\frac{\mu(t)}{t-x}dt, \tag{8}$$

and

$$\hat{K}\mu(x) \equiv \int_{-1}^{1} k(x,t)\mu(t)\,dt \tag{9}$$

defines a compact operator from L_Z to $L_{Z^{-1}}$. We assume that $\hat{S}\mu + \hat{K}\mu = 0$ has only the zero solution so that $(\hat{S}+\hat{K})^{-1} : L_{Z^{-1}} \to L_Z$ is bounded by the Fredholm alternative. Following [**?**] we introduce the "regularizing" operator S^* on H^* by

$$S^*\rho(x) \equiv a(x)\rho(x) - \frac{b(x)Z(x)}{\pi}\int_{-1}^{1}\frac{\rho(t)}{Z(t)(t-x)}dt, \tag{10}$$

where $Z(x)$ represents the fundamental function of the *SIE*, see [**?**]. There is a relationship between the fundamental function and the canonical function $X(z)$ of the *SIE*, see e.g. [**?**]. Indeed letting

$$\theta(t) = -\frac{1}{2\pi i}\ln\frac{a(t)-ib(t)}{a(t)+ib(t)} = \frac{1}{\pi}\arctan\frac{b(t)}{a(t)} + N(t),$$

where the function $N(t)$ takes integer values, we have

$$\begin{aligned} X(z) &= (1-z)^{-\kappa}\exp\left\{-\int_{-1}^{1}\frac{\theta(t)\,dt}{t-z}\right\}, \quad z\notin[-1,1], \\ Z(x) &= (1-x)^{-\kappa}\exp\left\{-\int_{-1}^{1}\frac{\theta(t)\,dt}{t-x}\right\}, \quad x\in(-1,1). \end{aligned} \tag{11}$$

Here κ denotes the index of the equation, to be defined below. Observe that $Z(x)$ is positive and bounded in each closed subinterval of $(-1,1)$, since in our case it can be written as follows, [**?**]

$$Z(x) = (1-x)^{\gamma_1-\theta(1)}(1+x)^{\gamma_2+\theta(-1)}\omega(x), \tag{12}$$

with $-1<\gamma_1-\theta(1), \gamma_2+\theta(-1)<1$, and $\omega(x)$ a nonvanishing and bounded function. The integers γ_1 and γ_2 are related to its index, see [**?**, **?**, **?**],

$$\kappa = -(\gamma_1+\gamma_2).$$

Note that $Z\in H^*$and $1/Z\in H$.

The classical theory [**?**] shows that if $\kappa\geq 0$, the dominant equation (**??**) where $\hat{K}\equiv 0$, is solvable for any right hand side f. Uniqueness however is not guaranteed. To ensure it, we need the supplementary conditions

$$\frac{1}{\pi}\int_{-1}^{1} t^{\kappa-1-l}\mu(t)\,dt = C_l, \quad l=0,1,..,\kappa-1, \tag{13}$$

where C_l denote constants. For $\kappa<0$, the dominant equation has a unique solution if and only if f satisfies the $-\kappa$ orthogonality conditions

$$\int_{-1}^{1}\frac{t^l}{Z(t)}f(t)\,dt = 0, \qquad l=0,1,\ldots,-\kappa-1. \tag{14}$$

Let $\{\phi_n\}_{n=0}^{\infty}$ be the set of monic polynomials orthogonal with respect to the weight function Z, and let $\{\psi_n\}_{n=0}^{\infty}$ be the corresponding set of monic polynomials orthogonal with respect to $1/Z$. There is a special relationship between the two sets of orthogonal polynomials given by the dominant operator $\hat{S}$ of (**??**). Let us recall the following two results, Theorems 3.1 and 3.2 of [**?**]:

Lemma 2.1. *Let Q, R be functions such that*
(i) $QX-R$ is analytic in the deleted complex plane and zero at infinity.
(ii) on $(-1,1)$, $Q^+(x)=Q^-(x)$, $R^+(x)=R^-(x)$ and the functions $aQZ-R$, bQZ are in the Hölder space H^*. Then for $-1<x<1$,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{b(t)\,Q(t)\,Z(t)}{t-x}\,dt = -a(x)\,Q(x)\,Z(x) + R(x).$$

LEMMA 2.2. *Let Q, R be functions such that*

(i) $QX^{-1} - R$ is analytic in the deleted complex plane and zero at infinity.

