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Bayes Estimation for the Rayleigh Failure Model

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Abstract

In this paper, we consider a hierarchical Bayes estimation of the parameter, the reliability and hazard rate function based on type-II censored samples from a Rayleigh failure model. Bayes calculations can be implemented easily by means of the Gibbs sampler. A numerical study is provided.

Key Words and Phrases: Rayleigh Failure Model, Reliability, Hazard Rate, Type-II Censoring, Hierarchical Bayes Model, Gibbs Sampler, Adaptive Rejection Sampling

1. Introduction

The Rayleigh distribution is widely used in reliability analysis, applied statistics and communication engineering. Siddiqui(1962) discussed the origin and properties of the Rayleigh distribution. Dyer and Whisenand(1973) mentioned the importance of the Rayleigh distribution in communication engineering. Sinha and Howlader(1983) obtained the Bayes estimator with respect to the Jeffreys' noninformative prior for the reliability under squared error loss function. They also proposed the Bayes credible sets and the highest posterior density credible intervals for reliability function. Howlader and Hossain(1995) derived the Bayes estimators and highest posterior density intervals for the Rayleigh parameter and its reliability

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based on type II censored data, and Bayes predictive estimators and highest posterior density prediction intervals for a future observation and also for the remaining $(n-r)$ failure times.

For Gibbs sampler approach, Dey and Lee(1992) considered Bayesian computation for the parameters and the reliability function of 2-parameter exponential distribution, and also constrained parameter and truncated data problem in multivariate life distributions. Tiwari, Yang and Zalkikar(1996) considered Bayesian estimation of the parameters and the reliability function based on type II censored samples from a Pareto failure model.

This paper considers the Gibbs sampler approach for the hierarchical Bayes analysis of the Rayleigh failure model under type-II censored data. In section 2, we give the model and describe the computation methods for Bayes estimation. In section 3, we implement the Rayleigh failure model with an illustration from the simulated data.

2. Hierarchical Bayes Model and Gibbs Sampler

In problems such as life-testing, the ordered observations are a common occurrence. In that case, time and cost can be saved by stopping the experiment after the r ordered observations have occurred, rather than waiting for all n failures. We assume that the Rayleigh model represent the life time of all items. The Rayleigh probability density function(pdf) with parameter σ^2 is given by

$$f(x|\sigma^2) = \frac{x}{\sigma^2}\exp(-\frac{x^2}{2\sigma^2}),\ x > 0. \tag{1}$$

A random sample of n items is drawn from the Rayleigh failure model (1) and is put on life test. The observed sample consists of, for a preassigned r, the ordered failure times, $x_1 < x_2 < \cdots < x_r$ and $(n-r)$ survivors. The likelihood function of the censored sample is

$$f(S|\sigma^2) \propto \frac{n!}{(n-r)!}\frac{1}{(\sigma^2)^r}\exp(-\frac{S}{2\sigma^2}), \tag{2}$$

where $S = \sum_{i=1}^{r} x_i^2 + (n-r)x_r^2$. The reliability and hazard rate functions are, respectively, given by

$$R(t) = \exp(-\frac{t^2}{2\sigma^2}), \tag{3}$$

$$h(t) = \frac{t}{\sigma^2}. \tag{4}$$

The general hierarchical Bayes model is as follow:

I. $f(S|\sigma^2) \propto \frac{n!}{(n-r)!}\frac{1}{(\sigma^2)^r}\exp(-\frac{S}{2\sigma^2})$, where $S = \sum_{i=1}^r x_i^2 + (n-r)x_r^2$.

II. $\sigma^2|\alpha_1, \beta_1 \sim$ Inverted Gamma(α_1, β_1).

III. Marginally α_1 and β_1 are mutually independent with

$f(\alpha_1) \propto \exp(-\frac{\alpha_1}{c})$, where c is a known positive constant, and

$\beta_1 \sim$ Inverted Gamma (α_2, β_2), where α_2 and β_2 are known positive constants.

In II, The inverted gamma density is given by

$$f(\sigma^2|\alpha_1, \beta_1) = \frac{1}{\Gamma(\alpha_1)\beta_1^{\alpha_1}(\sigma^2)^{\alpha_1+1}}\exp(-\frac{1}{\beta_1\sigma^2}),\ \alpha_1 > 0, \beta_1 > 0. \tag{5}$$

We denote the above distribution as $\sigma^2|\alpha_1, \beta_1 \sim IG(\alpha_1, \beta_1)$. In III, we shall consider several choices of c while doing the data analysis in Section 3.

The Gibbs sampler is an iterative Monte Carlo integration method, developed formally by Geman and Geman(1984) in the context of image restoration. In statistical framework Tanner and Wong(1987) used essentially this algorithm in their substitution sampling approach. Gelfand and Smith(1990) developed the Gibbs sampler for fairly general parametric settings.

