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The Case of Proportional Cell Frequencies for the Two-Way Cross-Classification with Interaction [1]

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Abstract

The case of proportional cell frequencies for the two-way cross-classification with interaction is considered. Several types of hypotheses for the general unbalanced data that are commonly used in the literature are shown, and they are written out for this particular case. A reparameterized form of the cell means model is defined to establish the reparameterized model, and orthogonal property of the model is shown using the augmented matrix and the numerator sums of squares are computed. Different ways of producing the same analysis of variance tables are shown in both orthogonal and nonorthogonal situations.

Key Words and Phrases: proportional cell frequencies, hypothesis, reparameterization, orthogonality, nonorthogonality.

1. Introduction

Consider the two-way cross-classification with interaction. This may be written as

$$y_{ijk} = \mu_{ij} + e_{ijk}, \qquad i = 1, 2, \cdots, a, \ j = 1, 2, \cdots, b, \ k = 1, 2, \cdots, n_{ij} \tag{1}$$

where y_{ijk} is the kth observation at the ith level of A and the jth level of B and e_{ijk} is the error term. We assume that y_{ijk} is independently normally distributed with mean μ_{ij} and common variance σ^2. We further assume that the number of observations per cell, $n_{ij} > 0$ for all i, j, and $n_{ij} > 1$ for some i and j.

We now consider the special case in which the number of observations per cell is given by $n_{ij} = r_i c_j$ where r_i, $i = 1, 2, \cdots, a$ and c_j, $j = 1, 2, \cdots, b$ are known row

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and column constants. This is known as the case of proportional cell frequencies or proportional cell frequency model, and this model is known to have orthogonality property (see for example Seber 1964, John 1971, Winer et al. 1991). In this article we show several different expressions for the hypotheses in this particular case. We then consider a reparameterization of the cell means model and relate this to the overparameterized model. The results are compared and discussed.

2. General Discussion of Hypotheses for the Two-Way Cross-Classification

For the general unbalanced two-way cross-classification model with interaction given in (1), various hypotheses have been developed, and we list seven hypotheses for factors A, B and their interaction $A \times B$ that are more commonly used in the literature (e.g., Speed et al. 1978, Hocking 1985, Searle 1987).

$$\begin{aligned}
H_A &: \quad \bar{\mu}_{i\cdot} = \bar{\mu}_{a\cdot} \qquad i = 1,2,\cdots,a-1 \\
H_A^* &: \quad \sum_{j=1}^{b} n_{ij}\,\mu_{ij} = \sum_{j=1}^{b}\sum_{k=1}^{a} \frac{n_{ij}\,n_{kj}}{n_{\cdot j}}\mu_{kj} \qquad i = 1,2,\cdots,a-1 \\
H_A^{**} &: \quad \sum_{j=1}^{b} \frac{n_{ij}}{n_{i\cdot}}\mu_{ij} = \sum_{j=1}^{b} \frac{n_{aj}}{n_{a\cdot}}\mu_{aj} \qquad i = 1,2,\cdots,a-1 \\
H_B &: \quad \bar{\mu}_{\cdot j} = \bar{\mu}_{\cdot b} \qquad j = 1,2,\cdots,b-1 \\
H_B^* &: \quad \sum_{i=1}^{a} n_{ij}\,\mu_{ij} = \sum_{i=1}^{a}\sum_{k=1}^{b} \frac{n_{ij}\,n_{ik}}{n_{i\cdot}}\mu_{ik} \qquad j = 1,2,\cdots,b-1 \\
H_B^{**} &: \quad \sum_{i=1}^{a} \frac{n_{ij}}{n_{\cdot j}}\mu_{ij} = \sum_{i=1}^{a} \frac{n_{ib}}{n_{\cdot b}}\mu_{ib} \qquad j = 1,2,\cdots,b-1 \\
H_{AB} &: \quad \mu_{ij} - \bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot j} + \bar{\mu}_{\cdot\cdot} = 0 \qquad i = 1,2,\cdots,a-1,\ j = 1,2,\cdots,b-1
\end{aligned} \tag{2}$$

where

$$\bar{\mu}_{i\cdot} = \frac{1}{b}\sum_{j=1}^{b}\mu_{ij}\,, \quad \bar{\mu}_{\cdot j} = \frac{1}{a}\sum_{i=1}^{a}\mu_{ij}\,, \quad \bar{\mu}_{\cdot\cdot} = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\mu_{ij}\,,$$

$$n_{i\cdot} = \sum_{j=1}^{b} n_{ij}\,, \quad n_{\cdot j} = \sum_{i=1}^{a} n_{ij}\,.$$

We also list three types of ANOVA tables, which we shall call the sequential sum of squares, partially sequential sum of squares, and marginal means sum of squares. The reduction in sum of squares is denoted by $R()$-notation in fitting $E(y_{ijk}) = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$.

Table 1. Sequential sums of squares

Description	Hypothesis	SS($R()$-notation)
Row effect	H_A^{**}	$R(\alpha\|\mu)$
Column effect	H_B^*	$R(\beta\|\mu,\alpha)$
Interaction effect	H_{AB}	$R((\alpha\beta)\|\mu,\alpha,\beta)$
Residual		$Q(\hat{\mu})$

Table 2. Partially sequential sums of squares

Description	Hypothesis	SS($R()$-notation)
Row effect	H_A^*	$R(\alpha\|\mu,\beta)$
Column effect	H_B^*	$R(\beta\|\mu,\alpha)$
Interaction effect	H_{AB}	$R((\alpha\beta)\|\mu,\alpha,\beta)$
Residual		$Q(\hat{\mu})$

Table 3. Marginal means sums of squares

Description	Hypothesis	SS($R()$-notation)
Row effect	H_A	$R(\alpha\|\mu,\beta,(\alpha\beta))$
Column effect	H_B	$R(\beta\|\mu,\alpha,(\alpha\beta))$
Interaction effect	H_{AB}	$R((\alpha\beta)\|\mu,\alpha,\beta)$
Residual		$Q(\hat{\mu})$

3. Hypotheses for the Proportional Frequency Case

Assuming the proportional frequency model, the cell frequencies are $n_{ij} = r_i c_j$ for $i = 1, 2, \cdots, a,\ j = 1, 2, \cdots, b$. We want to write out the different hypotheses of the two-way cross-classification with interaction for this particular case.

Theorem 1. For the proportional frequency model the null hypotheses in (2) are as follows:

$$H_A \ : \ \sum_{j=1}^{b} \mu_{ij} = \sum_{j=1}^{b} \mu_{aj} \qquad i = 1, 2, \cdots, a-1$$

$$H_A^* = H_A^{**} \ : \ \sum_{j=1}^{b} c_j\, \mu_{ij} = \sum_{j=1}^{b} c_j\, \mu_{aj} \qquad i = 1, 2, \cdots, a-1$$

$$H_B \ : \ \sum_{i=1}^{a} \mu_{ij} = \sum_{i=1}^{a} \mu_{ib} \qquad j = 1, 2, \cdots, b-1$$

$$\begin{aligned} H_B^* = H_B^{**} \quad &: \quad \sum_{i=1}^{a} r_i\, \mu_{ij} = \sum_{i=1}^{a} r_i\, \mu_{ib} \qquad j = 1, 2, \cdots, a-1 \\ H_{AB} \quad &: \quad \mu_{ij} - \overline{\mu}_{i\cdot} - \overline{\mu}_{\cdot j} + \overline{\mu}_{\cdot\cdot} = 0 \qquad i = 1, 2, \cdots, a-1 \\ & \qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \cdots, b-1 \end{aligned}$$