(ii) on $(-1,1)$, $Q^{+}(x) = Q^{-}(x)$, $R^{+}(x) = R^{-}(x)$ and the functions $aQ/Z - R$, bQ/Z are in the Hölder space H^{*}. Then for $-1 < x < 1$,

$$\frac{1}{\pi}\int_{-1}^{1} \frac{b(t)\, Q(t)}{Z(t)(t-x)}\, dt = \frac{a(x)\, Q(x)}{Z(x)} - R(x).$$

For our analysis the important tool is given by Theorem 9.14 of [?]. See also theorems 3.2 and 3.4 of [?]. In our notation, recalling that $N = \deg b(x)$, (??), it reads

THEOREM 2.3. *Let p_n be a polynomial of degree n. Then the function q_n defined by $q_n = \left(aZI + b\hat{S}ZI\right) p_n$ is a polynomial of degree at most* $\max\{n-\kappa, N-1\}$ *and of degree $n-\kappa$ if $n-\kappa > N-1$. If p_n is an orthogonal polynomial of degree n with respect to the weight Z and if $n-\kappa > N-1$, then $q_n = \left(aZI + b\hat{S}ZI\right) p_n$ is an orthogonal polynomial of degree $n-\kappa$ with respect to the weight* $1/Z$. *Moreover if $n-\kappa > N-1$ then*

$$\|p_m\|_Z = \|q_m\|_{Z^{-1}}.$$

From now on assume that $n-\kappa > N-1$. Recall that κ is the index of $\hat{S}$. By applying the above result to the two families of monic polynomials $\{\phi_n\}_{n=0}^{\infty}$ and $\{\psi_n\}_{n=0}^{\infty}$, orthogonal with respect to the weights Z and Z^{-1} respectively, we have

$$a(x)\phi_n(x)Z(x) + \frac{b(x)}{\pi}\int_{-1}^{1} \frac{Z(t)\phi_n(t)}{t-x}\, dt = (-1)^{\kappa}\, \psi_{n-\kappa}(x), \tag{15}$$

$$\frac{a(x)\psi_n(x)}{Z(x)} - \frac{b(x)}{\pi}\int_{-1}^{1} \frac{\psi_n(t)}{Z(t)(t-x)} dt = (-1)^{\kappa}\, \phi_{n+\kappa}(x), \tag{16}$$

where $n = \kappa + N, \kappa + N + 1, \ldots$. Also $\phi_{-1} = \psi_{-1} \equiv 0$.

Note that two sets of polynomials mentioned above can be taken normalized because the operator $\hat{S}$ is unitary if the index of $\hat{S}$ is zero [?, ?].

In what follows we will consider the polynomial $b(x)$ evaluated at the zeros t_i and s_j of the orthogonal polynomials ϕ_n and $\psi_{n-\kappa}$ respectively. Since b has N zeros, we can always assume that by taking n large enough, the following conditions are satisfied

$$b(t_i) \neq 0,\ i = 1, ..., n;\quad b(s_j) \neq 0,\ j = 1, ..., n-\kappa. \tag{17}$$

3. Quadrature Methods in SIE. To set up a numerical method we define a new unknown u using the fundamental function Z

$$\mu(t) = Z(t)u(t). \tag{18}$$

Indeed, the function $Z(t)$ contains the "bad features" of the unknown, and $u(t)$ is smooth. With this notation, we introduce simple operators by

$$\begin{aligned} Su &= \hat{S}Zu = \hat{S}\mu \\ Ku &= \hat{K}Zu = \hat{K}\mu. \end{aligned} \tag{19}$$

It is also necessary to introduce the new function Λ_n associated with ϕ_n

$$\Lambda_n(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{Z(t)\,\phi_n(t)}{t-x}\,dt. \tag{20}$$

As quadrature formula we use Hunter's method

$$\frac{1}{\pi}\int_{-1}^{1} \frac{Z(t)\,u(t)}{t-x}dt = \frac{1}{\pi}\sum_{i=1}^{n} \frac{w_i\,u(t_i)}{t_i-x} + \frac{\Lambda_n(x)}{\phi_n(x)}u(x) + \epsilon_H \tag{21}$$

where the w_i's are the Gaussian quadrature weights associated with the weight Z, the t_i's are the zeros of ϕ_n and ϵ_H represents the error term

$$\epsilon_H = \int_{-1}^{1} Z(t)\,R_n(x,t)\ \phi_n(t)\ dt,\ \ R_n(x,t) = \frac{1}{2\pi i}\int_C \frac{u(z)\,dz}{(z-x)\,(z-t)\ \phi_n(z)}\,, \tag{22}$$

where C denotes a contour in the complex plane, enclosing the interval $[-1,1]$. In this notation (**??**) becomes

