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프랙탈 이미지 압축을 위한 Coarseness에 대한 연구

A Study on the Coarseness for the Fractal Image Compression

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요 약

현재와 같이 그래픽을 기반으로 하는 멀티미디어 환경에서 이미지 데이터의 압축과 재생을 위한 이미지 처리 기법은 생성되는 커다란 크기의 데이터 처리를 고려해야 한다. 이를 위하여 여러 가지 기법이 제안되었고 Wavelet과 JPEG등과 같은 기법으로 바람직한 결과를 얻었다. 본 논문에서는 기존의 supremum metric보다 효과적인 root mean square metric에 대하여 연구하였고 이를 수학적으로 비교, 검증하였다.

Abstract

An image processing techniques for image compression and its reconstruction are to be considered seriously for the generated huge size data as in the present multimedia environment based on the graphics. So far some approaches for this matter are proposed such as a Wavelet and JPEG method and got desirable result. In this paper, we have studied for the rms(root mean square) metric which is more effective than the existing sup(supremum)metric, and compared with sup metric by means of mathematical verification.

1. Overview

The basic goal of image compression is to preserve the best possible fidelity between the original image and compressed image via binary coding processing. For this goal, M.F.Barnsley suggested using fractals to encode natural images. Whereas Voss used fractals to generate a natural looking image, Barnsly suggested the fractal algorithms approximating the image should be sought for the given image.

Since simple fractal algorithms can typically generate very complex images, Barnsley suggest to store the relevant parameters only, resulting in reduced memory requirement for the storage of images. Some advantages for compression are that Compression conserves storage space which allows larger inventories and reduces the data rate[4].

Also compression technique permit a progressive transmission, and producing an increreasingly good reconstruction image as bits arrive.

We have a question. " How to find a set of transformations that encode an arbitrary image ?" for a difficulty to extend to apply to encode general images.

The image compression scheme used to encode an image shares many features in common with simple fractal generating algorithm. The decoded images can have fractal characteristics are also generated by that scheme. Finding a good fractal algorithms and parametrizing them for image compression has known as the inverse problem. The first published automated scheme to solve the inverse problem was presented by A.Jacquin in his PhD dissertation[2].

2. IFS (Iterated Function System)

The IFS consists of a collection of contractive transformations $\{ w_i : R^2 \rightarrow R^2 \}$.

The IFS is defined by the map[1]. :

$$W(\cdot) = \bigcup_{i=1}^{n} w(\cdot) .$$

In IFS, when the w_i are contractive in the plane, W is contractive in a space of closed bounded compact subsets of the plane. This is proved by Hutchinson[6].

DEFINITION 1

Let X be a metric space with metric δ. A map w : X → X is *Lipschitz* with Lipschitz factors s if there exists a s such that $\delta(w(x),w(y)) \le s\delta(x,y)$,

$\forall x, y \in$ X. w is said to be *contractive* when $s < 1$ [2].

The term 'contractivity' means the infimum oner the values. The contractivity s measures the resemblance between 2 images. The main concept of fractal compression is that the image of a set can be reconstructed from any set of transformations of any shapes [6]. And even this may take less memory to store than the original image of the set.

For a given arbitrary set f , it is not possible to cover itself exactly with a finite number of transformations of itself. Then what happens if the covering $W(f)$ is approximate to original set ? An answer which is called Collage Theorem has given by Barnsley.

THEOREM 1 (Collage theorem)

Let $W : F \rightarrow F$ be a contraction with contractivity s and let $f \in F$, then

$$\delta(|W|, f) \le (1-s)^{-1}\delta(W(f), f) .$$

PROOF. : See page 94 of [5].

The bound of error precalculated by the collage theorem is much larger than the errors from the encoding images empirically. So the collage and generalized collage theorems just provide a motivation for a good encoding can be found.

3. PIFS
(Partitioned Iterated Function system)

The major disadvantage of general IFS for practical is large number of computation to find information for image processing. Computing every possible combination of transformations and domain block to find a best approximation is inefficient. To avoid this kind of problem and get a more practical effectiveness, PIFS is used as an extension of the iterated function system.

DEFINITION 2

Let X be a complete metric space, and let $D_i \subset X$ for $i = 1,\ldots,n$. A *Partitioned iterated system* is a collection of contractive maps $w_i : D_i \rightarrow X$, for $i = 1,\ldots,n$ [2].

In PIFS, the maps $w_1,...,w_n$ are applied to restricted domains by the contrast with IFS. An image is partitioned into parts that can be approximated by other parts after some scaling operations. This encoding process is to be a set of transformations with a fixed point which approximates the target image by the iteration upon any initial image.

