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The Range of confidence Intervals for σ_A^2/σ_B^2 in Two-Factor Nested Variance Component Model

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Abstract

The two-factor nested variance component model with equal numbers in the cells are given by $y_{ijk} = \mu + A_i + B_{ij} + C_{ijk}$ and the confidence intervals for the ratio of variance components, σ_A^2/σ_B^2 are obtained in various forms by many authors. This article shows the probability ranges of these confidence intervals on σ_A^2/σ_B^2 proved by the mathematical computation.

Key Words and Phrases: Variance component, confidence interval.

1. Introduction

Consider the two-factor nested variance component model given by $y_{ijk} = \mu + A_i + B_{ij} + C_{ijk}$, where $i = 1, 2, ..., I, j = 1, 2, ..., J$ and $k = 1, 2, ..., K$. A_i, B_{ij} and C_{ijk} are independent unobservable random variables, and $A_i \sim N(0, \sigma_A^2)$, $B_{ij} \sim N(0, \sigma_B^2)$ and $C_{ijk} \sim N(0, \sigma_C^2)$. μ is an unknown parameter and the y_{ijk} are observable random variables. An analysis of variance for the model is displaced in Graybill(1976) and confidence intervals of these function were studied by many authors. Broemeling(1969) found out the confidence intervals on σ_A^2/σ_C^2 and σ_B^2/σ_C^2. Wang(1978) found out the confidence intervals on σ_A^2/σ_C^2 and σ_A^2/σ_B^2 with the better range of confidence intervals. As well as, Graybill and Wang(1979) found out the ratio of total variances of confidence intervals on $\sigma_A^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)$, $\sigma_B^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)$ and $\sigma_C^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)$. Burdick and Graybill(1985) showed the range of confidence intervals $\sigma_A^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)$, $\sigma_B^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)$ and $\sigma_C^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)$ of Graybill and Wang were better than those of Satterthwate's. But these problems are still being studied and must continue to be studied also in the case of these that have been determined to discover whether they fit exactly the given confidence coefficient $1 - \alpha$ and to determine confidence intervals closer to the $100(1 - \alpha)$ percent confidence coefficient.

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Lately, Burch (1996) developed procedures for selecting the best "LaMotte-McMhorter type" confidence interval for a ratio of variance components in a mixed linear model with two sources of variation based on the unbiasedness and expected length of the intervals. Ofversten (1993) presented a method for deriving exact procedures for testing variance components in unbalanced mixed linear models by the so-called resampling method. Fayyad et al. (1996) derived an equality to place a bound on the power of the resampling test. Christensen (1996) proposed an unified treatment of Ofversten's method with Wald's test. Ueng (1997) improved and extended in this problems. We consider the Broemeling's asymptotic confidence intervals and Wang's confidence intervals on σ_A^2/σ_B^2 by simulation studies. Then we wlll show that the probability range of these confidence intervals on σ_A^2/σ_B^2 are equals and whether grate than $1-\alpha$ or not.

2. Variance Component Model

We consider the following variance component model.

(1). Two-factor nested variance model data structure :

A_1	A_2	...	A_I
$B_{11}B_{12}..B_{1J}$	$B_{21}B_{22}..B_{2J}$	...	$B_{I1}B_{I2}..B_{IJ}$
$y_{111}y_{121}..y_{1J1}$	$y_{211}y_{221}..y_{2J1}$	...	$y_{I11}y_{I21}..y_{IJ1}$
$y_{112}y_{122}..y_{1J2}$	$y_{212}y_{222}..y_{2J2}$	...	$y_{I12}y_{I22}..y_{IJ2}$
.....	.....	...	.....
$y_{11K}y_{12K}..y_{1JK}$	$y_{21K}y_{22K}..y_{2JK}$	...	$y_{I1K}y_{I2K}..y_{IJK}$

(2) ANOVA table

Source	*D.F.*	*S.S.*	*M.S.*	*E.M.S.*
Σ	IJK	$\Sigma_{i=1}^{I}\Sigma_{j=1}^{J}\Sigma_{k=1}^{K}Y_{ijk}^2$		
Mean	1	$IJK\bar{Y}^2$		
A	$n_1 = I-1$	$\Sigma_{i=1}^{I}\Sigma_{j=1}^{J}\Sigma_{k=1}^{K}(\bar{y}_{i..}-\bar{y}_{...})^2$	S_1^2	θ_1
B	$n_2 = I(J-1)$	$\Sigma_{i=1}^{I}\Sigma_{j=1}^{J}\Sigma_{k=1}^{K}(\bar{y}_{ij.}-\bar{y}_{i..})^2$	S_2^2	θ_2
C	$n_3 = IJ(K-1)$	$\Sigma_{i=1}^{I}\Sigma_{j=1}^{J}\Sigma_{k=1}^{K}(y_{ijk}-\bar{y}_{ij.})^2$	S_3^2	θ_3