Proof. We shall first prove for factor A. We denote $r_{\cdot} = \sum_{i=1}^{a} r_i$ and $c_{\cdot} = \sum_{j=1}^{b} c_j$. For H_A, we have

$$H_A : \ \overline{\mu}_{i\cdot} = \overline{\mu}_{a\cdot} \qquad i = 1, 2, \cdots, a-1$$

since this hypothesis is independent of n_{ij}. This is equivalent to

$$H_A : \ \sum_{j=1}^{b} \mu_{ij} = \sum_{j=1}^{b} \mu_{aj} \qquad i = 1, 2, \cdots, a-1 \, .$$

For H_A^{**}, substituting $n_{ij} = r_i\, c_j$ and $n_{i\cdot} = r_i\, c_{\cdot}$ into (3) we directly have

$$H_A^{**} : \ \sum_{j=1}^{b} c_j\, \mu_{ij} = \sum_{j=1}^{b} c_j\, \mu_{aj} \qquad i = 1, 2, \cdots, a-1 \, .$$

For H_A^*, substituting $n_{ij} = r_i\, c_j$ and $n_{\cdot j} = r_{\cdot}\, c_j$ we have

$$H_A^* : \ \sum_{j=1}^{b} c_j \mu_{ij} = \sum_{j=1}^{b} \sum_{k=1}^{a} \frac{r_k\, c_j^2}{r_{\cdot}\, c_j} \mu_{kj} = \sum_{k=1}^{a} \frac{r_k}{r_{\cdot}} \sum_{j=1}^{b} c_j\, \mu_{kj} \qquad i = 1, 2, \cdots, a-1$$

We note that $i = 1, 2, \cdots, a$ is implied for all the three types of hypotheses.

We shall now show that H_A^* is equivalent to H_A^{**}. We have $H_A^* \Rightarrow H_A^{**}$ since the right hand side is constant. To show this in detail, we let

$$\sum_{j=1}^{b} c_j\, \mu_{ij} = S \qquad i = 1, \cdots, a-1 \, .$$

Thus

$$S = \sum_{k=1}^{a-1} \frac{r_k}{r_{\cdot}} S + \frac{r_a}{r_{\cdot}} S_a$$

where $S_a = \sum_{j=1}^{b} c_j\, \mu_{aj}$. But

$$\sum_{k=1}^{a-1} \frac{r_k}{r_{\cdot}} = 1 - \frac{r_a}{r_{\cdot}} \, .$$

So

$$S = \left(1 - \frac{r_a}{r.}\right) S + \frac{r_a}{r.} S_a$$
$$S = S_a$$

which is

$$\sum_{j=1}^{b} c_j \, \mu_{ij} = \sum_{j=1}^{b} c_j \, \mu_{aj} \qquad i = 1, \cdots, a-1 \; .$$

To show that $H_A^{**} \Rightarrow H_A^*$ we write

$$\sum_{j=1}^{b} c_j \, \mu_{kj} = \sum_{j=1}^{b} c_j \, \mu_{ij}$$
$$\sum_{k=1}^{a} r_k \sum_{j=1}^{b} c_j \, \mu_{kj} = \sum_{k=1}^{a} r_k \sum_{j=1}^{b} c_j \, \mu_{ij}$$
$$\frac{1}{r.} \sum_{k=1}^{a} r_k \sum_{j=1}^{b} c_j \, \mu_{kj} = \sum_{j=1}^{b} c_j \, \mu_{ij} \; .$$

For the $A \times B$ interaction effect we clearly have

$$H_{AB}: \; \mu_{ij} - \overline{\mu}_{i\cdot} - \overline{\mu}_{\cdot j} + \overline{\mu}_{\cdot\cdot} = 0 \qquad i = 1, 2, \cdots, a-1, \; j = 1, 2, \cdots, b-1$$

since it is independent of n_{ij} . Q.E.D.

Thus we see that H_A^* and H_A^{**} are equivalent but that H_A is not uniformly equivalent $(i.e. \forall \mu)$ to H_A^* or H_A^{**} unless $c_j = 1, \; j = 1, \cdots, b$. Similar statement can be made for factor B.

4. Reparameterization

We consider a reparameterization of the two-factor cell means model for the proportional frequency case.

Theorem 2. Let the parameters μ, α_i, β_j and $(\alpha\beta)_{ij}$ be defined by

$$\mu = \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{r_i \, c_j}{r. \, c.} \mu_{ij}$$
$$\alpha_i = \sum_{j=1}^{b} \frac{c_j}{c.} \mu_{ij} - \mu \qquad i = 1, 2, \cdots, a-1$$

$$\begin{aligned}\beta_j &= \sum_{i=1}^{a}\frac{r_i}{r.}\mu_{ij}-\mu \qquad j=1,2,\cdots,b-1\\ (\alpha\beta)_{ij} &= \mu_{ij}-\sum_{j=1}^{b}\frac{c_j}{c.}\mu_{ij}-\sum_{i=1}^{a}\frac{r_i}{r.}\mu_{ij}+\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{r_i\,c_j}{r.\,c.}\mu_{ij} \qquad i=1,2,\cdots,a-1\\ & \qquad\qquad j=1,2,\cdots,b-1\end{aligned}$$

Then we have

$$E(y_{ijk})=\begin{cases}\mu+\alpha_i+\beta_j+(\alpha\beta)_{ij} & i=1,2,\cdots,a-1\\ & j=1,2,\cdots,b-1\\ \mu-\alpha_i-\sum_{j=1}^{b-1}\frac{c_j}{c_b}\beta_j-\sum_{j=1}^{b-1}\frac{c_j}{c_b}(\alpha\beta)_{ij} & i=1,2,\cdots,a-1\\ & j=b\\ \mu-\sum_{i=1}^{a-1}\frac{r_i}{r_a}\alpha_i+\beta_j-\sum_{i=1}^{a-1}\frac{r_i}{r_a}(\alpha\beta)_{ij} & i=a\\ & j=1,2,\cdots,b-1\\ \mu-\sum_{i=1}^{a-1}\frac{r_i}{r_a}\alpha_i-\sum_{j=1}^{b-1}\frac{c_j}{c_b}\beta_j-\sum_{i=1}^{a-1}\frac{r_i}{r_a}(\alpha\beta)_{ib} & i=a\\ & j=b\end{cases} \tag{3}$$

Proof. For $i=1,\cdots,a-1$, $j=1,\cdots,b-1$, we have

$$\begin{aligned}\mu_{ij} &= \Big(\sum_{j=1}^{b}\frac{c_j}{c.}\mu_{ij}-\mu\Big)+\Big(\sum_{i=1}^{a}\frac{r_i}{r.}\mu_{ij}-\mu\Big)-\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{r_i\,c_j}{r.\,c.}\mu_{ij}+2\,\mu+(\alpha\beta)_{ij}\\ &= \mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}\,.\end{aligned}$$

