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Linear Operators which Preserve Pairs on which the Rank is Additive

LeRoy B. Beasley*

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Abstract

Let A and B be $m \times n$ matrices. A linear operator T preserves the set of matrices on which the rank is additive if $\text{rank}(A+B) = \text{rank}(A) + \text{rank}(B)$ implies that $\text{rank}(T(A) + T(B)) = \text{rank}T(A) + \text{rank}T(B)$. We characterize the set of all linear operators which preserve the set of pairs of $n \times n$ matrices on which the rank is additive.

1 Introduction.

Let $\mathcal{M}_{m,n}(\mathbb{F})$ denote the set of all $m \times n$ matrices over the field $\mathbb{F}$, and let $T : \mathcal{M}_{m,n}(\mathbb{F}) \longrightarrow \mathcal{M}_{m,n}(\mathbb{F})$ be a linear operator. A common problem considered in linear algebra is called a preserver problem, that is, characterize those linear operators which "preserve" a function or a set. We say

that T preserves a subset $\mathcal{K}$ of $\mathcal{M}_{m,n}(\mathbb{F})$ whenever $A \in \mathcal{K}$ implies that $T(A) \in \mathcal{K}$; we say that T preserves a function $f : \mathcal{M}_{m,n}(\mathbb{F}) \to \mathbb{F}$ whenever $f(T(A)) = f(A)$ for all $A \in \mathcal{M}_{m,n}(\mathbb{F})$; and we say that T preserves a subset $\mathcal{P}$ of $\mathcal{M}_{m,n}(\mathbb{F}) \times \mathcal{M}_{m,n}(\mathbb{F})$ whenever $(A, B) \in \mathcal{P}$ implies that $(T(A), T(B)) \in \mathcal{P}$. For example, if $\mathcal{P}$ is the set $\{(A, B) : AB = BA\}$ and T preserves $\mathcal{P}$ then we say that T preserves the set of commuting pairs. The set of all linear operators which preserve the set of commuting pairs was characterized in [1, 8]. In this paper we shall characterize the set of all linear operators which preserve the set of pairs on which the rank is additive.

The classification of preserves began about 100 years ago. In 1897, Frobenius [4] characterized the linear operators on $\mathcal{M}_{n,n}(\mathbb{F})$ which preserve certain matrix functions: those linear operators on $\mathcal{M}_{n,n}(\mathbb{F})$ that preserve the determinant and those that preserve the characteristic polynomial.

After half a century of relative inactivity there was renewed interest in preserver problems. That interest was sparked by the investigation of rank preservers in 1959 by Marcus and Moyls [5]. They proved:

*Department of Mathematics, Utah State University, Logan, Utah 84322-3900, U.S.A. (lbeasley@math.usu.edu)

If $\mathbb{F}$ is a algebraically closed field and of characteristic 0 and T is a rank preserver, then there exist $m \times m$ and $n \times n$ matrices U and V, respectively, such that either

$$T(A) = UAV \text{ for all } A \in \mathcal{M}_{m,n}(\mathbb{F}) \tag{1}$$

or

$$m = n \text{ and } T(A) = UA^tV \text{ for all } A \in \mathcal{M}_{n,n}(\mathbb{F}) \tag{2}$$

where X^t denotes the transpose operator.

Also in 1959, Marcus and Moyls [6] found that T is a rank preserver if and only if T is a rank-1 preserver, that is, T preserves the set of matrices whose rank is 1. In 1967, Westwick [9] generalized these results to matrices over arbitrary algebraically closed fields.

Characterizations of preservers have been appearing regularly over the past thirty years and an excellent summary of nearly all characterizations of linear preservers can be found in a special issue of *Linear and Multilinear Algebra*, edited by S. Pierce with input by leaders in the field [7]. This article includes a list of over 200 articles written on the subject.

2 Additive Rank Preservers.

Let $\rho : \mathcal{M}_{m,n}(\mathbb{F}) \longrightarrow \{0, 1, 2, \cdots, \min(m, n)\}$ be the rank function and let $\mathcal{P} = \{(A, B) : A, B \in \mathcal{M}_{m,n}(\mathbb{F}) \text{ and } \rho(A + B) = \rho(A) + \rho(B)\}$. Let $L_{U,V}$ denote the linear operator of the form (1), and when $m = n$, let $L^t{}_{U,V}$ denote the linear operator of the form (2) above. It is obvious that $L_{U,V}$ and $L^t{}_{U,V}$ (when $m = n$) preserve the set $\mathcal{P}$ for any nonsingular U and V.