where $\theta_1 = \sigma_C^2 + K\sigma_B^2 + JK\sigma_A^2$, $\theta_2 = \sigma_C^2 + K\sigma_B^2$ and $\theta_3 = \sigma_C^2$. And $Y_{ijk} = \bar{y}_{...} + (\bar{y}_{i..}-\bar{y}_{...})+(\bar{y}_{ij.}-\bar{y}_{i..})+(y_{ijk}-\bar{y}_{ij.})$, $\hat{\sigma_C^2} = S_3^2$, $\hat{\sigma_B^2} = \frac{1}{K}(S_2^2-S_3^2)$ and $\hat{\sigma_A^2} = \frac{1}{JK}(S_1^2-S_2^2)$ are the best unbiased estimates of σ_C^2, σ_B^2 and σ_A^2 respectively. Let $\frac{n_i S_i^2}{\theta_i}$ be U_i, where $i = 1,2,3$ and U_i are independent chi-square distribution, respectively. And let ${\chi^2}_{\alpha,n_i}$ be $F_{\alpha,n_i,\infty}$, where $i = 1,2,3$ and F_α is the upper α probability point of Snedecor's F distribution.

Example. A transistor manufacturing plant has many production lines, and each production line has many machines. Transistor K was manufactured by the machine J on the production line I. The model of the observation value of transistor k manufactured by the machine j on the production line i is represented by $Y_{ijk} = \mu + A_i + B_{ij} + C_{ijk}$.

3. The Range of confidence Intervals

Wang(1978)'s results were obtained by using the Broemeling's method for the Kimball's confidence intervals for σ_A^2/σ_B^2 in the two-factor nested variance component model. In this model, our results of the range of confidence intervals for σ_A^2/σ_B^2 are obtained as following .

Theorem 1. The range of the Kimball's confidence intervals for σ_A^2/σ_B^2 in the two-factor nested variance component model by using the Broemeling's method are obtained as follows.

(1). $P[L \leq \frac{\sigma_A^2}{\sigma_B^2}] \geq 1-\alpha$, where $L = \dfrac{\frac{S_1^2 F_{1-\alpha,n_2,n_3}}{S_2^2 F_{\alpha,n_1,n_3}} - 1}{J(1-\frac{S_3^2 F_{1-\alpha,n_2,n_3}}{S_2^2})}$ and $\frac{S_2^2}{S_3^2} > F_{1-\alpha,n_2,n_3}$.

(2). $P[\frac{\sigma_A^2}{\sigma_B^2} \leq U] \geq 1-\alpha$, where $U = \dfrac{\frac{S_1^2 F_{\alpha,n_2,n_3}}{S_2^2 F_{1-\alpha,n_1,n_3}} - 1}{J(1-\frac{S_3^2 F_{\alpha,n_2,n_3}}{S_2^2})}$ and $\frac{S_2^2}{S_3^2} > F_{\alpha,n_2,n_3}$.

(3). $P[L \leq \frac{\sigma_A^2}{\sigma_B^2} \leq U] \geq 1-2\alpha$, where L and M are those of (1) and (2).

Proof. We can change (1) into $\frac{J\sigma_A^2}{\sigma_B^2} = \frac{\theta_1-\theta_2}{\theta_2-\theta_3} \geq 0$, and let $a = \frac{n_1 S_2^2 F_{\alpha,n_1,n_2}}{\theta_2}$, $y = \frac{n_1(\theta_1-\theta_2)F_{\alpha,n_1,n_2}}{\theta_1(\theta_2-\theta_3)}(\frac{\theta_3 S_2^2}{\theta_2} - S_3^2 F_{1-\alpha,n_2,n_3})$ and $U_1 = \frac{n_1 S_1^2}{\theta_1}$, (1) is $P[U_1 \leq a+y]$. If y is zero, then $P[U_1 \leq a] = 1-\alpha$. If y is non-zero, in the $P[a < U_1 \leq a+y]$,

$$\begin{aligned} EY &= \frac{n_1(\theta_1-\theta_2)S_3^2 F_{\alpha,n_1,n_2}}{\theta_1(\theta_2-\theta_3)}\{\frac{\theta_3 E(S_2^2)}{\theta_2} - F_{1-\alpha,n_2,n_3}E(S_3^2)\} \\ &= \frac{n_1\theta_3(\theta_1-\theta_2)F_{\alpha,n_1,n_2}}{\theta_1(\theta_2-\theta_3)}(1-F_{1-\alpha,n_2,n_3}) > 0. \end{aligned}$$

Using Kang, since $\frac{\theta_1-\theta_2}{\theta_2-\theta_3} \geq 0$ there exist a constant ν and a function f_1 such that $P[a < U_1 \leq a+y] = f_1(\nu)EY \geq 0$. Thus $P[U_1 \leq a+y] \geq 1-\alpha$. Therefore (1) holds.

