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Fuzzy Linear Regression with the Weakest t-norm

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Abstract

In this paper a fuzzy regression model based on the weakest t-norm is introduced. The model shows a regression model which has fuzzy coefficients and fuzzy variables.

Key Words and Phrases : fuzzy number, weakest t-norm, fuzzy regression.

1. Introduction and Preliminaries

Tanaka and Watada(1988) illustrated possibilistic linear systems based on sup-min convolution and formulated a fuzzy linear regression. Their results help us to evaluate possibilistic linear systems or fuzzy linear regression when coefficients are symmetric fuzzy numbers and variables are real numbers. However, the difficulty of multiplication with symmetric fuzzy numbers based on minimum t-norm prevents to analyze possibilistic linear system with fuzzy coefficients and fuzzy variables. Hence, in this paper, our main interests are to illustrate a fuzzy linear regression with fuzzy coefficients and fuzzy variables.

Fuzzy sets can be regarded as a possibility distribution which is a fuzzy restriction. Given a fuzzy set A whose membership function $\mu_A(x)$ is normal, a possibility distribution $\pi(x)$ is defined as $\pi(x)\overset{\triangle}{=}\mu_A(x)$. A possibility measure of a set E is defined as $\pi(E) = \sup_{x\in E}\ \pi_X(x)$.

A possibility measure of fuzzy set A is defined as $\pi(A) = \sup_x\ (\mu_A(x) * \pi_X(x))$, where $*$ is a triangular norm(t-norm) that satisfies (i) $0*0=0,\ \ a*1=1*a=a$ (ii) $a*b \le c*d$ whenever $a \le c,\ b \le d$ (iii) $a*b=b*a$ (iv) $(a*b)*c = a*(b*c)$.

$$Min(a,b),\ a\cdot b,\ Max(0,a+b-1),\ and\ T_w(a,b) = \begin{cases} 0 & ,\ max(a,b)<1 \\ min(a,b) & ,\ otherwise \end{cases}$$

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are examples of t-norms and T_w is called as the Weakest t-norm. We define a fuzzy number A as

$$\mu_A(x)=\begin{cases}1 & ,\ a\le x\le b\\ L\left(\frac{a-x}{\alpha}\right) & ,\ a-\alpha\le x\le a,\ \ \alpha>0\\ R\left(\frac{x-b}{\beta}\right) & ,\ b\le x\le b+\beta,\ \ \beta>0\end{cases}$$

where $L(\cdot)$ and $R(\cdot)$ are shape functions, i.e. non-increasing continuous mapping from $[0,1]$ onto $[0,1]$ with $L(0)=R(0)=1$ and $L(1)=R(1)=0$. These fuzzy numbers are called fuzzy numbers of LR type and denoted by $A=(a,b,\alpha,\beta)_{LR}$. When $a=b$, A is called a triangular fuzzy number and denoted by $A=(a,\alpha,\beta)_{LR}$. When $a=b$ and $L(\cdot)=R(\cdot)$, A is called a symmetric fuzzy number and denoted by $A(a,\alpha)_L$. If T is a t-norm and A_1,A_2 are fuzzy sets of the real line R, then their T-sum $C=A\oplus B$ and T-multiplication $D=A\otimes B$ are defined by the generalized extension principle of Zadeh as

$$\begin{aligned}\mu_C(z) &= \sup_{z=x+y} T(\mu_A(x),\mu_B(y)),\quad z\in R,\\ \mu_D(z) &= \sup_{z=x\cdot y} T(\mu_A(x),\mu_B(x)),\quad z\in R.\end{aligned}$$

2. T_w-based addition and multiplication of fuzzy numbers

Let $A_1(a_1,b_1,\alpha_1,\beta_1)_{LR}$ and $A_2(a_2,b_2,\alpha_2,\beta_2)_{LR}$. Then Mesiar(1997) showed that T_w sum

$$A_1\oplus A_2=(a_1+a_2,b_1+b_2,max(\alpha_1,\alpha_2),max(\beta_1,\beta_2))_{LR}.$$

Now we will show T_w-subtraction and T_w-multiplication.

Lemma 2.1 Let $A_1(a_1,b_1,\alpha_1,\beta_1)_{LR}$ and $A_2(a_2,b_2,\alpha_2,\beta_2)_{RL}$.
Then $A_1\ominus A_2=(a_1-b_2,b_1-a_2,max(\alpha_1,\beta_2),max(\alpha_2,\beta_1))_{LR}$.
Proof. For $z\le a_1-b_2$

$$\begin{aligned}\mu_C(z) &= \sup_{x-y=z} T_w(\mu_{A_1}(x),\mu_{A_2}(y))\\ &= max(\mu_{A_1}(b_2+z),\mu_{A_2}(a_1-z))\\ &= max\left(L\left(\frac{a_1-b_2-z}{\alpha_1}\right),L\left(\frac{a_1-b_2-z}{\beta_2}\right)\right)\\ &= L\left(\frac{a_1-b_2-z}{max(\alpha_1,\beta_2)}\right)\quad for\ \ z\ge a_1-b_2-max(\alpha_1,\beta_2)\end{aligned}$$

Similarly, for $z\ge b_1-a_2$
$\mu_c(z)=R\left(\frac{z-b_1+a_2}{max(\alpha_2,\beta_1)}\right)$ for $z\le b_1-a_2+max(\alpha_2,\beta_1)$ and for $a_1-b_2<z<b_1-a_2$, $\mu_c(z)=1$ and it is zero otherwise.

