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AMLE for Normal Distribution under Progressively Censored Samples

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Abstract

By assuming a progressively censored sample, we propose the approximate maximum likelihood estimator (AMLE) of the location and the scale parameters of the two-parameter normal distribution and obtain the asymptotic variances and covariance of the AMLEs. An example is given to illustrate the methods of estimation discussed in this paper.

Key Words and Phrases: Approximate maximum likelihood estimator, Normal distribution, Progressively censored sample

1. Introduction

In life and dosage response studies, progressively censored samples frequently occur, when at various stages of an experiment, some though not all of the survivors are removed from observation. When test facilities are limited and when prolonged tests are expensive, the early censoring of a substantial number of sample specimens leaves test failites free for other tests, while specimens that remain on test until subsequent failure provide information on a limited number of the more extreme sample values.

Cohen and Norgaard (1977) discussed the inference problems for a wide range of distributions under this progressive censoring scheme. These developments were summarized by Cohen and Whitten (1988), and more by Cohen (1991). Viveros and Balakrishnan (1994) developed exact conditional inference based on progressive Type-II censored samples. Balakrishnan and Sandhu (1996) obtained the best linear unbiased estimators (BLUEs) and the maximum likelihood estimators (MLE) for exponential distribution under general progressive Type-II censored sample. Kang

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and Cho (1998) derived the minimum risk estimator (MRE) of the location and the scale parameter of the two-parameter exponential distribution under general progressive Type-II censored sample.

In progressively censored samples, these samples consist of a total of N observations. they are right censored at points $T_1 < T_2 < \ldots < T_j \ldots < T_k$. At $x = T_j$, c_j observations are censored and for these it is known only that $x > T_j$. The total number of censored observations is $\sum_{j=1}^{k} c_j$, and the number of fully measured observations is $n = N - \sum_{j=1}^{k} c_j$.

Let $f(x;\theta_1,\theta_2,\ldots,\theta_r)$ and $F(x;\theta_1,\theta_2,\ldots,\theta_r)$ designate the probability density function (pdf) and the cumulative distribution function (cdf) of an unrestriced (i.e., complete) distribution with parameters $\theta_1,\theta_2,\ldots,\theta_r$. The Likelihood function of progressively censored at T_j $(j = 1, 2, \ldots, k)$ is given by

$$L(\ ) = K \prod_{j=1}^{k} [1 - F(T_j)]^{c_j} \prod_{i=1}^{n} f(x_i), \tag{1}$$

where K denotes ordering constants that do not depend on the parameters.

In this paper, we propose the AMLEs of the location parameter and the scale parameter of the two-parameter normal distribution with progressively censored samples and obtain the asymptotic variances and the covariance of the AMLEs. We will calculate the AMLEs of parameters and the standard errors (SE) of the AMLEs through example.

2. Estimation for parameter

The random variable X has a two-parameter normal distribution if it has a pdf of form;

$$f(x;\mu,\ \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma}},\ -\infty < x < \infty, \tag{2}$$

where σ is the scale and μ is the location parameter. The pdf and the cdf of the standard normal distribution $Z = (X - \mu)/\sigma$ become

$$\begin{aligned} \phi(z;0,\ 1) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \\ \Phi(z;0,\ 1) &= \int_{-\infty}^{z} \phi(y;0,\ 1)dy. \end{aligned}$$

From the likelihood function (1), the loglikelihood function of a progressively censored sample from a normal distribution (2) is given by

$$\ln L = -n\ \ln\sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 + \sum_{j=1}^{k} \ln(1 - \Phi(\xi_j)) + \text{cost.}, \tag{3}$$

where $\xi_j = (T_j - \mu)/\sigma$.

