J. KSIAM **Vol.2**, No.2, 31-40, 1998

SPECTRAL ANALYSIS FOR HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH A WEAKLY SINGULAR KERNEL

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Abstract. Spectral analysis by energy method is given for a fully discrete methods for hyperbolic integro-differential equations with a weakly singular kernel. Stability and error estimates in H^1-norm are derived.

1. Introduction.

We will consider an approximate solution using the spectral methods for the hyperbolic integro-differential equations with a singular kernel:

$$u_{tt} + Au = \int_0^t K(t-s)Bu(s)\,ds + f, \quad (x,t) \in \Omega \times (0,T] \tag{1.1a}$$

with a Dirichelet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{1.1b}$$

and initial conditions

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \tag{1.1c}$$

Here $\Omega = (0,\pi)^2$, $A(x,t)$ is a linear, positive, symmetric, uniformly elliptic operator and $B(x)$ is a general partial differential operator of second order with smooth coefficients. Given functions $u_0(x), u_1(x)$ and $f(x,t)$ are real-valued and sufficiently smooth. Further, $K(t)$ is a positive decreasing weakly singular kernel with the property:

$$K(t) \le Ct^{\alpha}, \quad -1 < \alpha < 0, \quad t > 0. \tag{1.2}$$

Integro-differential equation (1.1) arises in visco-elastic problems. For more references on problem of the type (1.1), we refer to Renardy, Hrusa and Nohel[13] and references therein. The problem (1.1) with smooth kernels has been studied in Dix and Torrejón[7], Torrejón and Yong[14], Yanik and Fairweather[15], where they showed

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1991 *Mathematics Subject Classification.* 65M70, 65M15.

Key words and phrases. Spectral methods, integro-differential equations, fully discrete scheme.
This work is partially supported by SNU-DAEWOO Research Fund under grant 96-03-1057.

existence and uniqueness of solutions for smooth initial data using energy estimates. In [10], Hrusa and Renardy showed existence of discontinuous solution for nonsmooth initial data. Local and global weak solutions of (1.1) with singular kernels have been studied in Engler [8] and Hrusa and Renardy[9] using the limit of solutions with smoothing kernels. Galerkin approximate solutions of (1.1) with weakly singular kernels have been discussed in Choi and MacCamy[6], in which error estimates are given for the semidiscrete scheme. For the fully discrete scheme, Galerkin solutions have been discussed in Pani, Thomée and Wahlbin[12] in case of the kernel $K(t) \equiv 1$, where they also considered storage reduction. In spite of many works on (1.1) with singular kernels, to our best knowledge, the finite element solutions of (1.1) with time stepping(fully discrete scheme) appear to be untouched, even though it is crucial for real computation. In this paper, we discuss error estimates of backward Euler's fully discrete spectral approximate solutions for (1.1) with a weakly singular kernel. In section 2, error estimates for several projections, like L^2-projection, Ritz projection and Ritz-Volterra projection, will be discussed. In section 3, error estimates for finite element solutions by spectral methods with time stepping will be discussed, where H^1-error estimates of $O(k+N^{-2})$ are shown.

2. Error Estimates for Projections.

Let $V_N = span\{\psi_{ij} = sinix_1 sinjx_2 : i,j = 1,2,\ldots,N\}$ be the subspace of the usual Sobolev space $V = H_0^1(\Omega)$. The weak solution for (1.1) is defined as a function $u_N\colon (0,T] \longrightarrow V_N$ such that for all $\chi \in V_N$

$$(u_{Ntt},\chi) + a(u_N,\chi) = \int_0^t K(t-s)b(u_N(s),\chi)\,ds + (f,\chi), \tag{2.1a}$$

$$(u_N(x,0),\chi) = (u_0,\chi), \tag{2.1b}$$

$$(u_{Nt}(x,0),\chi) = (u_1,\chi). \tag{2.1c}$$

Here $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bilinear forms on $H_0^1(\Omega)\times H_0^1(\Omega)$ associated with differential operators A and B, respectively. The inner product $(\cdot,\cdot) : H_0^1 \times H_0^1 \to R$ is defined as

$$(\phi,\psi) = \int_\Omega \phi(x)\psi(x)dx, \qquad \phi,\psi \in H_0^1(\Omega).$$

Define a L^2-projection $P_N\colon L^2(\Omega) \longrightarrow V_N$ by $P_N v = \sum_{i,j}^N a_{ij}(t)\psi_{ij}$ for $v(x,t) = \sum_{i,j}^\infty a_{ij}(t)\psi_{ij}$. Then we have

$$(v - P_N v,\chi) = 0, \quad \forall \chi \in V_N.$$

That is, P_N is an orthogonal projection operator. It is well known that the following error estimate holds for the the L^2-projection(see [4]). Hereafter, a constant C will be used as a generic constant independent of N and mesh k. For notational convenience, we omit dependent variables x and t if there is no confusion.