In implementing Gibbs sampler, we follow the recommendation of Gelman and Rubin (1992) and run $m(\geq 2)$ parallel chains, each for $2d$ iterations with starting points drawn from an overdispersed distribution. But to diminish the effects of the starting distributions, the first d iterations of each chain are discarded. After d iterations, all the subsequent iterates are retained for finding the desired posterior distributions, posterior mean and variance, as well as for monitoring the convergence of the Gibbs sampler. The convergence monitoring is discussed in detail in Section 3.

To implement the Gibbs sampler, we need to calculate the full conditional distributions. From the model, the joint posterior density of σ^2, α_1 and β_1 is

$$\begin{aligned} & f(\sigma^2, \alpha_1, \beta_1|S) \\ \propto\ & f(S|\sigma^2)f(\sigma^2|\alpha_1, \beta_1)f(\alpha_1)f(\beta_1) \end{aligned}$$

$$\propto \exp[-\frac{S}{2\sigma^2} - \frac{1}{\beta_1\sigma^2} - \frac{1}{\beta_2\beta_1} - \frac{\alpha_1}{c}]\frac{1}{\Gamma(\alpha)}\frac{1}{(\sigma^2)^{\alpha_1+r+1}}\frac{1}{\beta_1^{\alpha_1+\alpha_2+1}}. \quad (6)$$

From (6), the full conditional distributions are given by

$$f(\sigma^2|\alpha_1,\beta_1,S) \propto \exp[-\frac{1}{\sigma^2}(\frac{S}{2}+\frac{1}{\beta_1})]\frac{1}{(\sigma^2)^{\alpha_1+r+1}},$$

$$\text{that is, } \sigma^2|\alpha_1,\beta_1,S \sim IG[\alpha_1+r,(\frac{S}{2}+\frac{1}{\beta_1})^{-1}], \quad (7)$$

$$f(\beta_1|\sigma^2,\alpha_1,S) \propto \exp[-\frac{1}{\beta_1}(\frac{1}{\sigma^2}+\frac{1}{\beta_2})]\frac{1}{\beta_1^{(\alpha_1+\alpha_2+1)}},$$

$$\text{that is, } \beta_1|\sigma^2,\alpha_1,S \sim IG[\alpha_1+\alpha_2,(\frac{1}{\sigma^2}+\frac{1}{\beta_2})^{-1}], \quad (8)$$

and

$$f(\alpha_1|\sigma^2,\beta_1,S) \propto \exp(-\frac{\alpha_1}{c})\frac{1}{(\beta_1\sigma^2)^{\alpha_1}}\frac{1}{\Gamma(\alpha_1)}. \quad (9)$$

In implementing the Gibbs sampler, one should be able to draw samples from the conditional densities given in (7)-(9). Simulation from the conditional densities (7) and (8) which are both inverted gamma densities can be done by standard methods. However, in order to simulate from the posterior density (9), one approach is to use the adaptive rejection sampling algorithm of Gilks and Wild(1992). Fortunately, the use of the adaptive rejection sampling algorithm of Gilks and Wild(1992) becomes simple for us because of the following result of lemma.

Lemma $f(\alpha_1|\sigma^2,\beta_1,S)$ is a log-concave function of α_1.

Proof. Consider

$$f(\alpha_1|\sigma^2,\beta_1,S) \propto \exp(-\frac{\alpha_1}{c})\frac{1}{(\beta_1\sigma^2)^{\alpha_1}}\frac{1}{\Gamma(\alpha_1)} \quad (10)$$

then

$$log f(\alpha_1|\sigma^2,\beta_1,S) = a - \alpha_1 log(\beta_1\sigma^2) - log\Gamma(\alpha_1) - \frac{\alpha_1}{c} \quad (11)$$

where a is the norming constant. Hence

$$\begin{aligned}\frac{\partial logf(\alpha_1|\sigma^2,\beta_1,S)}{\partial\alpha_1} &= -log(\beta_1\sigma^2)-\frac{1}{c}-\frac{d}{d\alpha_1}log(\frac{\Gamma(\alpha_1+1)}{\alpha_1}) \\ &= -log(\beta_1\sigma^2)-\frac{1}{c}+\frac{1}{\alpha_1}-\frac{\int_0^\infty logze^{-z}z^{\alpha_1}dz}{\int_0^\infty e^{-z}z^{\alpha_1}dz}. \qquad (12)\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{\partial^2 logf(\alpha_1|\sigma^2,\beta_1,S)}{\partial\alpha_1^2} \\ =& -\frac{1}{\alpha_1^2}-\left[\frac{\int_0^\infty(logz)^2e^{-z}z^{\alpha_1}dz}{\int_0^\infty e^{-z}z^{\alpha_1}dz}-\left(\frac{\int_0^\infty logze^{-z}z^{\alpha_1}dz}{\int_0^\infty e^{-z}z^{\alpha_1}dz}\right)^2\right] \\ =& -\frac{1}{\alpha_1^2}-Var(logz)<0, \text{ since } z\sim \text{Gamma}(1,\alpha_1+1). \qquad (13)\end{aligned}$$

So all required random variated generation is straightforward.