For $i=1,\cdots,a-1$, $j=b$, we obtain:

$$\begin{aligned}(\alpha\beta)_{ij} &= \mu_{ij}-\alpha_i-\beta_j-\mu\\ \sum_{j=1}^{b-1}\frac{c_j}{c_b}(\alpha\beta)_{ij} &= \sum_{j=1}^{b-1}\frac{c_j}{c_b}\mu_{ij}-\sum_{j=1}^{b-1}\frac{c_j}{c_b}\alpha_i-\sum_{j=1}^{b-1}\frac{c_j}{c_b}\beta_j-\sum_{j=1}^{b-1}\frac{c_j}{c_b}\mu\\ &= \Big(\sum_{j=1}^{b}\frac{c_j}{c_b}\mu_{ij}-\mu_{ib}\Big)-\Big(\sum_{j=1}^{b}\frac{c_j}{c_b}\alpha_i-\alpha_i\Big)-\sum_{j=1}^{b-1}\frac{c_j}{c_b}\beta_j-\Big(\sum_{j=1}^{b}\frac{c_j}{c_b}\mu-\mu\Big)\end{aligned}$$

$$\mu_{ib}=\mu+\alpha_i-\sum_{j=1}^{b-1}\frac{c_j}{c_b}\beta_j-\sum_{j=1}^{b-1}\frac{c_j}{c_b}(\alpha\beta)_{ij}+\sum_{j=1}^{b-1}\frac{c_j}{c_b}\mu_{ij}-\frac{c.}{c_b}\alpha_i-\frac{c.}{c_b}\mu\,.$$

The last three terms vanish since

$$\frac{c.}{c_b}\alpha_i=\frac{c.}{c_b}\Big(\sum_{j=1}^{b}\frac{c_j}{c.}\mu_{ij}-\mu\Big)=\sum_{j=1}^{b}\frac{c_j}{c_b}\mu_{ij}-\frac{c.}{c_b}\mu\,.$$

Thus we have

$$\mu_{ib} = \mu + \alpha_i - \sum_{j=1}^{b-1} \frac{c_j}{c_b}\beta_j - \sum_{j=1}^{b-1} \frac{c_j}{c_b}(\alpha\beta)_{ij}\,.$$

For $i = a,\ j = 1, \cdots, b-1$, we obtain:

$$\begin{aligned}
(\alpha\beta)_{ij} &= \mu_{ij} - \alpha_i - \beta_j - \mu \\
\sum_{i=1}^{a-1} \frac{r_i}{r_a}(\alpha\beta)_{ij} &= \sum_{i=1}^{a-1} \frac{r_i}{r_a}\mu_{ij} - \sum_{i=1}^{a-1} \frac{r_i}{r_a}\alpha_i - \sum_{i=1}^{a-1} \frac{r_i}{r_a}\beta_j - \sum_{i=1}^{a-1} \frac{r_i}{r_a}\mu \\
&= \left(\sum_{i=1}^{a} \frac{r_i}{r_a}\mu_{ij} - \mu_{aj}\right) - \sum_{i=1}^{a-1} \frac{r_i}{r_a}\alpha_i - \left(\sum_{i=1}^{a} \frac{r_i}{r_a}\beta_j - \beta_j\right) - \left(\sum_{i=1}^{a} \frac{r_i}{r_a}\mu - \mu\right)
\end{aligned}$$

$$\mu_{aj} = \mu - \sum_{i=1}^{a-1} \frac{r_i}{r_a}\alpha_i + \beta_j - \sum_{i=1}^{a-1} \frac{r_i}{r_a}(\alpha\beta)_{ij} + \sum_{i=1}^{a} \frac{r_i}{r_a}\mu_{ij} - \frac{r_.}{r_a}\beta_j - \frac{r_.}{r_a}\mu\,.$$

The last three terms vanish since

$$\frac{r_.}{r_a}\beta_j = \frac{r_.}{r_a}\left(\sum_{i=1}^{a} \frac{r_i}{r_.}\mu_{ij} - \mu\right) = \sum_{i=1}^{a} \frac{r_i}{r_a}\mu_{ij} - \frac{r_.}{r_a}\mu\,.$$

Thus we have

$$\mu_{aj} = \mu - \sum_{i=1}^{a-1} \frac{r_i}{r_a}\alpha_i + \beta_j - \sum_{i=1}^{a-1} \frac{r_i}{r_a}(\alpha\beta)_{ij}\,.$$

For $i = a$ and $j = b$, we obtain:

$$\mu_{ab} = \mu - \sum_{i=1}^{a-1} \frac{r_i}{r_a}\alpha_i - \sum_{j=1}^{b-1} \frac{c_j}{c_b}\beta_j - \sum_{i=1}^{a-1} \frac{r_i}{r_a}(\alpha\beta)_{ij}\,. \qquad \text{Q.E.D.}$$

To express these results in matrix form, we write the parameter vector

$$\underline{\theta} = \begin{bmatrix} \mu \\ \underline{\alpha} \\ \underline{\beta} \\ (\underline{\alpha\beta}) \end{bmatrix}$$

where

$$\begin{aligned}
\underline{\alpha} &= (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{a-1})' \\
\underline{\beta} &= (\beta_1,\ \beta_2,\ \cdots,\ \beta_{a-1})' \\
(\underline{\alpha\beta}) &= ((\alpha\beta)_{11}, \cdots, (\alpha\beta)_{1,b-1},\ (\alpha\beta)_{21}, \cdots, (\alpha\beta)_{2,b-1},\ \cdots\cdots, \\
&\qquad (\alpha\beta)_{a-1,1}, \cdots, (\alpha\beta)_{a-1,b-1})'\,.
\end{aligned}$$

Examining the structure of (3) carefully, we can write the reparameterized model in matrix form as follows:

$$E(\mathbf{y}) = \mathbf{W}\mathbf{X}_0\mathbf{P}\underline{\theta}\,, \tag{4}$$

where $\mathbf{W}$ is the cell frequency matrix, $\mathrm{Diag}(\mathbf{1}_{r_i c_j})$, and the basic design matrix is

$$\mathbf{X}_0 = (\mathbf{1}_{ab},\ \mathbf{I}_a \otimes \mathbf{1}_b,\ \mathbf{1}_a \otimes \mathbf{I}_b,\ \mathbf{I}_a \otimes \mathbf{I}_b),$$

and the parameter matrix is

$$\mathbf{P} = \mathrm{Diag}(1,\ \mathbf{\Delta}'_a,\ \mathbf{\Delta}'_b,\ \mathbf{\Delta}'_a \otimes \mathbf{\Delta}'_b),$$

where $\otimes$ denotes the Kronecker product defined as $(a_{ij}\mathbf{B})$ for $\mathbf{A} \otimes \mathbf{B}$, and

$$\begin{aligned}
&\mathbf{1}_a = (1, 1, \cdots, 1)',\ \ a \times 1 \text{ vector of ones},\\
&\mathbf{I}_a = \mathrm{Diag}(1, 1, \cdots, 1),\ \ a \times a \text{ identity matrix},\\
&\mathbf{\Delta}_a = (\mathbf{I}_{a-1}\,, \mathbf{s}_{a-1}), \quad \mathbf{\Delta}_b = (\mathbf{I}_{b-1}\,, \mathbf{s}_{b-1}),\\
&\mathbf{s}_{a-1} = -\frac{1}{r_a}\begin{bmatrix} r_1 \\ \vdots \\ r_{a-1} \end{bmatrix}, \quad \mathbf{s}_{b-1} = -\frac{1}{c_b}\begin{bmatrix} c_1 \\ \vdots \\ c_{b-1} \end{bmatrix}.
\end{aligned}$$

We can approach this reparameterized model from a different direction; that is, we begin with the classical overparameterized model,

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \qquad \begin{aligned} i &= 1, \cdots, a \\ j &= 1, \cdots, b \\ k &= 1, \cdots, n_{ij} \end{aligned} \tag{5}$$

and then impose the conditions

$$\begin{aligned}
&\sum_{i=1}^{a} r_i\alpha_i = 0\,, \quad \sum_{j=1}^{b} c_j\beta_j = 0\,, \quad \sum_{j=1}^{b} c_j(\alpha\beta)_{ij} = 0\,, \quad i = 1, \cdots, a-1,\\
&\sum_{i=1}^{a} r_i(\alpha\beta)_{ij} = 0\,, \quad j = 1, \cdots, b
\end{aligned} \tag{6}$$

which lead to a full rank model. We then have the following results.