$$b(x)\,\Lambda_n(x) = (-1)^{\kappa}\,\psi_{n-\kappa}(x) - a(x)\ Z(x)\ \phi_n(x) \tag{23}$$

where κ is the index of the singular integral equation. Note that (**??**) is exact when u is a polynomial of degree less than $2n+1$, see [**?**]. In a similar way by introducing the function

$$\tilde{\Lambda}_n(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\psi_n(t)}{Z(t)\,(t-x)}\,dt, \tag{24}$$

we can also write the "dual form" of Hunter's quadrature (**??**), which is obtained by using the weight function $1/Z$ in place of Z, the weights w_j^* in place of w_j and where the polynomials $\psi_{n-\kappa}(x)$ and their zeros replace $\phi_n(x)$ and their zeros, and similar changes take place in the error term $\tilde{\epsilon}_H$,

$$\frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{Z(t)\,(t-x)}dt = \frac{1}{\pi}\sum_{j=1}^{n-\kappa} \frac{w_j^*\,u(s_j)}{s_j-x} + \frac{\tilde{\Lambda}_{n-\kappa}(x)}{\psi_{n-\kappa}(x)}u(x) + \tilde{\epsilon}_H. \tag{25}$$

Shifting the indices in (**??**) and rewriting it in this notation

$$-b(x)\,\tilde{\Lambda}_{n-\kappa}(x) = (-1)^{\kappa}\,\phi_n(x) - \frac{a(x)}{Z(x)}\,\psi_{n-\kappa}(x)\,. \tag{26}$$

For $i = 1, 2, \ldots, n$, the Christoffel number w_i associated with the zeros t_i of ϕ_n, from (3.4.3) in [?] is

$$w_i = \int_{-1}^{1} \frac{Z(t)\ \phi_n(t)}{\phi_n'(t_i)\ (t - t_i)} dt = \pi \frac{\Lambda_n(t_i)}{\phi_n'(t_i)}. \tag{27}$$

It follows then from (??)

$$b(t_i)\, w_i = \frac{\pi}{\phi_n'(t_i)} \left[(-1)^{\kappa} \psi_{n-\kappa}(t_i) - a(t_i)\, Z(t_i)\, \phi_n(t_i)\right] \equiv \pi\,(-1)^{\kappa} \frac{\psi_{n-\kappa}(t_i)}{\phi_n'(t_i)}. \tag{28}$$

Similarly from (??) the value for the Christoffel numbers w_j^* associated with the zeros s_j of the orthogonal polynomials $\psi_{n-\kappa}$ is

$$-b(s_j)\, w_j^* \equiv -b(s_j)\, \pi \frac{\tilde{\Lambda}_{n-\kappa}(s_j)}{\psi_{n-\kappa}'(s_j)} = \pi\,(-1)^{\kappa} \frac{\phi_n(s_j)}{\psi_{n-\kappa}'(s_j)} \quad \text{for } j = 1, 2, \ldots, n-\kappa. \tag{29}$$

Recall that the Christoffel numbers are all positive [?]. Using this fact with (??), (??) and (??), it follows then that $\{t_i\} \cap \{s_j\} = \emptyset$.

Recalling (??) and (??), since $t_i \in (-1, 1)$, we define $K_n u$ by discretizing Ku by means of a Gaussian quadrature with weight $Z(t)$ as follows

$$Ku(x) = \int_{-1}^{1} Z(t)\, k(x,t)\, u(t)\ dt \simeq \sum_{i=1}^{n} w_i k(x, t_i)\, u(t_i) \equiv K_n u(x). \tag{30}$$

We now consider the regularized equation. Recalling (??), the *FIE* equivalent to the *SIE* (??) is given by (107.15) of [?],

$$\mu(x) + S^* K \mu(x) = S^* f(x) + Z(x)\, b(x)\, \tilde{p}_{\kappa-1}(x), \tag{31}$$

where $\tilde{p}_{\kappa-1}$ represents an arbitrary polynomial of degree not greater than $\kappa - 1$, which is identically zero for nonpositive index.