Using the Gibbs sampler, the posterior distribution of σ^2 given S is approximated by

$$f(\sigma^2|S)\approx(md)^{-1}\sum_{k=1}^{m}\sum_{l=d+1}^{2d}f(\sigma^2|\alpha_1=\alpha_{1kl},\beta_1=\beta_{1kl},S).$$

Also following Gelfand and Smith(1991), Rao-Blackwellized estimates of posterior mean and variance of the σ^2 are given by

$$E(\sigma^2|S)\approx(md)^{-1}\sum_{k=1}^{m}\sum_{l=d+1}^{2d}\frac{1}{\alpha_{1kl}+r-1}(\frac{S}{2}+\frac{1}{\beta_{1kl}}) \qquad (14)$$

and

$$\begin{aligned}Var(\sigma^2|S) \approx& (md)^{-1}\sum_{k=1}^{m}\sum_{l=d+1}^{2d}\frac{1}{(\alpha_{1kl}+r-1)^2(\alpha_{1kl}+r-2)}(\frac{S}{2}+\frac{1}{\beta_{1kl}})^2 \\ +& (md)^{-1}\sum_{k=1}^{m}\sum_{l=d+1}^{2d}\frac{1}{(\alpha_{1kl}+r-1)^2}(\frac{S}{2}+\frac{1}{\beta_{1kl}})^2 \\ -& [(md)^{-1}\sum_{k=1}^{m}\sum_{l=d+1}^{2d}\frac{1}{\alpha_{1kl}+r-1}(\frac{S}{2}+\frac{1}{\beta_{1kl}})]^2. \qquad (15)\end{aligned}$$

3. A Numerical Example

In this Section, an illustrative example is represented by simulated data. In our simulation data, we take $n = 20$, $r = 10$, $\sigma^2 = 5.0$ and generate the observations x_j from the Rayleigh failure model with σ^2. For hierarchical Bayesian analysis, using Gibbs sampler, we need the marginal posterior densities which are as follows:

$$\sigma^2|\alpha_1, \beta_1, S \sim IG[\alpha_1 + r, (\frac{S}{2} + \frac{1}{\beta_1})^{-1}], \tag{16}$$

$$\beta_1|\sigma^2, \alpha_1, S \sim IG[\alpha_1 + \alpha_2, (\frac{1}{\sigma^2} + \frac{1}{\beta_2})^{-1}], \tag{17}$$

$$f(\alpha_1|\sigma^2, \beta_1, S) \propto \exp(-\frac{\alpha_1}{c})\frac{1}{(\beta_1\sigma^2)_1^{\alpha}}\frac{1}{\Gamma(\alpha_1)}. \tag{18}$$

We place vague second-stage prior on β_1 , both having prior mean and prior standard deviation equal to 100,000, that is, $\alpha_2 = 3$ and $\beta_2 = 5 \times 10^{-6}$.

To implement and monitor the convergence of the Gibbs sampler, we follow the basic approach of Gelman and Rubin(1992). We consider 10 independent sequences each with a sample of size 1,000, and with a burn-in sample of another 1,000.

To monitor the convergence of the Gibbs sampler for σ^2, the parameter of interest, we follow Gelman and Rubin(1992). Compute $B/1,000$ = the variance between the 10 sequence means $\bar{\sigma}_g^2$ each based on 1,000 values. Also, let W denoted the average of the 10 within-sequence variance. Then find

$$\widehat{V} = \frac{1,000-1}{1,000}W + \frac{1}{1,000}B + \frac{1}{10 \times 1,000}B \tag{19}$$

Finally, find $\sqrt{\widehat{R}} = \widehat{V}/W$. If $\widehat{R}$ is near 1 for the scalar estimands $\widehat{\sigma}^2$ of interest, then this suggests that the desired convergence is achieved in the Gibbs sampler.