Theorem 3. For the above reparameterized model, the hypotheses H_A^* (or H_A^{**}), H_B^* (or H_B^{**}), and H_{AB} for the case of proportional frequencies are given by

$$\begin{aligned}
H_A^* = H_A^{**} &: \ \underline{\alpha} = \mathbf{0}\\
H_B^* = H_B^{**} &: \ \underline{\beta} = \mathbf{0}\\
H_{AB} &: \ \underline{(\alpha\beta)} = \mathbf{0}\,.
\end{aligned}$$

Proof. H_A^* or H_A^{**} is given by

$$\sum_{j=1}^{b} c_j\,\mu_{ij} = \sum_{j=1}^{b} c_j\,\mu_{aj} \qquad i = 1,\cdots,a-1\,.$$

Now

$$\begin{aligned}\sum_{j=1}^{b} c_j\,\mu_{ij} &= \sum_{j=1}^{b} c_j(\mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}) \\ &= c.\,\mu + c.\,\alpha_i + \sum_{j=1}^{b} c_j\,\beta_j + \sum_{j=1}^{b} c_j\,(\alpha\beta)_{ij} \\ &= c.\,\mu + c.\,\alpha_i\,,\end{aligned}$$

since the last two terms vanish. So H_A^* or H_A^{**} is

$$\begin{aligned} & c.\,\mu + c.\,\alpha_i = c.\,\mu + c.\,\alpha_a \qquad i = 1,\cdots,a-1 \\ \Leftrightarrow\quad & \alpha_i = \alpha_a \qquad i = 1,\cdots,a-1\,.\end{aligned}$$

But since $\sum_{i=1}^{a} r_i\,\alpha_i = 0$ this implies

$$\alpha_i = 0 \qquad i = 1,\cdots,a-1\,.$$

Hence, we have

$$H_A^* = H_A^{**}:\ \alpha_i = 0 \qquad i = 1,\cdots,a-1\,.$$

Similarly, we have

$$H_B^* = H_B^{**}:\ \beta_j = 0 \qquad j = 1,\cdots,b-1\,.$$

Using the no interaction constraint,

$$\mu_{ij} - \mu_{sj} - \mu_{it} + \mu_{st} = 0 \quad \forall\ \ i,s = 1,\cdots,a \qquad \forall\ \ j,t = 1,\cdots,b\,,$$

we have

$$\sum_{i=1}^{a}\frac{r_i}{r.}\sum_{j=1}^{b}\frac{c_j}{c.}(\mu_{ij}-\mu_{sj}-\mu_{it}+\mu_{st}) = 0$$

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{r_i c_j}{r.c.}\mu_{ij} - \sum_{j=1}^{b}\frac{c_j}{c.}\mu_{sj} - \sum_{i=1}^{a}\frac{r_i}{r.}\mu_{it} + \mu_{st} = 0$$

$$\mu - (\mu+\alpha_s) - (\mu+\beta_t) + (\mu+\alpha_s+\beta_t+(\alpha\beta)_{st}) = 0$$

$$(\alpha\beta)_{st} = 0, \qquad s = 1, \cdots, a\,,\ t = 1, \cdots, b\,.$$

Hence we obtain

$$(\alpha\beta)_{ij} = 0, \qquad i = 1, \cdots, a-1,\ \ j = 1, \cdots, b-1\,. \qquad \text{Q.E.D.}$$

5. Orthogonality of the Reparameterized Model

Using the reparameterized model (4) we have the following augmented matrix.

$$\begin{bmatrix} \mathbf{P}'\mathbf{X}_0'\mathbf{W}'\mathbf{W}\mathbf{X}_0\mathbf{P} & \mathbf{P}'\mathbf{X}_0'\mathbf{W}'\mathbf{y} \\ \mathbf{y}'\mathbf{W}\mathbf{X}_0\mathbf{P} & \mathbf{y}'\mathbf{y} \end{bmatrix}.$$

Now

$$\mathbf{X}_0'\mathbf{W}'\mathbf{W}\mathbf{X}_0 = \begin{bmatrix} r.\,c. & c.\mathbf{1}_a'\mathbf{R} & r.\mathbf{1}_b'\mathbf{C} & (\mathbf{1}_a'\mathbf{R}) \otimes (\mathbf{1}_b'\mathbf{C}) \\ c.\mathbf{R}\mathbf{1}_a & c.\mathbf{R} & (\mathbf{R}\mathbf{1}_a) \otimes (\mathbf{1}_b'\mathbf{C}) & \mathbf{R} \otimes (\mathbf{1}_b'\mathbf{C}) \\ r.\mathbf{C}\mathbf{1}_b & (\mathbf{1}_a'\mathbf{R}) \otimes (\mathbf{C}\mathbf{1}_b) & r.\mathbf{C} & (\mathbf{1}_a'\mathbf{R}) \otimes \mathbf{C} \\ (\mathbf{R}\mathbf{1}_a) \otimes (\mathbf{C}\mathbf{1}_b) & \mathbf{R} \otimes (\mathbf{C}\mathbf{1}_b) & (\mathbf{R}\mathbf{1}_a) \otimes \mathbf{C} & \mathbf{R} \otimes \mathbf{C} \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{R} &= \mathrm{Diag}(r_1, \cdots, r_a) \\ \mathbf{C} &= \mathrm{Diag}(c_1, \cdots, c_b)\,. \end{aligned}$$

Thus

$$\mathbf{P}'\mathbf{X}_0'\mathbf{W}'\mathbf{W}\mathbf{X}_0\mathbf{P} = \begin{bmatrix} r.\,c. & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & c.\,\boldsymbol{\Delta}_a\mathbf{R}\boldsymbol{\Delta}_a' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & r.\,\boldsymbol{\Delta}_b\mathbf{C}\boldsymbol{\Delta}_b' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\boldsymbol{\Delta}_a\mathbf{R}\boldsymbol{\Delta}_a') \otimes (\boldsymbol{\Delta}_b\mathbf{C}\boldsymbol{\Delta}_b') \end{bmatrix}$$

which is block diagonal. Also

$$\mathbf{P}'\mathbf{X}_0'\mathbf{W}'\mathbf{y} = \begin{bmatrix} y_{\cdot\cdot\cdot} & \\ y_{i\cdot\cdot} - \frac{r_i}{r_a} y_{a\cdot\cdot} & i = 1, \cdots, a-1 \\ y_{\cdot j\cdot} - \frac{c_j}{c_b} y_{\cdot b\cdot} & j = 1, \cdots, b-1 \\ y_{ij\cdot} - \frac{c_j}{c_b} y_{ib\cdot} - \frac{r_i}{r_a} y_{aj\cdot} + \frac{r_i c_j}{r_a c_b} y_{ab\cdot} & i = 1, \cdots, a-1 \\ & j = 1, \cdots, b-1 \end{bmatrix}.$$