We need some results which are used for last part of the paper. From [?], the canonical function $X(z)$ and its reciprocal have the expansions

$$X(z) = (-1)^{\kappa} \sum_{j=-\infty}^{-\kappa} \alpha_j\, z^j \qquad \text{and} \qquad X^{-1}(z) = (-1)^{\kappa} \sum_{j=-\infty}^{\kappa} \beta_j\, z^j, \tag{32}$$

where the coefficients α_j are given by

$$\alpha_j = \frac{1}{\pi} \int_{-1}^{1} \tau^{-j-1} b(\tau)\, Z(\tau)\ d\tau \qquad \text{for} \quad j \le \min(-\kappa, -1). \tag{33}$$

The other coefficients can be obtained from $X\, X^{-1} \equiv 1$, which gives

$$\sum_{l=0}^{-j} \alpha_{-\kappa+j+l}\, \beta_{\kappa-l} = \begin{cases} 0 & j < 0 \\ 1 & j = 0. \end{cases} \tag{34}$$

Note that the canonical function for large x behaves like $(1-x)^{-\kappa}$, (**??**). Then

$$X(x) \sim (-1)^{\kappa} x^{-\kappa},$$

and from this we have $\alpha_{-\kappa} = 1$, so that $\beta_\kappa = 1$ as well. We define the polynomial $\hat{p}_\kappa(x)$ as follows

$$\hat{p}_\kappa(x) = pp\left((-1)^{\kappa} X^{-1}\right) = \sum_{i=0}^{\kappa} \beta_i x^i. \tag{35}$$

Moreover in (2.11) of [**?**] it is observed that the function R of lemma **??** represents the principal part of the canonical function, so that

$$R(x) = \hat{p}_\kappa(x). \tag{36}$$

Theorem 3.4.

$$\frac{1}{\pi}\sum_{j=1}^{n-\kappa} \frac{w_j^* b(s_j)}{(t_l - s_j)(t_i - s_j)} + \hat{p}_\kappa[t_i, t_l] = \begin{cases} 0 & i \neq l \\ \dfrac{\pi}{b(t_l)\, w_l} & i = l \end{cases}$$

where the last term represents the first divided difference of a certain polynomial $\hat{p}_\kappa(x)$, of degree κ, which for $\kappa = 0$ becomes identically 0*.*

Proof. Using partial fractions, if $l \neq i$

$$\frac{1}{\pi}\sum_{j=1}^{n-\kappa} \frac{w_j^* b(s_j)}{(t_l - s_j)(t_i - s_j)} = \frac{1}{\pi(t_i - t_l)} \sum_{j=1}^{n-\kappa} w_j^* b(s_j)\left[\frac{1}{t_l - s_j} - \frac{1}{t_i - s_j}\right]. \tag{37}$$

Now we need to evaluate the right hand side of (**??**). Since $b(x)$ is a polynomial of fixed degree N, for n large enough the above quadrature is exact. The second term in the right hand side of (**??**) can now be rewritten using (**??**), while for the term on the left hand side we can use lemma **??** with $Q \equiv 1$ to get

$$\begin{aligned} \frac{1}{\pi}\sum_{j=1}^{n-\kappa} \frac{w_j^* b(s_j)}{s_j - x} &= \frac{a(x)}{Z(x)} - R(x) + \frac{1}{\psi_{n-\kappa}(x)}\left[(-1)^{\kappa}\phi_n(x) - \frac{a(x)}{Z(x)}\psi_{n-\kappa}(x)\right] \\ &= -R(x) + \frac{(-1)^{\kappa}\phi_n(x)}{\psi_{n-\kappa}(x)}. \end{aligned} \tag{38}$$

Use of (**??**) and collocation of (**??**) at the nodes t_l yields

$$\frac{1}{\pi}\sum_{j=1}^{n-\kappa} \frac{w_i^* b(s_j)}{s_j - t_l} = -\hat{p}_\kappa(t_l). \tag{39}$$

On using this result twice in (**??**) we have the first claim. For the case, $i = l$, we start by differentiating (**??**) with respect to x, to get

$$\frac{1}{\pi}\sum_{j=1}^{n-\kappa} \frac{w_j^* b(s_j)}{(s_j - x)^2} = -R'(x) + (-1)^{\kappa}\left[\frac{\phi_n'(x)}{\psi_{n-\kappa}(x)} - \frac{\psi_{n-\kappa}'(x)\,\phi_n(x)}{\psi_{n-\kappa}(x)^2}\right].$$

Upon collocation at t_l, we have then

$$\frac{1}{\pi}\sum_{j=1}^{n-\kappa}\frac{w_j^* b\left(s_j\right)}{\left(s_j-t_l\right)^2}=-\hat{p}_\kappa'\left(t_l\right)+(-1)^\kappa\,\frac{\phi_n'\left(t_l\right)}{\psi_{n-\kappa}\left(t_l\right)}.$$

The claim follows then using equation (**??**). ∎

We can now characterize the coefficients of the arbitrary polynomial $\tilde{p}_{\kappa-1}\left(x\right)$ of (**??**) by using the supplementary normalization conditions (**??**).