An inspection of table 1 reveals that the hierarchical Bayes procedure is not sensitive to the choice of "c" as different choices of "c" can lead to almost same point estimates of $\widehat{\sigma}^2$. So we use the $c = 100$ and hence $\widehat{\sigma}^2 = 5.5234$. For censored and complete observations, the figures 1, 2 and 3 are graphs of $\widehat{f}(\sigma^2|S)$, $\widehat{f}(h(t)|S)$, $\widehat{f}(R(t)|S)$ at a mission time $t = 2$, respectively. From the Gibbs sampler, $R_1^{(t)}, \cdots, R_m^{(t)}$ is a sample from $f(R(t)|S)$ and $h_1^{(t)}, \cdots, h_m^{(t)}$ is a sample from $f(h(t)|S)$. The 90% credible intervals are $(R_{(0.05m)}^{(t)}, R_{(0.95m)}^{(t)})$ and $(h_{(0.05m)}^{(t)}, h_{(0.95m)}^{(t)})$. $(0.05m)$ and $(0.95m)$ are the $0.05m^{th}$ and $0.95m^{th}$ order statistics. For various mission time t, tables 2 and 3 give the true values, the posterior means, the residuals and 90% credible intervals for $R(t)$ and $h(t)$ on same censored data.

Table 1: Posterior Mean and Standard Deviation of σ^2

C	0.01	0.1	1	10	100
$\hat{\sigma}^2$	5.5504	5.5378	5.5259	5.5259	5.5234
$\widehat{SD}(\sigma^2)$	1.9614	1.9548	1.9483	1.9482	1.9469

Table 2: Posterior Mean and 90% Credible Interval of $R(t)$

t	$R(t)$	$\hat{R}(t)$	$R(t) - \hat{R}(t)$	90% Credible Interval
1.5	0.7985	0.8019	-0.0034	(0.6997,0.8847)
2.0	0.6703	0.6776	-0.0073	(0.5301,0.8043)
2.5	0.5353	0.5479	-0.0126	(0.3709,0.7116)
3.0	0.4066	0.4250	-0.0185	(0.2397,0.6126)
3.5	0.2938	0.3173	-0.0236	(0.1431,0.5133)
4.5	0.1320	0.1598	-0.0278	(0.0402,0.3320)

Table 3: Posterior Mean and 90% Credible Interval of $h(t)$

t	$h(t)$	$\hat{h}(t)$	$h(t) - \hat{h}(t)$	90% Credible Interval
1.5	0.3000	0.2975	0.0025	(0.1633,0.4761)
2.0	0.4000	0.3967	0.0033	(0.2178,0.6348)
2.5	0.5000	0.4958	0.0042	(0.2722,0.7934)
3.0	0.6000	0.5950	0.0050	(0.3267,0.9521)
3.5	0.7000	0.6942	0.0058	(0.3811,1.1108)
4.5	0.9000	0.8925	0.0075	(0.4900,1.4282)

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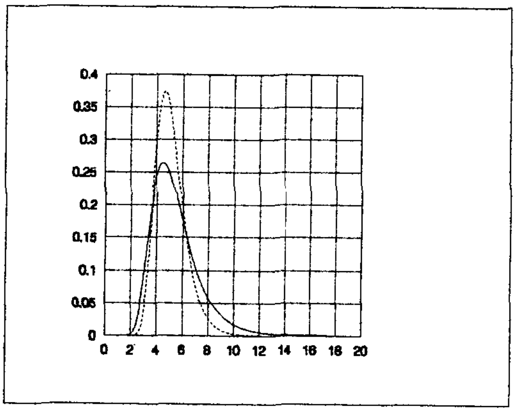

Figure 1: Estimated pdf of $\sigma^2|S$; Dotted line - complete data, Solid line - censored data

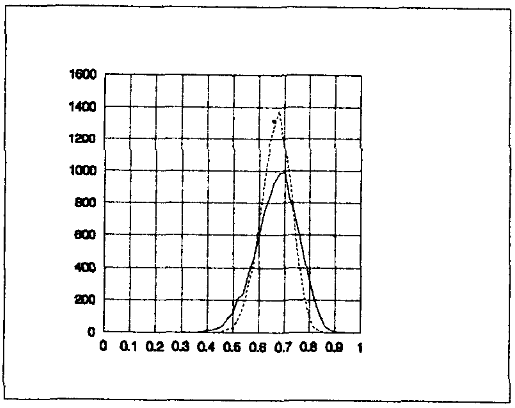

Figure 2: Estimated pdf of $R(t)|S$; Dotted line - complete data, Solid line - censored data

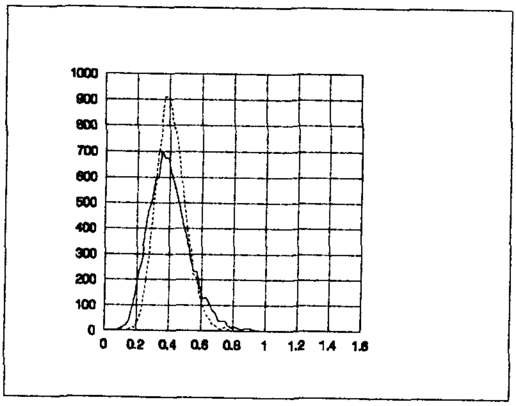

Figure 3: Estimated pdf of $h(t)|S$; Dotted line - complete data, Solid line - censored data