We differentitate the loglikelihood function (3) with respect to μ and σ and equate to zero to obtain the maximum likelihood estimating equations;

$$\begin{aligned}\frac{\partial \ln L}{\partial \mu} &= \frac{n}{\sigma}\Big[\frac{\bar{x}-\mu}{\sigma} + \sum_{j=1}^{k}\frac{c_j}{n}Q_j\Big] = 0,\\ \frac{\partial \ln L}{\partial \sigma} &= \frac{n}{\sigma}\Big[\frac{s^2+(\bar{x}-\mu)^2}{\sigma^2} - 1 + \sum_{j=1}^{k}\frac{\xi_j c_j}{n}Q_j\Big] = 0,\end{aligned} \tag{4}$$

where $Q_j = Q(\xi_j) = \phi(\xi_j)/(1-\Phi(\xi_j))$ and where $\bar{x}$ and s^2 are the mean and the variance of the n complete sample observations. The pair of equations (4) may be simplifed to the form

$$\begin{aligned}\bar{x}-\mu &= -\sigma\sum_{j=1}^{k}\frac{c_j}{n}Q_j,\\ s^2+(\bar{x}-\mu)^2 &= \sigma^2\Big[1-\sum_{j=1}^{k}\frac{\xi_j c_j}{n}Q_j\Big].\end{aligned} \tag{5}$$

Any one of various standard iterative procedures may be used to solve the pair of equations (5) for the maximum likelihood estimators $\widehat{\mu}_{MLE}$ and $\widehat{\sigma}_{MLE}$. The asymptotic variance-covariance matrix of $\widehat{\mu}_{MLE}$ and $\widehat{\sigma}_{MLE}$ is given as

$$\begin{bmatrix} E\Big(-\frac{\partial^2 \ln L}{\partial \mu^2}\Big) & E\Big(-\frac{\partial^2 \ln L}{\partial \mu \partial \sigma}\Big) \\ E\Big(-\frac{\partial^2 \ln L}{\partial \mu \partial \sigma}\Big) & E\Big(-\frac{\partial^2 \ln L}{\partial \sigma^2}\Big)\end{bmatrix}^{-1} \tag{6}$$

The MLEs of μ and σ do not exist in an explicit form and have to be determined from the likelihood equation by a numerical method. But we can expand the function Q_j, appearing in (4) to Taylor series around the point $a_j = (T_j - \bar{x})/s$ and then approximate it by

$$\frac{\phi(\xi_j)}{1-\Phi(\xi_j)} \simeq \alpha_j + \beta_j \xi_j \tag{7}$$

where

$$\alpha_j = \frac{\phi(a_j)}{1-\Phi(a_j)} - \Big[\frac{-a_j\phi(a_j)[1-\Phi(a_j)] + \phi(a_j)^2}{[1-\Phi(a_j)]^2}\Big]a_j$$

and

$$\beta_j = \frac{-a_j\phi(a_j)[1-\Phi(a_j)] + \phi(a_j)^2}{[1-\Phi(a_j)]^2}.$$

Now making use of the approximate expression in (7), we obtain the approximate likelihood equation of (4) as follows;

$$\frac{\partial \ln L}{\partial \mu} \simeq \frac{\partial \ln L^*}{\partial \mu} = \frac{n}{\sigma}\Big[\frac{\bar{x}-\mu}{\sigma} + \sum_{j=1}^{k}\frac{c_j}{n}(\alpha_j + \beta_j\xi_j)\Big] = 0, \tag{8}$$

$$\frac{\partial \ln L}{\partial \sigma} \simeq \frac{\partial \ln L^*}{\partial \sigma} = \frac{n}{\sigma}\Big[\frac{s^2 + (\bar{x}-\mu)^2}{\sigma^2} - 1 + \sum_{j=1}^{k}\frac{\xi_j c_j}{n}(\alpha_j + \beta_j\xi_j)\Big] = 0. \tag{9}$$

From solving equations (8) and (9) for μ and σ, we derive the AMLEs of μ and σ as follows;

$$\widehat{\mu}_{AMLE} = a + b\,\widehat{\sigma}_{AMLE} \tag{10}$$

$$\widehat{\sigma}_{AMLE} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{11}$$

where

$$a = \frac{\sum_{i=1}^{n} x_i + \sum_{j=1}^{k} c_j\beta_j T_j}{n + \sum_{j=1}^{k} c_j\beta_j},$$

$$b = \frac{\sum_{j=1}^{k} c_j\alpha_j}{n + \sum_{j=1}^{k} c_j\beta_j},$$

$$A = -1 + b^2 + \frac{b}{n}\Big(b\sum_{j=1}^{k} c_j\beta_j - \sum_{j=1}^{k} c_j\alpha_j\Big),$$