In PIFS, a transformation w_i is specified by an affine map and the domain to which w_i is applies. For an IFS with an expansive w_i(collection of w_i of Lipschitz maps in a complete metric space X), the iterates $W^{\circ n}$ will not converge to a compact fixed point and the expansive part will cause the limit set to be infinitely stretched along some direction. PIFS also can contain expansive transformations but still have a bounded attractor. In PIFS, domains are restricted so that the initial point is very important to prevent from empty set after iteration. This is the reason why the contractivity to be checked first. It is need to consider the overlapping problem when partitioning.

4. Fractal Image Encoding

The most fractal-based algorithms for image compression and reconstruction has been studied by 2 major aspect. One is the study to formulate a consistent mathematical theory of generalized fractal transforms. Another one concerns the inverse problem of image approximation. First, as a part of inverse problem, we studied the metric relative to generated error when image is processed.

The fractal image encoding is to store an image as a map $W: F \rightarrow F$ from a complete metric space of image F to itself with fixed point property. The space F can be taken to be any reasonable image model. In order to assure that a fixed point of W exists, the transformation is constructed so that W or $W^{\circ m}$ is contractive. The contractive mapping fixed point theorem ensures convergence to a fixed point upon iteration from any initial image. Insisting that $W^{\circ m}$ be contractive is less restrictive than requiring that W be contractive. The goal is to construct the map W with fixed point close to a given image for compact store of W.

For evaluating the closeness or measure the collage distance, a simple but theoretical motivate method is supremum metric[3]. Sup metric for error is impractical. :

$$\delta_{\sup}(f,g) = \sup_{(x,y)\in I^2} |f(x,y) - g(x,y)|$$

With this metric, a map is contractive if it contracts in the z-direction.

Other metric which is more complicated but more useful - is the rms(root mean square) metric[3]. :

$$\delta_{\mathrm{rms}}(f,g) = [\int_{I^2} (f(x,y) - g(x,y))^2 dxdy]^{\frac{1}{2}}$$

The rms metric is more convenient to use than other metric if we use the standard inner product by < · , · >. But this need more complicated contractivity requirements.

DEFINITION 3

Let f be a Lipschitz function. If there is a number n such that $f^{\circ n}$ is contractive, f is called *eventually contractive.*

The collage theorem can be generalized for the case of eventually contractive mappings because of all contractive maps are eventually contractive(not vice versa). Let s be the contractivity of W and let σ be the contractivity of $W^{\circ m}$.

THEOREM 2

(Generalized Collage Theorem)

For $f \in F$ and $W : F \to F$ eventually contractive,

$$\delta(|W|,f) \leq \frac{1}{1-\sigma}\frac{1-s^m}{1-s}\delta(W(f),f)$$

PROOF. : See page 36 of [2].

It is important that a mapping W which is contractive for δ_{sup} may only be eventually contractive for δ_{rms}. The condition $s_i < 1$ for the contractivity of sup metric is not sufficient to ensure contractivity for the rms metric. This conditions is only sufficient to ensure eventual contractivity for the rms metric. Since the goal of the encoding is compression, another concern is the compact specification of the map w_i.

$$w_i \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_i & b_i & 0 \\ c_i & d_i & 0 \\ 0 & 0 & s_i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} e_i \\ f_i \\ o_i \end{bmatrix} .$$

To limit memory, only w_is are used. The values of s_i and o_i are computed by least squares regression to minimize expression $\delta(f \cap (R_i \times I), w_i(f))$ using the rms metric, and they are stored using a fixed number of bits. A significant improvement can be obtained if only discretized s_i and o_i values are used when computing the error during encoding.

The selection of the transformations w_i affects whether W is contractive or not. To check eventual contractivity for the metric δ_{sup}, iterate W until $\sigma < 1$ if a map is eventually contractive. A procedure to determine eventual contractivity for the rms metric is follows.

1°) Use the concepts of the convergence rate of repeated substitution.

2°) Let w_i be a fixed point of a function

$$R = [\int_{I^2}(f(x,y)-g(x,y))^2 dxdy]^{1/2}$$

and suppose $w_k = R(w_{k-1})$ for $k = 1,2,..$

3°)Then, if $R(w)$ is continuously differentiable in a neighborhood of w_i and $R'(w_i) \neq 0$, $\Delta w_{k+1} \approx R'(w_i)\Delta w_k$ for $w_k \approx w_i$.