PROPOSITION 3.5. *We have*

$$\tilde{p}_{\kappa-1}\left(x\right)=\sum_{i=0}^{\kappa-1}\xi_i\,x^i$$

where

$$\xi_i=C_i+\sum_{l=1}^{\kappa-1-i}\beta_{\kappa-l}\,C_{i+l},$$

*and the C_i's are given by (**??**).*

Proof. From the proof of Theorem 4.4 of [**?**] and by taking into account limits of sequences of functions, Theorem 5.3 of [**?**], $\tilde{p}_{\kappa-1}\left(x\right)$ has the following coefficients. For $0\le i\le\kappa-1$, on using (**??**)

$$\begin{aligned}\xi_i &= \frac{1}{\pi}\int_{-1}^{1}\tau^{\kappa-1-i}\,\phi\left(\tau\right)\,d\tau-\sum_{l=1}^{\kappa-1-i}\xi_{i+l}\,\frac{1}{\pi}\int_{-1}^{1}\tau^{\kappa-1+l}\,b\left(\tau\right)\,Z\left(\tau\right)\,d\tau \\ &= C_i-\sum_{l=1}^{\kappa-1-i}\xi_{i+l}\,\alpha_{-\kappa-l}.\end{aligned}\tag{40}$$

Introduce the vectors $\boldsymbol{\xi}=\left(\xi_0,\xi_1,\ldots,\xi_{\kappa-1}\right)^T$, $\mathbf{C}=\left(C_0,C_1,\ldots,C_{\kappa-1}\right)^T$and the $\kappa\times\kappa$ matrix $A=\left(A_{ij}\right)$with entries

$$A_{ij}=\begin{cases}0 & i>j\\ 1 & i=j\\ \alpha_{-\kappa-j+i} & i<j.\end{cases}$$

To get ξ_i, rearrange (**??**) and solve the triangular system $A\,\boldsymbol{\xi}=\mathbf{C}$. The triangular matrix A is nonsingular since it has nonzero diagonal elements. Hence the inverse matrix $A^{-1}=B=\left(B_{ij}\right)$ exists. By considering (**??**), we easily find its elements

$$B_{ij}=\begin{cases}0 & i>j\\ 1 & i=j\\ \beta_{-\kappa-j+i} & i<j.\end{cases}\tag{41}$$

We have then $\boldsymbol{\xi} = B\,\mathbf{C}$, and use of (**??**) leads to the claim. □

Remark. These proofs in a certain sense use a "dual" argument of the one used in the proof of theorem 2.1 of [**?**].

We consider the case only the index $\kappa = 0$ for the remainder of the paper. In this case, the regularizing operator S^* becomes the inverse S^{-1} of the dominant operator S because $\hat{p}_{\kappa-1}$ becomes zero in (**??**).

4. The Galerkin Scheme. Approximating u by a polynomial u_n of degree $\leq n$ and setting the residual $r_n = Su_n + Ku_n - f$ orthogonal to SU_n where U_n is the set of polynomials of degree $\leq n$,which has a basis $\{\phi_l\}_{l=0}^{n}$,

$$u_n = \sum_{l=0}^{n} \varkappa_l \phi_l,$$

we obtain the linear equation to determine $\{\varkappa_l\}_{l=0}^{n}$,

$$\sum_{l=0}^{n} < S\phi_l, S\phi_j >_{Z^{-1}} \varkappa_l + \sum_{l=0}^{n} < K\phi_l, S\phi_j >_{Z^{-1}} \varkappa_l =< f, S\phi_j >_{Z^{-1}} \tag{42}$$

where $\{S\phi_l\}_{l=0}^{n}$ are orthonormal with respect to $1/Z$, (**??**) because $\kappa = 0$.

Let P_n be the operator of orthogonal projection onto $V_n = SU_n$. Then u_n satisfies

$$P_n S u_n = -P_n K u_n + P_n f. \tag{43}$$

Since $Su_n \in V_n$, $P_n S u_n = S u_n$,

$$S u_n = -P_n K u_n + P_n f. \tag{44}$$

From [**?**, **?**, **?**], the unique existence of u_n for n large enough and the convergence of u_n to u in L_Z are obtained with

$$||u - u_n||_Z \leq C_g\, ||Su - P_n S u||_{Z^{-1}}$$

where C_g is constant. For f and $k \in C^{r,\alpha}$ which is the Hölder space of order $0 \leq \alpha < 1$ for the rth derivative and $r > 0$,