Suppose $\mathbb{F}$ is algebraically closed, $T : \mathcal{M}_{m,n}(\mathbb{F}) \longrightarrow \mathcal{M}_{m,n}(\mathbb{F})$ preserves $\mathcal{P}$ and that T is singular. Without loss of generality we may assume that $T\begin{pmatrix} I_s & O \\ O & O \end{pmatrix} = O$, for some s such that $s \geq 1$. Then, given any r with $1 \leq r \leq s$, since $\begin{pmatrix} I_s & O \\ O & O \end{pmatrix} = \begin{pmatrix} I_r & O & O \\ O & O & O \\ O & O & O \end{pmatrix} + \begin{pmatrix} O & O & O \\ O & I_s - r & O \\ O & O & O \end{pmatrix}$ and T preserves $\mathcal{P}$, we have

$$0 = \rho\left(T\begin{pmatrix} I_s & O \\ O & O \end{pmatrix}\right) = \rho\left(T\begin{pmatrix} I_r & O & O \\ O & O & O \\ O & O & O \end{pmatrix}\right) + \rho\left(T\begin{pmatrix} O & O & O \\ O & I_{s-r} & O \\ O & O & O \end{pmatrix}\right).$$

Thus, $T\begin{pmatrix} I_r & O & O \\ O & O & O \\ O & O & O \end{pmatrix} = T\begin{pmatrix} O & O & O \\ O & I_{s-r} & O \\ O & O & O \end{pmatrix} = 0$ for all r with $1 \leq r \leq s$.

Now, if X is in the image of T and $\rho(X) = r \leq s$, then there are nonsingular U and V such that $UXV = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$. Thus $T \circ L_{U,V} \circ T$ preserves $\mathcal{P}$ and the image of $T \circ L_{U,V} \circ T$ has strictly lower dimension than does T. Thus, suppose that T preserves $\mathcal{P}$, and that T has the lowest dimension image of any operator in the semigroup generated by T and the set $\{L_{U,V} : U, V \text{ are nonsingular}\}$ (together with $\{{L^t}_{U,V} : U, V \text{ are nonsingular}\}$ if $m = n$). From the above observations we have that if s is the largest integer such that there is a matrix of rank s in the kernel of T, then the image of T has no element of rank 1 through rank s. I.e., if $X \in ImT$, then $\rho(X) > s$ or $X = O$. Thus, the dimension of the image is at most $(n - s)(m - s)$ (since $\mathbb{F}$ is algebraically closed, see Westwick, Theorem 2.1 [10]). Further, $\ker T$ is a space of matrices of rank at most s. By [3], $\dim \ker T \leq ns$. It follows that $nm = (n - s)(m - s) + ns$. This is possible only if $s = 0$ or $s = m$. Thus, unless T maps all of $\mathcal{M}_{m,n}(\mathbb{F})$ to $\{O\}$, T must be nonsingular.

We have established:

LEMMA 1. If $\mathbb{F}$ is algebraically closed, and T is a linear mapping preserving $\mathcal{P}$, then either T is the zero map or T is nonsingular.

(Note that in the above argument the only requirement on the field is that the maximum dimension of a subspace of $\mathcal{M}_{m,n}(\mathbb{F})$ all of whose nonzero elements have rank at least $s + 1$ is strictly less than $n(m - s)$.)

LEMMA 2. If T is a nonsingular linear operator on $\mathcal{M}_{m,n}(\mathbb{F})$ preserving $\mathcal{P}$ then T preserves the set of matrices of rank at most $m - 1$.

Proof. Suppose that A is a matrix of rank $k \leq m-1$ such that $\rho(T(A)) = m$. Without loss of generality, we may assume that $A = \begin{pmatrix} I_k & O \\ O & O \end{pmatrix}$. Then, since T preserves $\mathcal{P}$ we must have that

$$\rho\left(T\begin{pmatrix} I_m & O \end{pmatrix}\right) = \rho(T(A)) + \rho\left(T\begin{pmatrix} O & O & O \\ O & I_{m-k} & O \end{pmatrix}\right)$$

$$= m + \rho\left(T\begin{pmatrix} O & O & O \\ O & I_{m-k} & O \end{pmatrix}\right) > m$$

since T is nonsingular. But this is impossible, consequently, if $\rho(A) < m$, we must have $\rho(T(A)) < m$. ∎

The following result of Chan and Lim is needed:

LEMMA 3. [2] If T preserves the set of all matrices of rank at most k then either the image of T has no matrix of rank greater than k or there exist nonsingular matrices U and V such that $T = L_{U,V}$ or, when $m = n$, $T = {L^t}_{U,V}$.

THEOREM. If $T : \mathcal{M}_{m,n}(\mathbb{F}) \longrightarrow \mathcal{M}_{m,n}(\mathbb{F})$ is a linear operator and whenever $\rho(A+B)=\rho(A)+\rho(B)$ we have that $\rho(T(A+B))=\rho(T(A))+\rho(T(B))$, then either $T(X)=O$ for all $X \in \mathcal{M}_{m,n}(\mathbb{F})$, or there exist nonsingular matrices $U \in \mathcal{M}_m(\mathbb{F})$ and $V \in \mathcal{M}_n(\mathbb{F})$ such that $T(X)=UXU$ for all $X \in \mathcal{M}_{m,n}(\mathbb{F})$ or, $m=n$ and $T(X)=UX^tU$ for all $X \in \mathcal{M}_n(\mathbb{F})$.

Proof. Suppose $T \neq O$. By Lemma 1, T is nonsingular. By Lemma 2, T preserves the set of matrices of rank at most $m-1$. By Lemma 3 the theorem follows. ■

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