Letting

$$\begin{aligned}
\mathbf{u} &= y_{i\cdot\cdot} - \frac{r_i}{r_a} y_{a\cdot\cdot} \qquad i = 1, \cdots, a-1 \\
\mathbf{v} &= y_{\cdot j\cdot} - \frac{c_j}{c_b} y_{\cdot b\cdot} \qquad j = 1, \cdots, b-1 \\
\mathbf{w} &= y_{ij\cdot} - \frac{c_j}{c_b} y_{ib\cdot} - \frac{r_i}{r_a} y_{aj\cdot} + \frac{r_i c_j}{r_a c_b} y_{ab\cdot} \qquad i = 1, \cdots, a-1\,, \quad j = 1, \cdots, b-1\,,
\end{aligned}$$

we have the matrix to sweep as follows:

$$\begin{bmatrix}
r.c. & \mathbf{0} & \mathbf{0} & \mathbf{0} & y_{\cdots} \\
\mathbf{0} & c.\boldsymbol{\Delta}_a \mathbf{R} \boldsymbol{\Delta}_a' & \mathbf{0} & \mathbf{0} & \mathbf{u} \\
\mathbf{0} & \mathbf{0} & r.\boldsymbol{\Delta}_b \mathbf{C} \boldsymbol{\Delta}_b' & \mathbf{0} & \mathbf{v} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & (\boldsymbol{\Delta}_a \mathbf{R} \boldsymbol{\Delta}_a') \otimes (\boldsymbol{\Delta}_b \mathbf{C} \boldsymbol{\Delta}_b') & \mathbf{w} \\
y_{\cdots} & \mathbf{u}' & \mathbf{v}' & \mathbf{w}' & \mathbf{y}'\mathbf{y}
\end{bmatrix}$$

Since we have a block diagonal matrix, parameter estimates across blocks are independent. This implies that the decrease in the residual sum of squares(SSE) due to adding a block is invariant to what is in the model and that the parameter estimates of each block are invariant to what is in the model. Thus we obtain the following result.

Theorem 4. The parameter estimates of the reparameterized model are as follows:

$$\begin{aligned}
&\hat{\mu} = \overline{y}_{\cdots} \\
&\hat{\alpha}_i = \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdots} \qquad i = 1, \cdots, a-1 \\
&\hat{\beta}_j = \overline{y}_{\cdot j\cdot} - \overline{y}_{\cdots} \qquad j = 1, \cdots, b-1 \\
&(\widehat{\alpha\beta})_{ij} = \overline{y}_{ij\cdot} - \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot j\cdot} + \overline{y}_{\cdots} \qquad i = 1, \cdots, a-1, \;\; j = 1, \cdots, b-1
\end{aligned}$$

Proof. For μ, we have

$$\hat{\mu} = \frac{1}{r.\,c.} y_{\cdots} \; = \; \overline{y}_{\cdots}\,.$$

For α_i, we have

$$\underline{\hat{\alpha}} = \frac{1}{c.} (\boldsymbol{\Delta}_a \mathbf{R} \boldsymbol{\Delta}_a')^{-1} \mathbf{u}\,,$$

where

$$\begin{aligned}
\boldsymbol{\Delta}_a \mathbf{R} \boldsymbol{\Delta}_a' &= \mathbf{R}_{a-1} + r_a\, \mathbf{s}_{a-1}\, \mathbf{s}_{a-1}'\,, \\
\mathbf{R}_{a-1} &= \mathrm{Diag}(r_1, \cdots, r_{a-1})\,.
\end{aligned}$$

So

$$\begin{aligned}
(\mathbf{\Delta}_a \mathbf{R} \mathbf{\Delta}_a')^{-1} &= (\mathbf{R}_{a-1} + r_a \, \mathbf{s}_{a-1} \, \mathbf{s}_{a-1}')^{-1} \\
&= \mathbf{R}_{a-1}^{-1} - \frac{r_a}{1 + r_a \, \mathbf{s}_{a-1}' \mathbf{R}_{a-1}^{-1} \mathbf{s}_{a-1}} \mathbf{R}_{a-1}^{-1} \mathbf{s}_{a-1} \mathbf{s}_{a-1}' \mathbf{R}_{a-1}^{-1} \\
&= \mathbf{R}_{a-1}^{-1} - \frac{r_a^2}{r.} \mathbf{R}_{a-1}^{-1} \mathbf{s}_{a-1} \mathbf{s}_{a-1}' \mathbf{R}_{a-1}^{-1} \\
&= \mathbf{R}_{a-1}^{-1} - \frac{1}{r.} \mathbf{1}_{a-1} \mathbf{1}_{a-1}'
\end{aligned}$$

and

$$\begin{aligned}
\underline{\hat{\alpha}} &= \frac{1}{c.} \Big(\mathbf{R}_{a-1}^{-1} - \frac{1}{r.} \mathbf{1}_{a-1} \mathbf{1}_{a-1}' \Big) \mathbf{u} \\
&= \frac{1}{c.} \mathbf{R}_{a-1}^{-1} \, \mathbf{u} - \frac{1}{c.\, r.} \mathbf{1}_{a-1} \mathbf{1}_{a-1}' \, \mathbf{u} \, .
\end{aligned}$$

Then the ith value of the vector $\underline{\hat{\alpha}}$ is:

$$\begin{aligned}
\hat{\alpha}_i &= \frac{1}{c.\, r_i} \Big(y_{i\cdot\cdot} - \frac{r_i}{r_a} y_{a\cdot\cdot} \Big) - \frac{1}{c.\, r.} \Big(y_{\cdot\cdot\cdot} - \frac{r.}{r_a} y_{a\cdot\cdot} \Big) \\
&= \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot} \qquad i = 1, \cdots, a-1 \, .
\end{aligned}$$

By symmetry, we have

$$\hat{\beta}_j = \overline{y}_{\cdot j \cdot} - \overline{y}_{\cdot\cdot\cdot} \qquad j = 1, \cdots, b-1 \, .$$

Now for $(\widehat{\alpha\beta})_{ij}$,

$$\begin{aligned}
(\underline{\widehat{\alpha\beta}}) &= \Big((\mathbf{\Delta}_a \mathbf{R} \mathbf{\Delta}_a') \otimes (\mathbf{\Delta}_b \mathbf{C} \mathbf{\Delta}_b') \Big)^{-1} \mathbf{w} \\
&= \Big[(\mathbf{\Delta}_a \mathbf{R} \mathbf{\Delta}_a')^{-1} \otimes (\mathbf{\Delta}_b \mathbf{C} \mathbf{\Delta}_b')^{-1} \Big] \mathbf{w} \\
&= \Big[\Big(\mathbf{R}_{a-1}^{-1} - \frac{1}{r.} \mathbf{1}_{a-1} \mathbf{1}_{a-1}' \Big) \otimes \Big(\mathbf{C}_{b-1}^{-1} - \frac{1}{c.} \mathbf{1}_{b-1} \mathbf{1}_{b-1}' \Big) \Big] \mathbf{w} \\
&= \left[\begin{pmatrix} \frac{1}{r_1}\mathbf{Z} & & \\ & \ddots & \\ & & \frac{1}{r_{a-1}}\mathbf{Z} \end{pmatrix} - \frac{1}{r.} \begin{pmatrix} \mathbf{Z} & \cdot & \cdot & \mathbf{Z} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \mathbf{Z} & \cdot & \cdot & \mathbf{Z} \end{pmatrix} \right] \begin{bmatrix} w_{11} \\ \vdots \\ w_{1,b-1} \\ \cdot \\ \cdot \\ \cdot \\ w_{a-1,1} \\ \vdots \\ w_{a-1,b-1} \end{bmatrix}
\end{aligned}$$

where

$$\mathbf{Z} = \mathbf{C}_{b-1}^{-1} - \frac{1}{c.}\mathbf{1}_{b-1}\mathbf{1}_{b-1}' .$$