$$B = -2(\bar{x}-a)b + \frac{1}{n}\Big(\sum_{j=1}^{k} c_j\alpha_j(T_j - a) - 2b\big(\sum_{j=1}^{k} c_j\beta_j(T_j - a)\big)\Big),$$

and

$$C = s^2 + (\bar{x}-a)^2 + \frac{1}{n}\sum_{j=1}^{k} c_j\beta_j(T_j - a)^2.$$

3. Asymptotic Properties of the proposed estimator

The conditional bias of $\widehat{\mu}_{AMLE}$ can be computed exactly from (10). But it is not possible to determine the conditional bias of $\widehat{\sigma}_{AMLE}$ exactly, it may be evaluated approximately by $E(\partial \text{ln}L^*/\partial\sigma)/E(-\partial^2\text{ln}L^*/\partial\sigma^2)$ (see Kendall and Stuart (1973)). Moreover, we obtain from equations (7) and (8) that

$$E\Big(-\frac{\partial^2 \text{ln}L^*}{\partial\mu^2}\Big) = \frac{nV_1}{\sigma^2},$$
$$E\Big(-\frac{\partial^2 \text{ln}L^*}{\partial\mu\partial\sigma}\Big) = \frac{nV_2}{\sigma^2},$$

and

$$E\Big(-\frac{\partial^2 \text{ln}L^*}{\partial\sigma^2}\Big) = \frac{nV_3}{\sigma^2}.$$

where

$$V_1 = 1+\frac{1}{n}\sum_{j=1}^{k} c_j\beta_j,$$
$$V_2 = \frac{1}{n}(\sum_{j=1}^{k} c_j\alpha_j + 2\sum_{j=1}^{k} c_j\beta_j\xi_j),$$

and

$$V_3 = -1+\frac{1}{n}\Big(3\frac{n^2+n-1}{(n-1)} + 2\sum_{j=1}^{k} c_j\alpha_j c_j + 3\sum_{j=1}^{k} c_j\beta_j\xi_j^2\Big).$$

From the above expressions, we can ontain the asymptotic variances and covariance of the AMLEs as follows;

$$\text{Var}(\widehat{\mu}_{AMLE}) \simeq \frac{\sigma^2 V_3}{n(V_1V_3 - V_2^2)},$$
$$\text{Var}(\widehat{\sigma}_{AMLE}) \simeq \frac{\sigma^2 V_1}{n(V_1V_3 - V_2^2)}, \quad (12)$$

and

$$\text{Cov}(\widehat{\mu}_{AMLE},\ \widehat{\sigma}_{AMLE}) \simeq -\frac{\sigma^2 V_2}{n(V_1V_3 - V_2^2)}. \quad (13)$$

In order to illustrate the calculations involved in partical applications of the probit technique, we use the example was given by Cohen (1963). A random sample of 300 units of a certain type of electronic component were place on test, and the life-span x was recorded for each item that failed. At the end of 1650 hours, 50 of the survivors were withdrawn (censored), and the test was terminated after 1735 hours with 95 survivors. The sample is summarized as : $N = 300,\ T_1 = 1650,\ T = 1735,\ n = 120 + 35 = 155, c_1 = 50, c_2 = 95, \bar{x} = 1544.8,$ and $s^2 = 17022$. From the simultaneous solution of these equations (5) and the asymptotic variance-covariance matrix (6), the MLEs and the approximate standard errors of the MLEs are given by

$$\widehat{\mu}_{MLE} = 1702.83,\ \widehat{\sigma}_{MLE} = 208.33,$$

and

$$\sigma_{\widehat{\mu}_{MLE}} = 14.56,\ \sigma_{\widehat{\sigma}_{MLE}} = 13.00.$$

From the estimators (10) and (11) and the asymptotic variances (12), we obtain the estimates and the approximate standard errors of the AMLEs as follows;

$$\widehat{\mu}_{AMLE} = 1690.23,\ \widehat{\sigma}_{AMLE} = 203.42,$$

and

$$\sigma_{\widehat{\mu}_{AMLE}} = 13.35,\ \sigma_{\widehat{\sigma}_{AMLE}} = 11.90.$$

From this example, the two estimators are not much different and the proposed estimators are more efficient than the MLE in the sense of SEs. The MLEs are very complicated but the proposed estimators have explicit form. So the proposed estimators are useful to estimate the location and scale parameter of the normal distribution under progressively censored samples.

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