Lemma 2.1. *There exists a constant C such that*

$$\|v - P_N v\| \le CN^{-2}\|v\|_2.$$

The following version of Gronwall's lemma will be frequently used for error estimates, whose proof can be found in Chen, Thomée and Wahlbin [5].

Lemma 2.2. *Assume that y is a nonnegative function in $L_1(0,T)$ and satisfies*

$$y(t) \le z(t) + \beta \int_0^t K(t-s)y(s)\,ds, \quad 0 < t \le T,\ -1 < \alpha < 0,$$

where $z(t) \ge 0, \beta \ge 0$. Then there is a constant C_T such that

$$y(t) \le z(t) + C_T \int_0^t K(t-s)z(s)\,ds, \quad 0 < t \le T.$$

We now introduce the standard Ritz projection operator $R_N : H_0^1(\Omega) \longrightarrow V_N(\Omega)$ with

$$a(v - R_N v, \chi) = 0, \quad \forall \chi \in V_N(\Omega). \tag{2.2}$$

The error estimate of Ritz projection can be found in Bressan and Quarteroni[3], Bernardi and Maday[2] for all $v \in V$ with $0 \le \mu \le r$ and $r \ge 1$ as

$$\|v - R_N v\|_\mu \le CN^{-(r-e(\mu))}\|v\|_r, \tag{2.3}$$

where

$$e(\mu) = \begin{cases} \mu, & \mu \le 1 \\ 2\mu - 1, & \mu > 1. \end{cases}$$

Further, we introduce the Ritz-Volterra projection operator $\Pi_N : V \longrightarrow V_N$ by

$$a((\Pi_N u - u)(t), \chi) = \int_0^t K(t-s)b((\Pi_N u - u)(s), \chi)\,ds, \quad \forall \chi \in V_N^0(\Omega), \tag{2.4}$$

as in Lin, Thomée and Wahlbin[11]. Then we have the following error estimate for the Ritz-Volterra projection.

Lemma 2.3. *There exists a constant C such that for $u \in V$.*

$$\|(\Pi_N u - u)(t)\| + N^{-1}\|(\Pi_N u - u)(t)\|_1 \le CN^{-2}\sup_{s \le t}\|u(s)\|_2.$$

Proof. Let $\rho_N = \Pi_N u - u$. We begin with an H^1 - estimate for ρ_N. ¿From (2.3), we obtain

$$\|R_N u - u\| + N^{-1}\|R_N u - u\|_1 \le CN^{-2}\|u\|_2.$$

It follows from the coercivity of $a(\cdot,\cdot)$ and the orthogonal projection that for $c_0 > 0$,

$$\begin{aligned} c_0\|\Pi_N u - R_N u\|_1^2 & \\ &\le a(\Pi_N u - R_N u, \Pi_N u - R_N u) \\ &= a(\rho_N, \Pi_N u - R_N u) + a(u - R_N u, \Pi_N u - R_N u) \\ &= a(\rho_N, \Pi_N u - R_N u) \\ &= \int_0^t K(t-s) b(\rho_N(s), (\Pi_N u - R_N u)(t))\, ds \\ &\le C\|\Pi_N u - R_N u\|_1 \int_0^t K(t-s)\|\rho_N(s)\|_1\, ds. \end{aligned}$$

Thus, we obtain

$$\|\Pi_N u - R_N u\|_1 \le C \int_0^t K(t-s)\|\rho_N(s)\|_1\, ds$$

and

$$\|\rho_N\|_1 \le \|\Pi_N u - R_N u\|_1 + \|R_N u - u\|_1 \le C \int_0^t K(t-s)\|\rho_N(s)\|_1\, ds + \|R_N u - u\|_1.$$

It follows from Lemma 2.2 that

$$\|\rho_N\|_1 \le C \sup_{s\le t} \|(R_N u - u)(s)\|_1 \le C N^{-1} \sup_{s \le t} \|u(s)\|_2.$$