The proof of (2) is similar to that of (1), we can change (2) into $\frac{J\sigma_A^2}{\sigma_B^2} = \frac{\theta_1-\theta_2}{\theta_2-\theta_3} \geq 0$, and let $x = \frac{n_1(\theta_1-\theta_2)F_{1-\alpha,n_1,n_2}}{\theta_1(\theta_2-\theta_3)}(S_3^2 F_{\alpha,n_2,n_3} - \frac{\theta_3 S_2^2}{\theta_2})$, $b = \frac{n_1 S_2^2 F_{1-\alpha,n_1,n_2}}{\theta_2}$, $U_1 = \frac{n_1 S_1^2}{\theta_1}$, and (2) is $P[-x+b \leq U_1]$. If y is zero, then $P[b \leq U_1] = 1-\alpha$. If y is non-zero, then $\frac{\theta_1-\theta_2}{\theta_2-\theta_3} \geq 0$ and

$$\begin{aligned} EY &= \frac{n_1(\theta_1-\theta_2)F_{1-\alpha,n_1,n_2}}{\theta_1(\theta_2-\theta_3)}\{F_{\alpha,n_2,n_3}E(S_3^2)-\frac{\theta_3 E(S_2^2)}{\theta_2}\} \\ &= \frac{n_1\theta_3(\theta_1-\theta_2)F_{1-\alpha,n_1,n_2}}{\theta_1(\theta_2-\theta_3)}(F_{\alpha,n_2,n_3}-1)>0, \end{aligned}$$

there exist a constant ν and a function f_1 such that $P[-x+b\le U_1<b]=f(\nu)EY\ge 0$. Thus $P[-x+b\le U_1]\ge 1-\alpha$. Therefore (2) holds.

Using the result of (1) and (2), the proof of (3) is $P[-x+b\le U_1\le a+y]=P[b\le U_1\le a]+P[-x+b\le U_1<b]+P[a<U_1\le a+y]\ge P[b\le U_1\le a]=1-2\alpha$. Therefore (3) holds.

Theorem 2. The range of Wang's confidence interval for σ_A^2/σ_B^2 in the two-factor nested variance component model are dbtained as follows.

(1). $P[L\le \frac{\sigma_A^2}{\sigma_B^2}]\ge 1-\alpha$, where $\frac{S_2^2}{S_3^2}>F_{1-\alpha,n_2,n_3}$, $\frac{S_1^2}{S_2^2}>F_{\alpha,n_1,n_2}$ and

$$L=\frac{\frac{S_1^2}{S_2^2}-F_{\alpha,n_1,n_2}}{JF_{\alpha,n_1,n_2}(1-\frac{S_3^2F_{1-\alpha,n_2,n_3}}{S_2^2})}.$$ If $\frac{S_2^2}{S_3^2}>F_{1-\alpha,n_2,n_3}$ and $\frac{S_1^2}{S_2^2}\le F_{\alpha,n_1,n_2}$, then $L=0$.

(2). $P[\frac{\sigma_A^2}{\sigma_B^2}\le U]\ge 1-\alpha$, where $\frac{S_2^2}{S_3^2}>F_{\alpha,n_2,n_3}$, $\frac{S_1^2}{S_2^2}>F_{1-\alpha,n_1,n_2}$ and

$$U=\frac{\frac{S_1^2}{S_2^2}-F_{1-\alpha,n_1,n_2}}{JF_{1-\alpha,n_1,n_2}(1-\frac{S_3^2F_{\alpha,n_2,n_3}}{S_2^2})}.$$ If $\frac{S_2^2}{S_3^2}>F_{\alpha,n_2,n_3}$ and $\frac{S_1^2}{S_2^2}\le F_{1-\alpha,n_1,n_2}$, then $U=0$.

(3). $P[L\le\frac{\sigma_A^2}{\sigma_B^2}\le U]\ge 1-2\alpha$, where L and U are those of (1) and (2).

Proof. The proof is similar to that of Theorem 1. We can change (1) into $\frac{J\sigma_A^2}{\sigma_B^2}=\frac{\theta_1-\theta_2}{\theta_2-\theta_3}\ge 0$, and let $U_1=\frac{n_1S_1^2}{\theta_1}$, $a=\frac{n_1S_2^2F_{\alpha,n_1,n_2}}{\theta_2}$ and $y=\frac{n_1(\theta_1-\theta_2)F_{\alpha,n_1,n_2}}{\theta_1(\theta_2-\theta_3)}(\frac{\theta_3S_2^2}{\theta_2}-S_3^2F_{1-\alpha,n_2,n_3})$, and (1) is $P[U_1\le a+y]$.