$$||u - u_n||_Z = O(n^{-(r+\alpha)}). \tag{45}$$

Discretizing (**??**) by n point quadrature rules with the weight Z^{-1} and (**??**), we obtain v_n, called a discrete Galerkin approximation to u, as the solution of the resulting functional equations and v_n is given by

$$v_n = \sum_{l=0}^{n} \tilde{\varkappa}_l \phi_l, \tag{46}$$

where the $\{\tilde{\varkappa}_l\}$ satisfies

$$\begin{aligned}(47)\qquad & \sum_{l=0}^{n}\left[\sum_{d=1}^{n} w_d^* \psi_l(s_d)\psi_j(s_d)\right]\tilde{\varkappa}_l \\ & +\sum_{l=0}^{n}\left[\sum_{d=1}^{n}\sum_{p=1}^{n} w_d^* w_p k(s_d,t_p)\psi_j(s_d)\phi_l(t_p)\right]\tilde{\varkappa}_l \\ = & \sum_{d=1}^{n} w_d^* f(s_d)\psi_j(s_d) \qquad \text{for } 0\le j\le n\end{aligned}$$

where the first term becomes $\tilde{\varkappa}_j||\psi_j||^2_{Z^{-1}}=\tilde{\varkappa}_j$ because of the precision of the quadrature rule.

For the convergence of the discrete Galerkin approximation v_n to u, we refer the following theorem[**?**, **?**].

THEOREM 4.6. *For n large enough, the discrete Galerkin approximation v_n exists uniquely and converge to u in L_Z if f and $k\in C^{r,\alpha}$, $r+\alpha>5/2$. Furthermore $||u-v_n||_Z=O(n^{-(r+\alpha)+1})$, and $||u-v_n||_\infty=O(n^{-(r+\alpha)+5/2})$.*

Let us define $\Pi_n: C[-1,1]\to V_n$ by

$$(48)\qquad \Pi_n u(x)=\sum_{k=0}^{n}\left[\sum_{j=1}^{n} w_j^* u(s_j)\psi_k(s_j)\right]\psi_k(x).$$

Then using (**??**) and the fact $\kappa=0$, we have

$$(49)\qquad Sv_n(x)=S\left[\sum_{k=0}^{n}\tilde{\varkappa}_k\phi_k(x)\right]=\sum_{k=0}^{n}\tilde{\varkappa}_k\psi_k(x)$$

and

$$(50)\qquad \Pi_n f(x)=\sum_{k=0}^{n}\left[\sum_{d=1}^{n} w_d^* f(s_d)\psi_k(s_d)\right]\psi_k(x).$$

Similarly

$$\begin{aligned}(51)\quad \Pi_n K_n v_n(x) &= \Pi_n\left[\sum_{p=1}^{n} w_p k(x,t_p)v_n(t_p)\right] \\ &= \sum_{p=1}^{n} w_p v_n(t_p)\Pi_n k(x,t_p) \\ &= \sum_{p=1}^{n} w_p v_n(t_p)\left[\sum_{k=0}^{n}\left[\sum_{d=1}^{n} w_d^* k(s_d,t_p)\psi_k(s_d)\right]\psi_k(x)\right] \\ &= \sum_{p=1}^{n} w_p\left(\sum_{l=0}^{n}\tilde{\varkappa}_l\phi_l(t_p)\right)\left[\sum_{k=0}^{n}\left[\sum_{d=1}^{n} w_d^* k(s_d,t_p)\psi_k(s_d)\right]\psi_k(x)\right] \\ &= \sum_{k=0}^{n}\left[\sum_{l=0}^{n}\left[\sum_{p=1}^{n}\sum_{d=1}^{n} w_d^* w_p k(s_d,t_p)\psi_k(s_d)\phi_l(t_p)\right]\tilde{\varkappa}_l\right]\psi_k(x).\end{aligned}$$

Here

$$\begin{aligned}
& Sv_n(x) + \Pi_n K_n v_n(x) - \Pi_n f(x) \\
= & \sum_{k=0}^{n} \left[\tilde{\varkappa}_k + \sum_{l=0}^{n} \left(\sum_{p=1}^{n} \sum_{d=1}^{n} w_d^* w_p k(s_d, t_p) \psi_k(s_d) \phi_l(t_p) \right) \tilde{\varkappa}_l - \sum_{d=1}^{n} w_d^* f(s_d) \psi_k(s_d) \right] \psi_k(x) \\
= & \ 0
\end{aligned}$$

since the inside term of the middle is exactly same to (**??**). Consequently, the discrete Galerkin approximation v_n satisfies