We now consider the L_2-estimate for ρ_N using the duality argument. For any $\phi \in L^2$, let ψ be the solution of

$$A\psi = \phi \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

Then ψ is a unique solution of (2.5) such that

$$\|\psi\|_2 \le C\|\phi\| = C.$$

Note that, for $\chi \in V_N$,

$$(\rho_N, \phi) = (\rho_N, A\psi) = a(\rho_N, \psi - \chi) + a(\rho_N, \chi). \tag{2.6}$$

The last term on the right hand side of (2.6) becomes

$$\begin{aligned} a(\rho_N, \chi) &= \int_0^t K(t-s) b(\rho_N(s), \chi)\, ds \\ &= \int_0^t K(t-s) b(\rho_N(s), \chi - \psi)\, ds + \int_0^t K(t-s)\,(\rho_N(s), B^*\psi)\, ds, \end{aligned}$$

where B^* is the adjoint of B. Replacing $\chi = R_N\psi$ and using (2.6), we obtain

$$\begin{aligned}(\rho_N, \phi) &\le C\{\|R_N\psi - \psi\|_1 \sup_{s\le t}\|\rho_N(s)\|_1 + \|\psi\|_2 \int_0^t K(t-s)\|\rho_N(s)\|ds\} \\ &\le C\{N^{-1}\|\psi\|_2 \cdot N^{-1}\sup_{s\le t}\|u(s)\|_2 + \|\phi\| \int_0^t K(t-s)\|\rho_N(s)\|ds\}.\end{aligned}$$

Replacing $\phi = \rho_N/\|\rho_N\|$ in the above inequality, we obtain

$$\|\rho_N\| \le C\{N^{-2}\sup_{s\le t}\|u(s)\|_2 + \int_0^t K(t-s)\|\rho_N(s)\|\, ds\}.$$

An application of Lemma 2.2 completes the proof. □

Following the proof of Lemma 2.3, we may also obtain

$$\|\rho_{N_t}\| + N^{-1}\|\rho_{N_t}\|_1 \le CN^{-2}\sup_{s\le t}\|u_t(s)\|_2 \tag{2.7}$$

and

$$\|\rho_{N_{tt}}\| + N^{-1}\|\rho_{N_{tt}}\|_1 \le CN^{-2}\sup_{s\le t}\|u_{tt}(s)\|_2. \tag{2.8}$$

3. Discretization in Time Direction.

Let M be a positive integer and $k = T/M$. We define difference operators

$$\bar\partial_k\phi^n = \frac{\phi^n - \phi^{n-1}}{k}, \quad \bar\partial_k^2\phi^n = \bar\partial_k(\bar\partial_k\phi^n)$$

and

$$\tilde\phi(s) = \frac{1}{k}[(t_j - s)\phi(t_{j-1}) + (s - t_{j-1})\phi(t_j)], \quad t_{j-1}\le s\le t_j.$$

Let $U_N^n \in V_N$ be a solution of

$$(\bar\partial_k^2 U_N^n, \chi) + a_n(U_N^n, \chi) = \int_0^{t_n} K(t_n - s) b(\tilde U_N(s), \chi)ds + (f^n, \chi), \quad n\ge 2, \tag{3.1a}$$

$$(U_N^0, \chi) = (u_0, \chi), \tag{3.1b}$$

$$(\bar\partial_k U_N^1, \chi) = (u_1, \chi), \tag{3.1c}$$

for all $\chi \in V_N$. We use the trapezoidal rule as in Weiss[16] and Atkinson[1] for integrations in (3.1). Then we obtain the quadrature error

$$\varepsilon_n(\phi) = \sum_{j=1}^n\{\nu_{n,j}\phi(t_{j-1}) + \mu_{n,j}\phi(t_j)\} - \int_0^{t_n} K(t_n - s)\phi(s)ds, \tag{3.2}$$

where

$$\nu_{n,j} = \int_{t_{j-1}}^{t_j}(t_j - s)K(t_n - s)ds, \tag{3.3a}$$

$$\mu_{n,j} = \int_{t_{j-1}}^{t_j}(s - t_{j-1})K(t_n - s)ds. \tag{3.3b}$$

Lemma 3.1. *There is a constant C such that if $\phi_t \in L_\infty(0,T;L_2)$, then*

$$k\sum_{n=1}^{M}\|\varepsilon_n(\phi)\| \le Ck^2\sum_{n=1}^{M}\max_{0\le s\le t_n}\|\phi_t(s)\|.$$