4°) If $0 < |R'(w_i)| < 1$, w_k converges to w_i.

5°) This can be regarded as a contractivity of rms metric which used for fractal compression.

The contractivity of rms metric is only determined eventual contractivity after an encoding when no evidence for contractivity is occured. However it is necessary to have a sufficient number of contractive maps practically for encoding that are not eventual contractivity with any significant probability.

5. A Comparison of Metrics

The sup metric which has the greatest vertical gap between functions is more accurate to compute than the rms metric. However for applying to the arbitrary interval, the sup metric is weaker than the rms because of computational difficulty. And at the point of the convergence, the rms metric has smaller area than that of the sup metric. Selecting a metric between blocks after non-overlapping partition is need to find a good PIFS.

LEMMA

Let e and d are metrics on a compact space X. For each e-open sphere , there exists a d-open sphere d such that $S_d(x,\delta) \subset S_e(x,\varepsilon)$ Then metric space induced by e metric is coarser than the metric space induced by d metric.

PROOF: Let U open sets, and M_d M_e are metric space induced by d and e respectively. By condition, $U \in M_e$, $x \in U$. So U is e-open. $\therefore \exists S_e$ such that $x \in S_e \subset U$.

$x \in S_d \subset S_e \subset U$.

$\therefore U = \bigcup \{ S_d(x,\varepsilon) \mid x \in U \}$.

U is the union of the open sphere. Therefore U is open. We get $M_e < M_d$ Q. E. D.

THEOREM 3

The metric space induced by the rms metric is coarser than the metric space induced by the sup metric.

PROOF: For the non-overlap interval, let [a, b] be inner measure M[a, b].

Let $g \in S_d$ and let $\delta = \dfrac{\varepsilon}{(b-a)}$.

Then sup $[f(x)-g(x)] < \delta$ .The e-open sphere $\delta_{rms} = [\int_a^b (f-g)^2 dxdy]^{1/2}$

$\leq \int_a^b |f-p|\, dxdy$

$\leq \int_a^b \sup |f-p|\, dxdy \quad \leq \int_a^b \dfrac{\varepsilon}{(b-a)} dx$

$= \varepsilon$

$\therefore \quad g \in S_e(x,\varepsilon)$. and $S_d \subset S_e$.

By the LEMMA, The metric space induced by the rms metric is coarser than the metric space induced by the sup metric. Q. E. D.

From our theorem, we can expect the more desirable result using the rms metric rather than the sup metric as a measure of collage distance for closeness of two images after processing. This also means that minimization problems become easy to solve by performing standard regression of the algorithm parameters because of ease of discretization. If we use the standard inner product, it is very convenient to work with IFS maps for the non-overlappings between partitioned subsets in domain which are need to be minimized for better approximation. But more complicated contractivity requireme-nts is needed as mentioned before.

6. Steps on Encoding/Decoding

The transformation W can be regarded as a blockwise transformation in PIFS. From Collage theorem ;

$\delta(\,|W|, f\,) \leq (1-s)^{-1}\delta(W(f), f)$,

actual minimization try is on the $\delta(W(f), f)$ not on the $\delta(\,|W|, f\,)$.

The minimization process to be described consists of finding a W which satisfies $|W| \cong W(|W|)$ To find a transformation W for encode/decode, do the following steps.

1) W maps the space into itself : $W : R^N \rightarrow R^N$

2) W is a contractive transformation :

$\exists s \in [0,1) \mid \forall x,y \in R^N$,

$\delta^{\infty}(W(x), W(y)) \leq s\delta^{\infty}(x,y)$

At this time, domain blocks are s times larger than range blocks.

3)To be a contraction in a complete metric space, W has a unique fixed point s.t. $x_w = W(x_w)$. W minimizes the distance between each fixed point x_w and $|W|$.

$\delta^{\infty}(|w|, x_w) = \min \delta^{\infty}(|W|, w_i)$

Since minimization is over many transformations, finding a transformation W is a hard task. To minimize the collage distance, there are 2 kind of approaches. One is direct approaching method. The other one is indirect method which uses a concept of moment space.

7. Conclusion

To achieve the goal of fractal image compression, first of all, the contractivity of transform should be checked before applying compression method. For this reason, we have verified the reason why we should use rms metric rather than sup metric. And for the better approximation, we have compared rms metric with existent sup metric for the radius of convergence. By THEOREM 3, it is verified that the rms metric is better. Further studies will be the concerning on the more precise verifications and the required conditions keeping the contractivity with the measure evaluations to get the better fractal compression method.

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