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A Note on Asymptotic Relative Efficiency of the Nonparametric Reliability Estimation for the Proportional Hazards Model

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Abstract

This paper presents the asymptotic relative efficiency of the nonparametric estimator relative to the parametric maximum likelihood estimator of the reliability function under the proportional hazards model of random censorship. Also we examine the efficiency loss due to censoring proportions and misson times.

Key Words and Phrases: asymptotic relative efficiency, proportional hazards model, random censorship

1. Introduction

The proportional hazards model(PHM) has become an important topic in industrial life testing in recent years. The PHM has received special attention since the paper of Koziol and Green(1976). Chen, Hollander, and Langberg(1982) computed the small sample bias and variance of the Kaplan-Meier estimator under PHM. Ebrahimi(1985) suggested the nonparametric estimator of reliability and studied the asymptotic properties of the nonparametric estimator. Under the PHM, the nonparametic estimation of the survival function has been considered by Cheng and Lin(1987), Hollander and Pena(1989), and Herbst(1993).

We are interested in by how much nonparametric approach relative to parametric approach under PHM will invoke the efficiency loss. This note examines the asymptotic relative efficiency of the nonparametric reliability estimation relative to the parametric reliability estimation under the PHM of random censorship.

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2. Asymptotic Relative Efficiency

In this section, we examine the efficiency loss of the nonparametric MLE relative to the parametric MLE of reliability function under the PHM with random censorship.

Let $T_1, \cdots, T_n$ be a random sample from reliability function $\overline{F}$ and let $C_1, \cdots, C_n$ be a random sample from reliability function $\overline{G}$. Then instead of observing a complete sample $T_1, \cdots, T_n$, we can only observe the pairs $Z_i = \min(T_i, C_i)$and $\delta_i = I(T_i = Z_i)$, $i = 1, \cdots, n$. We assume that $T_1, \cdots, T_n$ and $C_1, \cdots, C_n$are independent, $\overline{F}(0) = \overline{G}(0) = 1$, $\overline{F}(\infty) = \overline{G}(\infty) = 0$, $\overline{F}$ and $\overline{G}$ are continuous. For this random censoring scheme, we consider the proportional hazards model with $\overline{G} = \overline{F}^{\beta}$ and β is an unknown positive real number.

In reliability analysis, the Kaplan-Meier(1957) estimator(KME) plays an important role and has wide range of applications. The KME of the reliability function $\overline{F}(x)$ is defined by

$$\widehat{\overline{F}}_{KM}(t) = \prod_{i:Z_{(i)} \leq t} \left(\frac{n-i}{n-i+1}\right)^{\delta_{(i)}}. \tag{1}$$

It is known that the estimator $\widehat{\overline{F}}_{KM}$ is asymptotically unbiased and uniformly strongly consistent, and when properly normalized that they converge weakly to the Gaussian process. That is, the asymptotic distribution of the estimator $\widehat{\overline{F}}_{KM}$ is

$$N\Big(\overline{F}(t),\ \frac{\overline{F}(t)^2}{n} \int_0^t \frac{dF(u)}{(1-F(u))^2(1-G(u))}\Big)$$

Under this PHM, Ebrahimi(1985) proposed a nonparametric estimator $\widehat{\overline{F}}_n(t)$ for the reliability function $\overline{F}(t)$ as follows :

$$\widehat{\overline{F}}_n(t) = \overline{\delta}\exp\Big\{\overline{\delta}\ln\frac{1}{\overline{\delta}} + \overline{\delta}\ln\widehat{\overline{H}}(t)\Big\} + (1-\overline{\delta})\exp\Big\{\overline{\delta}\ln\frac{1}{(1-\overline{\delta})} + \overline{\delta}\ln\widehat{\overline{S}}(t)\Big\}, \tag{2}$$

where

$$\overline{\delta} = \frac{1}{n}\sum_{i=1}^{n}\delta_i, \quad \widehat{\overline{H}}(t) = \frac{1}{n}\sum_{i=1}^{n} I(Z_i > t, \delta_i = 1), \quad \text{and} \quad \widehat{\overline{S}}(t) = \frac{1}{n}\sum_{i=1}^{n} I(Z_i > t, \delta_i = 0).$$