Proof. Since we may rewrite $\phi(s)$ as

$$\phi(s) = \begin{cases} \phi(t_{j-1}) + (s-t_{j-1})\phi_t(\xi_{j-1}), & t_{j-1} < \xi_{j-1} < s \\ \phi(t_j) + (s-t_j)\phi_t(\zeta_j), & s < \zeta_j < t_j, \end{cases}$$

we obtain, from the Taylor's theorem,

$$\begin{aligned} |(\tilde{\phi}-\phi)(s)| &\le \frac{1}{k}(t_j - s)(s-t_{j-1})\{|\phi_t(\xi_{j-1})| + |\phi_t(\zeta_j)|\} \\ &\le k\{|\phi_t(\xi_{j-1})| + |\phi_t(\zeta_j)|\}. \end{aligned}$$

Since $\varepsilon_n(\phi) = \int_0^{t_n} K(t_n - s)(\tilde{\phi}-\phi)(s)\,ds$, we obatin

$$\begin{aligned} |\epsilon_n(\phi)| &\le 2k\sum_{j=1}^{n}\max_{t_{j-1}\le s\le t_j}|\phi_t(s)|\int_{t_{j-1}}^{t_j}K(t_n-s)\,ds \\ &\le Ck\max_{0\le s\le t_n}|\phi_t(s)|. \end{aligned}$$

This implies

$$\|\varepsilon_n(\phi)\| \le Ck\max_{0\le s\le t_n}\|\phi_t(s)\|.$$

Hence the required result holds from summation of the above inequality. □

The following lemma is a discrete version of Lemma 2.2, which will be used judiciously.

Lemma 3.2. *Let $\nu_{n,j}$ and $\mu_{n,j}$ be defined as in (3.3). Assume that $y_n \ge 0$, $z_n \ge 0$ and $\beta \ge 0$. If, either $x_{n,j} = \nu_{n,j}$ or $x_{n,j} = \mu_{n,j}$,*

$$y_n \le z_n + \beta\sum_{j=1}^{n}x_{n,j}y_j, \qquad n \ge 0,$$

holds, then there is a constant C such that

$$y_n \le z_n + C\sum_{j=1}^{n}x_{n,j}z_j, \quad n \ge 0.$$

We now state the stability of approximate solutions in terms of a discrete energy norm $|||\cdot|||$ defined by

$$|||\phi^n|||_1^2 = \|\bar{\partial}_k\phi^n\|^2 + \|\phi^n\|_1^2, \quad n \ge 1.$$

Theorem 3.1. *Let $U_N \in V_N$ be a solution of (3.1). Then there exists a constant C such that*

$$|||U_N^n|||_1 \le C\{|||U_N^1|||_1 + k\sum_{n=2}^{M}\|f^n\|\}.$$

Proof. Replacing $\chi = \bar\partial_k U_N^n$ in (3.1a), we obtain

$$(3.4)\qquad \begin{aligned}&(\bar\partial_k^2 U_N^n, \bar\partial_k U_N^n) + a_n(U_N^n, \bar\partial_k U_N^n)\\ &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} K(t_n - s) b(\tilde U_N(s), \bar\partial_k U_N^n)\, ds + (f^n, \bar\partial_k U_N^n)\\ &\equiv I_1^n + I_2^n.\end{aligned}$$

Note that

$$(\bar\partial_k^2 U_N^n, \bar\partial_k U_N^n) = \frac{1}{2}\bar\partial_k\|\bar\partial_k U_N^n\|^2 + \frac{k}{2}\|\bar\partial_k^2 U_N^n\|^2$$

and

$$a_n(U_N^n, \bar\partial_k U_N^n) = \frac{1}{2}\bar\partial_k(a_n(U_N^n, U_N^n)) - \frac{1}{2}(\bar\partial_k a_n)(U_N^{n-1}, U_N^{n-1}) + \frac{k}{2}a_n(\bar\partial_k U_N^n, \bar\partial_k U_N^n).$$