Since the ith block gives

$$\mathbf{Z}\begin{bmatrix} w_{i1} \\ \vdots \\ w_{i,b-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{c_1}w_{i1} - \frac{1}{c.}\sum_{j=1}^{b-1} w_{ij} \\ \vdots \\ \frac{1}{c_{b-1}}w_{i,b-1} - \frac{1}{c.}\sum_{j=1}^{b-1} w_{ij} \end{bmatrix}$$

we have

$$\begin{aligned}\begin{bmatrix} (\widehat{\alpha\beta})_{i1} \\ \vdots \\ (\widehat{\alpha\beta})_{i,b-1} \end{bmatrix} &= \frac{1}{r_i}\begin{bmatrix} \frac{1}{c_1}w_{i1} - \frac{1}{c.}\sum_{j=1}^{b-1} w_{ij} \\ \vdots \\ \frac{1}{c_{b-1}}w_{i,b-1} - \frac{1}{c.}\sum_{j=1}^{b-1} w_{ij} \end{bmatrix} - \frac{1}{r.}\sum_{i=1}^{a-1}\begin{bmatrix} \frac{1}{c_1}w_{i1} - \frac{1}{c.}\sum_{j=1}^{b-1} w_{ij} \\ \vdots \\ \frac{1}{c_{b-1}}w_{i,b-1} - \frac{1}{c.}\sum_{j=1}^{b-1} w_{ij} \end{bmatrix} \\ &= \begin{bmatrix} \overline{w}_{i1} - \overline{w}_{i\cdot} - \overline{w}_{\cdot 1} + \overline{w}_{\cdot\cdot} \\ \vdots \\ \overline{w}_{i,b-1} - \overline{w}_{i\cdot} - \overline{w}_{\cdot b-1} + \overline{w}_{\cdot\cdot} \end{bmatrix}\end{aligned}$$

where

$$\begin{aligned}\overline{w}_{ij} &= \frac{1}{r_i\, c_j} w_{ij} &= \overline{y}_{ij\cdot} - \overline{y}_{ib\cdot} - \overline{y}_{aj\cdot} - \overline{y}_{ab\cdot} \\ \overline{w}_{i\cdot} &= \frac{1}{r_i\, c.}\sum_{j=1}^{b-1} w_{ij} &= \overline{y}_{i\cdot\cdot} - \overline{y}_{ib\cdot} - \overline{y}_{a\cdot\cdot} - \overline{y}_{ab\cdot} \\ \overline{w}_{\cdot j} &= \frac{1}{r.\, c_j}\sum_{i=1}^{a-1} w_{ij} &= \overline{y}_{\cdot j\cdot} - \overline{y}_{\cdot b\cdot} - \overline{y}_{aj\cdot} - \overline{y}_{ab\cdot} \\ \overline{w}_{\cdot\cdot} &= \frac{1}{r.\, c.}\sum_{i=1}^{a-1}\sum_{j=1}^{b-1} w_{ij} &= \overline{y}_{\cdots} - \overline{y}_{\cdot b\cdot} - \overline{y}_{a\cdot\cdot} - \overline{y}_{ab\cdot} .\end{aligned}$$

Substitution yields

$$\begin{bmatrix} (\widehat{\alpha\beta})_{i1} \\ \vdots \\ (\widehat{\alpha\beta})_{i,b-1} \end{bmatrix} = \begin{bmatrix} \overline{y}_{i1\cdot} - \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot 1\cdot} + \overline{y}_{\cdots} \\ \vdots \\ \overline{y}_{i,b-1} - \overline{y}_{i\cdot\cdot} - \overline{w}_{\cdot b-1\cdot} + \overline{y}_{\cdots} \end{bmatrix} \qquad i = 1, \cdots, a-1 .$$

Thus we have

$$(\widehat{\alpha\beta})_{ij} = y_{ij\cdot} - \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot j\cdot} + \overline{y}_{\cdots} , \qquad \begin{aligned} i &= 1, \cdots, a-1 \\ j &= 1, \cdots, b-1 . \end{aligned}$$

Q.E.D.

We shall now show the numerator sum of squares for each hypothesis. Let N_A, N_B, and N_{AB} denote the numerators sums of squares due to A effect, B effect, and $A \times B$ effect, respectively. Due to the orthogonal property of the reparameterized model, we obtain the following results.

Theorem 5. The numerator sums of squares for the hypotheses H_A^* or H_A^{**}, H_B^* or H_B^{**}, and H_{AB} are as follows:

$$N_A^* = N_A^{**} = c_{\cdot} \sum_{i=1}^{a} r_i (\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})^2$$

$$N_B^* = N_B^{**} = r_{\cdot} \sum_{j=1}^{b} c_j (\overline{y}_{\cdot j\cdot} - \overline{y}_{\cdot\cdot\cdot})^2$$

$$N_{AB} = \sum_{i=1}^{a} \sum_{j=1}^{b} r_i c_j (\overline{y}_{ij\cdot} - \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot j\cdot} + \overline{y}_{\cdot\cdot\cdot})^2 .$$

Proof. If we bring μ, α, β and $(\alpha\beta)$ into the model (4), then the following reductions in sums of squares occur. The numerator sum of squares for H_A^{**} is

$$\begin{aligned}
& N_A^{**} = R(\underline{\alpha}|\mu) \\
= & \sum_{i=1}^{a-1} (\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})(y_{i\cdot\cdot} - \frac{r_i}{r_a} y_{a\cdot\cdot}) \\
= & \sum_{i=1}^{a-1} (\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})[r_i c_{\cdot}(\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot}) - (r_i c_{\cdot} \overline{y}_{a\cdot\cdot} - r_i c_{\cdot} \overline{y}_{\cdot\cdot\cdot})] \\
= & \sum_{i=1}^{a-1} r_i c_{\cdot} (\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})^2 - \sum_{i=1}^{a-1} (\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})(r_i c_{\cdot} \overline{y}_{a\cdot\cdot} - r_i c_{\cdot} \overline{y}_{\cdot\cdot\cdot}) \\
= & \sum_{i=1}^{a-1} r_i c_{\cdot} (\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})^2 - c_{\cdot}(\overline{y}_{a\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot}) \sum_{i=1}^{a} r_i (\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot}) + r_a c_{\cdot} (\overline{y}_{a\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})^2 \\
= & \sum_{i=1}^{a-1} r_i c_{\cdot} (\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})^2 + r_a c_{\cdot} (\overline{y}_{a\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})^2 ,
\end{aligned}$$

since the term $\sum_{i=1}^{a} r_i (\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})$ vanishes. Thus we have

$$N_A^{**} = \sum_{i=1}^{a} r_i c_{\cdot} (\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})^2 .$$

And since H_A^* and H_A^{**} are equivalent, $N_A^* = N_A^{**}$.