$$Sv_n + \Pi_n K_n v_n - \Pi_n f = 0. \tag{52}$$

5. The Sloan Iterate. In this section, we introduce the Sloan iterate [**?**, **?**, **?**] $\tilde{u}_n$ of u_n given by

$$S\tilde{u}_n = -Ku_n + f \tag{53}$$

where

$$\tilde{u}_n = -S^{-1}Ku_n + S^{-1}f.$$

Here applying the orthogonal projection operator P_n to (**??**), we have

$$P_n S\tilde{u}_n = -P_n K u_n + P_n f = P_n S u_n = S u_n.$$

Therefore $u_n = S^{-1}P_n S\tilde{u}_n = Q_n\tilde{u}_n$ where $Q_n = S^{-1}P_n S$ is the orthogonal projection operator onto U_n. Thus $\tilde{u}_n$ satisfies

$$S\tilde{u}_n = -KQ_n\tilde{u}_n + f. \tag{54}$$

LEMMA 5.7. *The equation (**??**) has a unique solution for all n large enough.*

Proof. Because Q_n is orthogonal, $||K - KQ_n|| \to 0$ as $n \to \infty$. Letting $T_n = S + KQ_n$ and $T = S + K$,we have $M = T - T_n = K - KQ_n$ and

$$\begin{aligned}
T_n &= T_n - T + T = T - (T - T_n) \\
&= T - M \\
&= T(I - T^{-1}M)\ .
\end{aligned}$$

Here $||T^{-1}M|| < ||T^{-1}||||K - KQ_n|| < 1/2$ for n large enough. Hence $T_n^{-1} = (S + KQ_n)^{-1}$ exists since $(I - T^{-1}M)^{-1}$ exists. ∎

For n sufficiently large,

$$\begin{aligned}
u - \tilde{u}_n &= (S+K)^{-1}f - (S+KQ_n)^{-1}f \\
&= (S+KQ_n)^{-1}(KQ_n - K)(S+K)^{-1}f \\
&= (S+KQ_n)^{-1}(KQ_n - K)u \\
&= (S+KQ_n)^{-1}K(Q_n - I)u \\
&= (S+KQ_n)^{-1}K(Q_n - I)(Q_n - I)(u - u_n) \\
&= (S+KQ_n)^{-1}(KQ_n - K)(u - u_n)
\end{aligned} \tag{55}$$

since $(Q_n - I)^2 = (Q_n - I)$ and $(I - Q_n)u_n = 0$. From (**??**), $||u - u_n||_Z = O(n^{-(r+\alpha)})$if k and f are $C^{r,\alpha}$. Also Jackson's theorem and the orthogonality of Q_n give $||K - KQ_n|| = O(n^{-(r+\alpha)})$for $k \in C^{r,\alpha}$. With these facts and (**??**), we have the following theorem.

THEOREM 5.8. *Let u_n be the solution of Galerkin method for n sufficiently large and the Sloan iterate $\tilde{u}_n$ be defined by (**??**). Then $\tilde{u}_n$ converges in L_Z to u and $||u - \tilde{u}_n||_Z = O(n^{-2(r+\alpha)})$if k and f are $C^{r,\alpha}$ functions.*

This theorem shows that the Sloan iterate converges at twice the rate of the Galerkin approximation if all integrals are evaluated exactly. This order of convergence can be obtained only theoretically.

In the remainder of the paper, the discrete Sloan iterate will be defined and compared with the discrete Galerkin approximation.

Ignoring the error term ϵ_H in (**??**) with (**??**), we define

$$
\begin{aligned}
S_n u(x) &\equiv a(x)Z(x)u(x) + b(x)\left[\frac{1}{\pi}\sum_{i=1}^{n}\frac{w_i u(t_i)}{t_i - x} + \frac{\Lambda_n(x)}{\phi_n(x)}u(x)\right] \\
&= a(x)Z(x)u(x) + \frac{b(x)}{\pi}\sum_{i=1}^{n}\frac{w_i u(t_i)}{t_i - x} \\
&\quad + \frac{u(x)}{\phi_n(x)}\left[(-1)^{\kappa}\psi_{n-\kappa}(x) - a(x)Z(x)\phi_n(x)\right] \\
&= \frac{b(x)}{\pi}\sum_{i=1}^{n}\frac{w_i u(t_i)}{t_i - x} + (-1)^{\kappa}\frac{\psi_{n-\kappa}(x)}{\phi_n(x)}u(x).
\end{aligned} \tag{56}
$$

And for the discrete Sloan iterate, we now define $\hat{v}_n$ by

$$S_n\hat{v}_n = -K_n v_n + f.$$

Applying the operator Π_n with (**??**), then

$$
\begin{aligned}
\Pi_n S_n \hat{v}_n &= -\Pi_n K_n v_n + \Pi_n f \\
&= S v_n.
\end{aligned}
$$

Hence $v_n = S^{-1}\Pi_n S_n \hat{v}_n$ so that

$$S_n\hat{v}_n + K_n S^{-1}\Pi_n S_n \hat{v}_n = f. \tag{57}$$

Now we give the main result showing that the discrete Sloan iterate becomes the Nyström quadrature approximation [**?**, **?**].