Multiplying both sides of (3.4) by $2k$ and summing from $n = 2$ to M, we obtain, for positive constants c_0 and C,

$$\begin{aligned}\|\bar\partial_k U_N^M\|^2 + c_0\|U_N^M\|_1^2 \le \|\bar\partial_k U_N^1\|^2 + C\|U_N^1\|_1^2 + k|\sum_{n=2}^{M}(\bar\partial_k a_n)(U_N^{n-1}, U_N^{n-1})|\\ + 2k|\sum_{n=2}^{M}(I_1^n + I_2^n)|.\end{aligned}$$

The above inequality can be rewritten using the discrete norm $||cdot|||$ as

$$(3.5)\qquad |||U_N^M|||_1^2 \le C\{|||U_N^1|||_1^2 + k\sum_{n=2}^{M}|||U_N^{n-1}|||_1^2 + k\sum_{n=2}^{M}(|I_1^n| + |I_2^n|)\}.$$

Let $|||U_N|||_{1;M} = \max_{0\le n\le M}|||U_N^n|||_1$. Then

$$k\sum_{n=2}^{m}|I_2^n| = k\sum_{n=2}^{M}\|f^n\|\cdot|||U_N|||_{1;M}.$$

On the other hand, we can rewrite I_1^n using notations in (3.2)–(3.3) as

$$\begin{aligned}I_1^n &= \nu_{n,n} b(U_N^{n-1}, \bar\partial_k U_N^n) + \mu_{n,n} b(U_N^n, \bar\partial_k U_N^n)\\ &+ \sum_{j=1}^{n-1}\bar\partial_k\{\nu_{n,j} b(U_N^{j-1}, U_N^n) + \mu_{n,j} b(U_N^j, U_N^n)\}\\ &- \sum_{j=1}^{n-1}\{(\bar\partial_k\nu_{n,j}) b(U_N^{j-1}, U_N^{n-1}) + (\bar\partial_k\mu_{n,j}) b(U_N^j, U_N^{n-1})\}.\end{aligned}$$

Summation both sides from $n = 2$ to M, we obtain

$$\begin{aligned} k\sum_{n=2}^{M} I_1^n &= \sum_{n=2}^{M}\{\mu_{n,n} b(U_N^n, U_N^n) + (\nu_{n,n} - \mu_{n,n}) b(U_N^n, U_N^{n-1}) - \nu_{n,n} b(U_N^{n-1}, U_N^{n-1})\} \\ &\quad + \sum_{j=1}^{M-1}\{\nu_{M,j} b(U_N^{j-1}, U_N^M) + \mu_{M,j} b(U_N^j, U_N^M)\} \\ &\quad - \sum_{j=1}^{M-1}\{\nu_{j,j} b(U_N^{j-1}, U_N^j) + \mu_{j,j} b(U_N^j, U_N^j)\} \\ &\quad - k\sum_{j=1}^{M-1}\sum_{n=j+1}^{M}\{(\bar\partial_k \nu_{n,j}) b(U_N^{j-1}, U_N^{n-1}) + (\bar\partial_k \mu_{n,j}) b(U_N^j, U_N^{n-1})\}. \end{aligned}$$

Thus

$$\begin{aligned} k\sum_{n=2}^{M} |I_1^n| &\le C\sum_{j=1}^{M}\{|\mu_{j,j}|\|U_N^j\|_1 + |\nu_{j,j} - \mu_{j,j}|\|U_N^j\|_1 + |\nu_{j,j}|\|U_N^{j-1}\|_1\}|||U_N|||_{1;M} \\ &\quad + C\sum_{j=1}^{M}\{(|\nu_{M,j}| + |\nu_{j,j}|)\|U_N^{j-1}\|_1 + (|\mu_{M,j}| + |\mu_{j,j}|)\|U_N^j\|_1\}|||U_N|||_{1;M} \\ &\quad + Ck\sum_{j=1}^{M-1}\{\|U_N^{j-1}\|_1\sum_{n=j+1}^{M}|\bar\partial_k \nu_{n,j}| + \|U_N^j\|_1\sum_{n=j+1}^{M}|\bar\partial_k \mu_{n,j}|\}|||U_N|||_{1;M}. \end{aligned}$$