In a similar way, we obtain the numerator sum of squares for H_B^* or H_B^{**} as

$$R(\underline{\beta}|\mu, \underline{\alpha}) = R(\underline{\beta}|\mu) = \sum_{j=1}^{b} r_{\cdot} c_j (\bar{y}_{\cdot j \cdot} - \bar{y}_{\cdots})^2 .$$

The numerator sum of squares for H_{AB} is given as follows:

$$\begin{aligned}
N_{AB} &= R((\underline{\alpha\beta}|\mu, \underline{\alpha}, \underline{\beta}) \\
&= \sum_{i=1}^{a-1}\sum_{j=1}^{b-1} (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots})(y_{ij\cdot} - \frac{c_j}{c_b} y_{ib\cdot} - \frac{r_i}{r_a} y_{aj\cdot} + \frac{r_i c_j}{r_a c_b} y_{ab\cdot}) \\
&= \sum_{i=1}^{a-1}\sum_{j=1}^{b-1} r_i c_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots})(\bar{y}_{ij\cdot} - \bar{y}_{ib\cdot} - \bar{y}_{aj\cdot} + \bar{y}_{ab\cdot}) \\
&= \sum_{i=1}^{a-1}\sum_{j=1}^{b-1} r_i c_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots})^2 \\
&\qquad - \sum_{i=1}^{a-1}\sum_{j=1}^{b-1} r_i c_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots})(\bar{y}_{ib\cdot} - \bar{y}_{i\cdot\cdot}) \\
&\qquad - \sum_{i=1}^{a-1}\sum_{j=1}^{b-1} r_i c_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots})^2(\bar{y}_{aj\cdot} - \bar{y}_{\cdot j\cdot}) \\
&\qquad + \sum_{i=1}^{a-1}\sum_{j=1}^{b-1} r_i c_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots})^2(\bar{y}_{ab\cdot} - \bar{y}_{\cdots}) .
\end{aligned}$$

Here,

$$\begin{aligned}
&\sum_{i=1}^{a-1}\sum_{j=1}^{b-1} r_i c_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots})(\bar{y}_{ib\cdot} - \bar{y}_{i\cdot\cdot}) \\
= &\sum_{i=1}^{a-1}\sum_{j=1}^{b} r_i c_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots})(\bar{y}_{ib\cdot} - \bar{y}_{i\cdot\cdot}) \\
&- \sum_{i=1}^{a-1} r_i c_b (\bar{y}_{ib\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot b\cdot} + \bar{y}_{\cdots})^2 - \sum_{i=1}^{a-1} r_i c_b (\bar{y}_{ib\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot b\cdot} + \bar{y}_{\cdots})(\bar{y}_{\cdot b\cdot} - \bar{y}_{\cdots}) \\
= &- \sum_{i=1}^{a-1} r_i c_b (\bar{y}_{ib\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot b\cdot} + \bar{y}_{\cdots})^2 + r_a c_b (\bar{y}_{ab\cdot} - \bar{y}_{a\cdot\cdot} - \bar{y}_{\cdot b\cdot} + \bar{y}_{\cdots})(\bar{y}_{\cdot b\cdot} - \bar{y}_{\cdots}) ,
\end{aligned}$$

since the first term vanishes, and in a similar way we have

$$\sum_{i=1}^{a-1}\sum_{j=1}^{b-1} r_i c_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots})(\bar{y}_{aj\cdot} - \bar{y}_{\cdot j\cdot})$$

$$= -\sum_{j=1}^{b-1} r_a c_j (\overline{y}_{aj\cdot} - \overline{y}_{a\cdot\cdot} - \overline{y}_{\cdot j\cdot} + \overline{y}_{\cdots})^2 + r_a c_b (\overline{y}_{ab\cdot} - \overline{y}_{a\cdot\cdot} - \overline{y}_{\cdot b\cdot} + \overline{y}_{\cdots})(\overline{y}_{a\cdot\cdot} - \overline{y}_{\cdots}),$$

and

$$\begin{aligned}
&\sum_{i=1}^{a-1}\sum_{j=1}^{b-1} r_i c_j (\overline{y}_{ij\cdot} - \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot j\cdot} + \overline{y}_{\cdots})(\overline{y}_{ab\cdot} - \overline{y}_{\cdots}) \\
&= \sum_{i=1}^{a-1}\sum_{j=1}^{b} r_i c_j (\overline{y}_{ij\cdot} - \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot j\cdot} + \overline{y}_{\cdots})(\overline{y}_{ab\cdot} - \overline{y}_{\cdots}) \\
&\quad - \sum_{i=1}^{a} r_i c_b (\overline{y}_{ib\cdot} - \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot b\cdot} + \overline{y}_{\cdots})(\overline{y}_{ab\cdot} - \overline{y}_{\cdots}) + r_a c_b (\overline{y}_{ab\cdot} - \overline{y}_{a\cdot\cdot} - \overline{y}_{\cdot b\cdot} + \overline{y}_{\cdots})(\overline{y}_{ab\cdot} - \overline{y}_{\cdots}) \\
&= r_a c_b (\overline{y}_{ab\cdot} - \overline{y}_{a\cdot\cdot} - \overline{y}_{\cdot b\cdot} + \overline{y}_{\cdots})^2 + r_a c_b (\overline{y}_{ab\cdot} - \overline{y}_{a\cdot\cdot} - \overline{y}_{\cdot b\cdot} + \overline{y}_{\cdots})(\overline{y}_{a\cdot\cdot} + \overline{y}_{\cdot b\cdot} - 2\overline{y}_{\cdots}),
\end{aligned}$$

since the first two terms vanish. Thus we have

$$N_{AB} = \sum_{i=1}^{a}\sum_{j=1}^{b} r_i c_j (\overline{y}_{ij\cdot} - \overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot j\cdot} + \overline{y}_{\cdots})^2. \qquad \text{Q.E.D.}$$

Although N_{AB} does not have a simple algebraic expression for general unequal cell frequencies, we see that N_{AB} for the proportional frequency case has the same algebraic expression as the equal cell frequency case.

We now show the numerator sums of squares for the hypotheses H_A and H_B.

Theorem 6. The numerator sum of squares for the hypothesis H_A is:

$$N_A = \sum_{i=1}^{a} r_i c^* \Big(\sum_{j=1}^{b} \overline{y}_{ij\cdot} - \overline{y}_A \Big)^2$$

where

$$\frac{1}{c^*} = \sum_{j=1}^{b} \frac{1}{c_j}, \quad \overline{y}_A = \frac{1}{r_\cdot} \sum_{i=1}^{a}\sum_{j=1}^{b} r_i \overline{y}_{ij\cdot}$$

For H_B, we have

$$N_B = \sum_{j=1}^{b} c_j r^* \Big(\sum_{i=1}^{a} \overline{y}_{ij\cdot} - \overline{y}_B \Big)^2$$

where

$$\frac{1}{r^*} = \sum_{i=1}^{a} \frac{1}{r_i}, \quad \overline{y}_B = \frac{1}{c_\cdot} \sum_{i=1}^{a}\sum_{j=1}^{b} c_j \overline{y}_{ij\cdot}$$

Proof. It is well known that for the general unbalanced data,

$$N_A = \sum_{i=1}^{a} \left(\frac{h_{r_i}}{b}\right) \Big(\sum_{j=1}^{b} \overline{y}_{ij} - \overline{y}_r\Big)^2$$

where

$$\overline{y}_r = \sum_{i=1}^{a} \left(\frac{h_{r_i}}{h_{r\cdot}}\right) \sum_{j=1}^{b} \overline{y}_{ij\cdot}\,, \qquad h_{r_i}^{-1} = \frac{1}{b}\sum_{j=1}^{b} n_{ij}^{-1}\,, \qquad h_{r\cdot} = \sum_{i=1}^{a} h_{r_i}\,.$$

Letting $n_{ij} = r_i\, c_j$, we have the following simplification:

$$h_{r_i}^{-1} = \frac{1}{b}\sum_{j=1}^{b}(r_i\, c_j)^{-1} = \frac{1}{b} r_i^{-1} \sum_{j=1}^{b} c_j^{-1}$$

$$\Rightarrow \quad h_{r_i} = b \frac{r_i}{\sum_{j=1}^{b} c_j^{-1}}$$

and

$$h_{r\cdot} = \sum_{i=1}^{a} b \frac{r_i}{\sum_{j=1}^{b} c_j^{-1}} = \frac{b\, r\cdot}{\sum_{j=1}^{b} c_j^{-1}}\,.$$