THEOREM 5.9. *The discrete Sloan iterate $\hat{v}_n$ of (**??**) also satisfies*

$$S_n\hat{v}_n + K_n\hat{v}_n = f$$

which is the equation representing the Nyström approximation of u.

Proof. Letting $h_n = S^{-1}\Pi_n S_n \hat{v}_n$, we have

$$\begin{aligned} K_n S^{-1}\Pi_n S_n \hat{v}_n(x) &= K_n h_n \\ &= \sum_{j=1}^{n} w_j k(x, t_j) h_n(t_j). \end{aligned}$$

Thus it suffices to show

$$h_n(t_l) = \hat{v}_n(t_l) \qquad \text{for } 1 \le l \le n.$$

Let $\tau = S_n \hat{v}_n$ and use (**??**) and (**??**) to have

$$\begin{aligned} h_n(x) &= S^{-1}\Pi_n \tau(x) \\ &= \frac{a(x)}{Z(x)}\Pi_n\tau(x) - \frac{b(x)}{\pi}\int_{-1}^{1} \frac{\Pi_n\tau(t)}{Z(t)(t-x)}dt \\ &= \frac{a(x)}{Z(x)}\Pi_n\tau(x) - b(x)\left[\frac{1}{\pi}\sum_{j=1}^{n}\frac{w_j^*\Pi_n\tau(s_j)}{s_j - x} + \frac{\tilde{\Lambda}_n(x)}{\psi_n(x)}\Pi_n\tau(x)\right] \end{aligned}$$

since Π_n is a polynomial. From (**??**),

$$\begin{aligned} h_n(x) &= \frac{a(x)}{Z(x)}\Pi_n\tau(x) - \frac{b(x)}{\pi}\sum_{j=1}^{n}\frac{w_j^*\Pi_n\tau(s_j)}{s_j - x} + \frac{\Pi_n\tau(x)}{\psi_n(x)}\left[\phi_n(x) - \frac{a(x)}{Z(x)}\psi_n(x)\right] \\ &= \frac{-b(x)}{\pi}\sum_{j=1}^{n}\frac{w_j^*\Pi_n\tau(s_j)}{s_j - x} + \frac{\phi_n(x)}{\psi_n(x)}\Pi_n\tau(x). \end{aligned}$$

Evaluating at the zero t_l of ϕ_n, we obtain

$$h_n(t_l) = \frac{-b(t_l)}{\pi}\sum_{j=1}^{n}\frac{w_j^*\Pi_n\tau(s_j)}{s_j - t_l}$$

since $\Pi_n\tau(s_j) = \tau(s_j)$($\Pi_n$ is the operator of polynomial interpolation on $\{s_j\}_{j=1}^n$),

$$= \frac{-b(t_l)}{\pi}\sum_{j=1}^{n}\frac{w_j^* S_n\hat{v}_n(s_j)}{s_j - t_l}.$$

.

Hence we have, from (**??**)

$$\begin{aligned} h_n(t_l) &= \frac{-b(t_l)}{\pi}\sum_{j=1}^{n}\frac{w_j^*}{s_j - t_l}\left[\frac{b(s_j)}{\pi}\sum_{i=1}^{n}\frac{w_i\hat{v}_n(t_i)}{t_i - s_j}\right] \\ &= \frac{b(t_l)}{\pi}\sum_{i=1}^{n} w_i\hat{v}_n(t_i)\left[\frac{1}{\pi}\sum_{j=1}^{n}\frac{w_j^* b(s_j)}{(t_i - s_j)(t_l - s_j)}\right] \\ &= \hat{v}_n(t_l). \end{aligned}$$

The last equality comes from the theorem **??**. □

The discrete Sloan iterate obtained from a polynomial Galerkin approximation for solving *SIEs* becomes the Nyström approximation when all integrals are calculated by Gaussian quadratures using zeros of basis polynomials as their nodes. This reads that the discrete Galerkin approximation v_n and the discrete Sloan iterate $\hat{v}_n$ agree at the quadrature nodes. Therefore we do not achieve the computational superconvergence in practice.

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