Since

$$\begin{aligned} |\bar\partial_k \nu_{n,j}| &= \frac{1}{k}\left|\int_{t_{j-1}}^{t_j}(t_j - s)[K(t_n - s) - K(t_{n-1} - s)]\,ds\right| \\ &\le \int_{t_{j-1}}^{t_j}[K(t_{n-1} - s) - K(t_n - s)]\,ds, \end{aligned}$$

we obtain, by interchanging summation with integration,

$$\begin{aligned} \sum_{n=j+1}^{M}|\bar\partial_k \nu_{n,j}| &\le \sum_{n=j+1}^{M}\int_{t_{j-1}}^{t_j}[K(t_{n-1} - s) - K(t_n - s)]\,ds \\ &= \int_{t_{j-1}}^{t_j}[K(t_j - s) - K(t_M - s)]\,ds \\ &\le C(\|K\|_{L_1(0,T)}). \end{aligned}$$

Similarly, we obtain

$$\sum_{n=j+1}^{M}|\bar\partial_k \mu_{n,j}| \le C(\|K\|_{L_1(0,T)}).$$

Noting that $|\nu_{n,j}| \leq k^{2+\alpha}$ and $|\mu_{n,j}| \leq k^{2+\alpha}$ for $j \leq n \leq M$ and dividing (3.5) by $|||U_N|||_{1;M}$, we obtain

$$|||U_N^M|||_1 \leq C\{|||U_N^1|||_1 + k\sum_{n=1}^{M} |||U_N^n|||_1 + k\sum_{n=2}^{M} \|f^n\|\}.$$

Hence, the discrete Gronwall's inequality completes the proof. □

Let $\theta^n = U_N^n - \Pi_N u(t_n)$. Then the error $e^n = U_N^n - u(t_n) = \theta^n + \rho_N^n$. Since we know the estimate of ρ_N^n, we have only to find the estimate for θ^n.

Theorem 3.2. *Let $u(t)$ be the solution of (1.1) and U_N^n be a solution of (3.1). Then there exists a constant C such that*

$$|||\theta^n|||_1 \leq C\{N^{-2} + k + |||\theta^0|||_1\}.$$

Proof. From (2.4) and (3.1), we obtain

$$(\bar{\partial}_k^2\theta^n, \chi) + a_n(\theta^n, \chi) = \int_0^{t_n} K(t_n - s)b(\tilde{\theta}^k, \chi)\, ds + (\sum_{i=1}^{3} J_i^n, \chi), \tag{3.5}$$

where

$$J_1^n = u_{tt}(t_n) - \bar{\partial}_k^2 u(t_n),$$
$$J_2^n = -\bar{\partial}_k^2 \rho_N^n,$$
$$J_3^n = \int_0^{t_n} K(t_n - s)B\{\tilde{\Pi_N} u(s) - \Pi_N u(s)\}ds.$$

It follows from Theorem 3.1 that

$$|||\theta^M|||_1 \leq C\{|||\theta^1|||_1 + k\sum_{n=2}^{M} \|J_i^n\|\}.$$

From the relation $J_3^n = \varepsilon_n(B\rho_N) + \varepsilon_n(Bu)$ and Lemma 3.1, we obtain

$$\|J_3^n\| \leq Ck\{\sup_{0\leq s\leq t_n} \|B\rho_{Nt}(s)\| + \sup_{0\leq s\leq t_n} \|Bu_t(s)\|\} \leq Ck\|u_t\|_2.$$

Hence, we obtain

$$k\sum_{n=2}^{M} \|J_3^n\| \leq Ck^2 \sum_{n=2}^{M} (\sup_{0\leq s\leq t_n} \|u_t\|_2) \leq Ck\int_0^{t_M} \|u_t(s)\|_2 ds.$$

Similarly, we obtain

$$k\sum_{n=2}^{M} \|J_1^n\| \leq Ck^2 \sum_{n=2}^{M} \int_{t_{n-1}}^{t_n} \|u_{tttt}\| ds = Ck^2 \int_{t_1}^{t_M} \|u_{tttt}\|\, ds.$$

For the estimate of J_2^n, it follows from (2.8) that

$$k\sum_{n=2}^{M} \|J_2^n\| \leq \sum_{n=2}^{M} \int_{t_{n-1}}^{t_n} \|\rho_{Ntt}\|\, ds \leq CN^{-2}.$$

These complete the proof. □

Remarks. Spectral analysis is discussed for a hyperbolic integro-differential equation with a weakly singular kernel and error estimates of the spectral method is given. Because of the memory term and global bases functions, a storage problem in computation arises in the method. In order to overcome this problem, we will give an improved method elsewhere.

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