If y is zero, then $P[U_1\le a]=1-\alpha$. If y is non-zero, then $\frac{\theta_1-\theta_2}{\theta_2-\theta_3}\ge 0$,

$$\begin{aligned} EY &= \frac{n_1(\theta_1-\theta_2)F_{\alpha,n_1,n_2}}{\theta_1(\theta_2-\theta_3)}\{\frac{\theta_3E(S_2^2)}{\theta_2}-E(S_3^2)F_{1-\alpha,n_2,n_3}\} \\ &= \frac{n_1\theta_3(\theta_1-\theta_2)F_{\alpha,n_1,n_2}}{\theta_1(\theta_2-\theta_3)}(1-F_{1-\alpha,n_2,n_3})>0. \end{aligned}$$

Using Kang, there exist a constant ν and a function f_1 such that $P[a<U_1\le a+y]=f_1(a+\nu)EY\ge 0$. Therefore (1) holds.

The proof is similar to that of (1), we can change (2) into $\frac{J\sigma_A^2}{\sigma_B^2} = \frac{\theta_1-\theta_2}{\theta_2-\theta_3} \geq 0$. And let $x = \frac{n_1(\theta_1-\theta_2)F_{1-\alpha,n_1,n_2}}{\theta_1(\theta_2-\theta_3)}(S_3^2 F_{\alpha,n_2,n_3} - \frac{\theta_3 S_2^2}{\theta_2}), b = \frac{n_1 S_2^2 F_{1-\alpha,n_1,n_2}}{\theta_2}$ and $U_1 = \frac{n_1 S_1^2}{\theta_1}$, and (2) is $P[-x+b \leq U_1]$. Since $P[b \leq U_1] = 1-\alpha$, $\frac{\theta_1-\theta_2}{\theta_2-\theta_3} \geq 0$,

$$\begin{aligned} EX &= \frac{n_1(\theta_1-\theta_2)F_{1-\alpha,n_1,n_2}}{\theta_1(\theta_2-\theta_3)}\{E(S_3^2)F_{\alpha,n_2,n_3} - \frac{\theta_3 E(S_2^2)}{\theta_2}\} \\ &= \frac{n_1\theta_3(\theta_1-\theta_2)F_{1-\alpha,n_1 n_2}}{\theta_1(\theta_2-\theta_3)}(F_{\alpha,n_2,n_3} - 1) > 0, \end{aligned}$$

and there exist a constant ν and a function f_1 such that $P[-x+b \leq U_1 < b] = f(b+\nu)EX \geq 0$. Thus $P[-x+b \leq U_1] \geq 1-\alpha$. Therefore (2) holds. Using the results of (1) and (2), the proof of (3) is $P[-x+b \leq U_1 \leq a+y] = P[b \leq U_1 \leq a] + P[-x+b \leq U_1 < b] + P[a < U_1 \leq a+y] \geq P[b \leq U_1 \leq a] = 1-2\alpha$. Therefore (3) holds.

Remark 1. In the Theorem 1, suppose there exists a constant $n(n \geq 2)$ and if y is non-zero, then we get $lim_{n\to\infty} P[U_1 \leq a + \frac{y}{n}] = P[U_1 \leq a] = 1-\alpha$. Since $P[U_1 \leq a] < P[U_1 \leq a+\frac{y}{n}] < P[U_1 \leq a+y]$, the confidence interval $P[U_1 \leq a+\frac{y}{n}]$ is better than $P[U_1 \leq a+y]$. Therefore, suitable $n(n \geq 2)$ in Theorem 1 and Theorem 2, we can find out the better confidence intervals than those of the given confidence intervals.

Remark 2. The simulation results of the confidence intervals for σ_A^2/σ_B^2 in the two-factor nested variance component model are equal or grater than $1-\alpha$. In case of $I = 3 \sim 7, J = 3 \sim 10, K = 3 \sim 10, n_1 = 2 \sim 6, n_2 = 6 \sim 45$ and $n_3 = 18 \sim 100$.

1) The range of Broemeling's method of confidence intervals $\alpha = 0.05$.
   Upper confidence interval : 0.9896 ~ 0.9960.
   Lower confidence interval : 0.9752 ~ 0.9907.
2) The probability range of Wang's confidence intervals of Theorem 2.

α	*the range of upper confidence interval*	*the range of lower confidence interval*
0.1	0.9010 – 0.9180	0.8863 – 0.9181
0.05	0.9480 – 0.9604	0.9383 – 0.9599
0.01	0.9868 – 0.9944	0.9859 – 0.9952

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