It is known that the $\widehat{\overline{F}}_n(t)$ is strongly consistent, and when properly normalized that this estimator converges weakly to the Gaussian process. That is, the asymptotic distribution of $\widehat{\overline{F}}_n(t)$ is

$$N\Big(\overline{F}(t), Var_{asy}(\widehat{\overline{F}}_n(t))\Big),$$

where

$$\begin{aligned}
Var_{asy}\Big(\widehat{\overline{F}}_n(t)\Big) &= \frac{1}{n}\Big[\frac{\beta}{(\beta+1)^2}\overline{F}^2(t)(\theta\ln\frac{1}{\theta}+\theta+(1-\theta)\ln\frac{1}{1-\theta}-1 \\
&+ \theta\ln\overline{H}(t)+(1-\theta)\ln\overline{S}(t))^2\Big]+\Big\{\overline{H}(t)(1-\overline{H}(t))\theta^4\frac{\overline{F}^2(t)}{\overline{H}^2(t)}\Big\} \\
&+ \Big\{\theta^2(1-\theta)^2\frac{\overline{F}^2(t)}{\overline{S}(t)}(1-\overline{S}(t))\Big\}
\end{aligned}$$

and $\theta = \frac{1}{1+\beta}$.

If the reliability function $\overline{F}(t)$ is one of a parametric family of distributions $\Big\{\overline{F}_\lambda(t), \lambda \in \Lambda\Big\}$ and $\widehat{\lambda}$ is MLE of λ, then associated MLE of $\overline{F}(t)$ is $\overline{F}_{\widehat{\lambda}}(t)$. Under standard regularity conditions (see Rao, 1973), the estimator $\overline{F}_{\widehat{\lambda}}(t)$ has asymptotic normal distribution with variance

$$Var_{asy}\Big\{\overline{F}_{\widehat{\lambda}}(t)\Big\} = \frac{1}{n}I^{-1}(\lambda)\Big(\frac{\partial\overline{F}_\lambda(t)}{\partial\lambda}\Big)^2 \tag{3}$$

where $I(\lambda) = \int_0^\infty\Big\{\frac{\partial\ln f_\lambda(u)}{\partial\lambda}\Big\}^2 f_\lambda(u)\{1-g(u)\}du + \int_0^\infty\Big\{\frac{\partial\ln\overline{F}_\lambda(u)}{\partial\lambda}\Big\}^2\overline{F}_\lambda(u)g(u)du,$
$f_\lambda(t) = -d\overline{F}_\lambda(t)/dt,$ and $g(t) = -d\overline{G}(t)/dt.$

Now, in order to examine the efficiency loss of the nonparametric MLE relative to the parametric MLE, we use the asymptotic relative efficiency $e(\widehat{\overline{F}}_n(t), \overline{F}_{\widehat{\lambda}}(t))$ as follows :

$$e(\widehat{\overline{F}}_n(t)\ ,\ \overline{F}_{\widehat{\lambda}}(t)\ ) = \frac{Var_{asy}(\overline{F}_{\widehat{\lambda}}(t)\ )}{Var_{asy}(\widehat{\overline{F}}_n\ (t)\ )}. \tag{4}$$

3. Specific Case

For the exponential model, $\overline{F}_\lambda(t) = \exp(-\lambda t)$ and $\overline{G}(t) = \exp(-\lambda\beta t)$, the calculation leads to

$$e\left(\widehat{\overline{F}}_n(t), \overline{F}_{\widehat{\lambda}}(t)\right) = \frac{(\lambda t)^2}{\theta^2(1-\theta)\left(\theta - 1 - \frac{\lambda t}{\theta}\right)^2 + \theta^3 e^{\frac{\lambda t}{\theta}} - \theta^3(1 - 2\theta + 2\theta^2)}. \qquad (5)$$

Table 1 presents the asymptotic relative efficiency of the nonparametric estimator $\widehat{\overline{F}}_n(t)$ at different reliability $\exp(-\lambda t)$ for several censoring rates $CP = \beta/(1+\beta)$. As the reliability becomes high, the efficiency drops to near zero. But, there is no simple relationship between the cases with and without censoring.

Table 1. Asymptotic relative efficiency $e(\widehat{\overline{F}}_n(t),\ \overline{F}_{\widehat{\lambda}}(t))$

	λt	.10536	.22314	.35667	.51083	.69315	.91629	1.20397	1.60944	2.30258
CP	SF	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0		.0999	.1992	.2968	.3914	.4805	.5597	.6212	.6476	.5891
20		.0444	.1391	.2520	.3668	.4733	.5623	.6221	.6308	.5318
33		.0514	.1592	.2828	.4022	.5060	.5846	.6256	.6054	.4640
50		.0716	.2034	.3378	.4545	.5446	.6002	.6082	.5406	.3378
66		.1101	.2723	.4110	.5138	.5764	.5896	.5327	.3731	.1239
75		.1416	.3215	.4586	.5476	.5846	.5544	.4300	.2127	.0330
80		.1723	.3650	.4975	.5704	.5769	.4931	.3036	.0942	.0066

where CP = censoring proportions, and SF = $\exp(-\lambda t)$.

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