Then we have

$$\overline{y}_r = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b} r_i \overline{y}_{ij\cdot}}{r\cdot}\,.$$

Also

$$\frac{h_{r_i}}{b} = \frac{r_i}{\sum_{j=1}^{b} c_j^{-1}}\,.$$

Substitution yields

$$N_A = \sum_{i=1}^{a} r_i\, c^* \Big(\sum_{j=1}^{b} \overline{y}_{ij\cdot} - \sum_{i=1}^{a}\sum_{j=1}^{b} \frac{r_i}{r\cdot}\overline{y}_{ij\cdot}\Big)^2 = \sum_{i=1}^{a} r_i\, c^* \Big(\sum_{j=1}^{b} \overline{y}_{ij\cdot} - \overline{y}_A\Big)^2$$

where

$$c^* = \Big(\sum_{j=1}^{b} \frac{1}{c_j}\Big)^{-1}, \qquad \overline{y}_A = \sum_{i=1}^{a}\sum_{j=1}^{b} \frac{r_i}{r\cdot}\overline{y}_{ij\cdot}\,.$$

In a similar way, we obtain

$$N_B = \sum_{j=1}^{b} c_j r^* \Big(\sum_{i=1}^{a} \overline{y}_{ij\cdot} - \overline{y}_B\Big)^2,$$

where

$$r^* = \Big(\sum_{i=1}^{a} \frac{1}{r_i}\Big)^{-1}, \quad \overline{y}_B = \sum_{i=1}^{a}\sum_{j=1}^{b} \frac{c_j}{c\cdot}\overline{y}_{ij\cdot}\,. \qquad \text{Q.E.D.}$$

Thus for the proportional frequency case the numerator sums of squares of the Table 1 through 3 are as follows:

Table 4. Sequential sum of squares and Partially sequential sum of squares

Hypotheis	Numerator sum of squares
$H_A^{**} = H_A^*$	$N_A^{**} = N_A^* = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2 = c.\sum_{i=1}^{a} r_i(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2$
H_B^*	$N_B^* = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot})^2 = r.\sum_{j=1}^{b} c_j(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot})^2$
H_{AB}	$N_{AB} = \sum_{i=1}^{a}\sum_{j=1}^{b} r_i c_j(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot})^2$
Residual	$Q(\hat{\mu}) = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}(y_{ijk} - \bar{y}_{ij\cdot})^2$

Table 5. Marginal means sum of squares

Hypotheis	Numerator sum of squares
H_A	$N_A = \sum_{i=1}^{a} r_i\Big(\sum_{j=1}^{b}\bar{y}_{ij\cdot} - \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{r_i}{r.}\bar{y}_{ij\cdot}\Big)^2\Big/\Big(\sum_{j=1}^{b}\frac{1}{c_j}\Big)$
H_B	$N_B = \sum_{j=1}^{b} c_j\Big(\sum_{i=1}^{a}\bar{y}_{ij\cdot} - \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{c_j}{c.}\bar{y}_{ij\cdot}\Big)^2\Big/\Big(\sum_{i=1}^{a}\frac{1}{r_i}\Big)$
H_{AB}	$N_{AB} = \sum_{i=1}^{a}\sum_{j=1}^{b} r_i c_j(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot})^2$
Residual	$Q(\hat{\mu}) = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}(y_{ijk} - \bar{y}_{ij\cdot})^2$

6. Discussion

Proportional cell frequency data is a special case of the general unbalanced data and allows us to explore the relation between parameter definitions and the analysis of variance.

Orthogonality occurs when we use the sequential method or the partially sequential method given in Table 1 or 2. The sums of squares for A, B, and $A \times B$

interaction for the case of proportional frequencies are obtained using the reparameterized model (4). They are given in Table 4, and they add up to the sum of squares for testing equality of means. These sums of squares are appropriate for testing the hypotheses, H_A^{**}, H_B^{**} (equivlaently H_A^*, H_B^*) and H_{AB}. This is equivalent to using the overparameterized model (5) and imposing conditions given in (6) to remove the redundancies. The normal equations take the following form:

$$
\begin{aligned}
&r.\,c.\,\mu + c.\sum_{i=1}^{a} r_i\,\alpha_i + r.\sum_{j=1}^{b} c_j\,\beta_j + \sum_{i=1}^{a}\sum_{j=1}^{b} r_i\,c_j\,(\alpha\beta)_{ij} = y_{\cdots} \\
&r_i\,c.\mu + r_i\,c.\alpha_i + r_i\sum_{j=1}^{b} c_j\,\beta_j + r_i\sum_{j=1}^{b} c_j\,(\alpha\beta)_{ij} = y_{i\cdot\cdot} \qquad i = 1,\cdots,a \\
&r.\,c_j\,\mu + c_j\sum_{i=1}^{a} r_i\,\alpha_i + r.\,c_j\,\beta_j + c_j\sum_{i=1}^{a} r_i\,(\alpha\beta)_{ij} = y_{\cdot j\cdot} \qquad j = 1,\cdots,b \\
&r_i\,c_j\,\mu + r_i\,c_j\,\alpha_i + r_i\,c_j\,\beta_j + r_i\,c_j\,(\alpha\beta)_{ij} = y_{ij\cdot} \qquad i = 1,\cdots,a \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1,\cdots,b.
\end{aligned}
$$

With the side conditions, it can be shown that they give the same estimates as those given in Theorem 4 with $i = a$ and $j = b$ included and give the same analysis as in Table 5.

Nonorthogonality occurs when we use the marginal means analysis. The sums of squares can be obtained using the marginal means parameters,

$$
\begin{aligned}
\alpha_i &= \overline{\mu}_{i\cdot} - \overline{\mu}_{\cdot\cdot} \qquad i = 1,\cdots,a-1 \\
\beta_j &= \overline{\mu}_{\cdot j} - \overline{\mu}_{\cdot\cdot} \qquad j = 1,\cdots,b-1 \\
(\alpha\beta)_{ij} &= \mu_{ij} - \overline{\mu}_{i\cdot} - \overline{\mu}_{\cdot j} + \overline{\mu}_{\cdot\cdot} = 0 \qquad i = 1,\cdots,a-1, \quad j = 1,\cdots,b-1\,,
\end{aligned}
$$

and applying the computing procedure in Table 3 to this reparameterized model. This is equivalent to using the overparameterized model (5) and imposing side conditions,

$$
\begin{aligned}
&\sum_{i=1}^{a}\alpha_i = 0\,, \quad \sum_{j=1}^{b}\beta_j = 0\,, \quad \sum_{j=1}^{b}(\alpha\beta)_{ij} = 0\,, \quad i = 1,\cdots,a-1\,, \\
&\sum_{i=1}^{a}(\alpha\beta)_{ij} = 0\,, \quad j = 1,\cdots,b\,,
\end{aligned}
$$

to remove the redundancies. These sums of squares are given in Table 5, which have been derived directly from the general results of the cell means model with unbalanced data. The main effects sums of squares differ from those of orthogonal case and the main effects and their interaction sums of squares do not add up to the sum of squares for testing equality of means.

Thus we see that orthogonality occurs when the sequential sum of squares or the partially sequential sum of squares is used and that nonorthogonality occurs when the marginal means sum of squares is used. It is often conceived that for the proportional frequency case the same analysis can be carried out as in the usual equal cell frequency case, but this is not exactly true when the marginal means analysis is used. We should be aware of what hypotheses being tested when we intetpret computer outputs since the hypotheses associated with the sums of squares depend on how the parameters are defined or what side conditions are imposed.

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