Lemma 2.2
(i) Let $A_1(a_1, \alpha_1, \beta_1)_{LR}$ and $A_2(a_2, \alpha_2, \beta_2)_{LR}$. Then for $a_1, a_2 > 0$
$A_1 \otimes A_2 = (a_1 a_2, max(\alpha_1 a_2, \alpha_2 a_1), max(\beta_1 a_2, \beta_2 a_1))_{LR}$.
(ii) Let $A_1(a_1, \alpha_1, \beta_1)_{RL}$ and $A_2(a_2, \alpha_2, \beta_2)_{LR}$. Then for $a_1 < 0, a_2 > 0$
$A_1 \otimes A_2 = (a_1 a_2, max(a_2 \alpha_1, -a_1 \beta_2), max(a_2 \beta_1, -a_1 \alpha_2))_{RL}$.
(iii) Let $A_1(a_1, \alpha_1, \beta_1)_{LR}$ and $A_2(a_2, \alpha_2, \beta_2)_{LR}$. Then for $a_1, a_2 < 0$
$A_1 \otimes A_2 = (a_1 a_2, max(-a_1 \beta_2, -a_2 \beta_1), max(-a_2 \alpha_1, -a_1 \alpha_2))_{RL}$.
Proof. (i) For $z \leq a_1 a_2$

$$\begin{aligned} \mu_{A_1 \otimes A_2}(z) &= \sup_{x \cdot y = z} T_w(\mu_{A_1}(x), \mu_{A_2}(y)) \\ &= max(\mu_{A_1}(z/a_2), \mu_{A_2}(z/a_1)) \\ &= max\left(L\left(\frac{a_1 a_2 - z}{a_2 \alpha_1}\right), L\left(\frac{a_1 a_2 - z}{a_1 \alpha_2}\right)\right) \\ &= L\left(\frac{a_1 a_2 - z}{max(a_2 \alpha_1, a_1 \alpha_2)}\right) \qquad for \quad z \geq a_1 a_2 - max(a_2 \alpha_1, a_1 \alpha_2). \end{aligned}$$

Similarly, for $z \geq a_1 a_2$

$$\mu_{A_1 \otimes A_2}(z) = R\left(\frac{z - a_1 a_2}{max(a_2 \beta_1, a_1 \beta_2)}\right) \quad for \quad z \leq a_1 a_2 + max(a_2 \beta_1, \alpha_1 \beta_2).$$

and it is zero otherwise.
(ii) and (iii). By the same procedure as (i), the results follow.

Lemma 2.3 Let $A_i = (a_i, \alpha_i, \beta_i)_{LR}$, $X_i(x_i, \gamma_i, \delta_i)_{LR}$, $i = 1, 2, \ldots, p$ and let $a_i > 0, x_i > 0, i = 1, 2, \ldots, p$. Then the possibilistic linear function with fuzzy parameters A_i and fuzzy variables $X_i, i = 1, 2, \cdots, p$, based on T_w, is

$$\begin{aligned} Y &= (A_1 \otimes X_1) \oplus (A_2 \otimes X_2) \oplus \cdots \oplus (A_p \otimes X_p) \\ &= \left(\sum_{i=1}^{p} a_i x_i, max_{1 \leq i \leq p}(a_i \gamma_i, x_i \alpha_i), max_{1 \leq i \leq p}(a_i \delta_i, x_i \beta_i)\right)_{LR.} \end{aligned}$$

Proof. For $a_i > 0$ and $x_i > 0, i = 1, 2, \ldots, p$, apply Lemma 2.2. Then
$A_i \otimes X_i = (a_i x_i, max(a_i \gamma_i, x_i \alpha_i), max(x_i \beta_i, a_i \delta_i))_{LR}$
and hence

$$\sum_{i=1}^{n} A_i \otimes X_i = (\sum_{i=1}^{p} a_i x_i, max(a_1 \gamma_1, \cdots, a_p \gamma_p, \alpha_1 x_1, \cdots, \alpha_p x_p),$$
$$max(a_1 \delta_1, \cdots, a_p \delta_p, \beta_1 x_1, \cdots, \beta_p x_p))_{LR.}$$

Lemma 2.4 Let $A_i = (a_i, \alpha_i)_L$, $X_i = (x_i, \gamma_i)_L$, $i = 1, 2, \cdots, p$. Then,
(i) for $a_i > 0, x_i > 0, i = 1, 2, \cdots, p$
the membership function of $Y = (A_1 \otimes X_1) \oplus \cdots \oplus (A_p \otimes X_p)$ is given by

$$\mu_Y(y) = L\left((y - \sum_{i=1}^{p} a_i x_i) / \max_{1 \le i \le p} (a_i \gamma_i, x_i \alpha_i)\right)$$

(ii) for $x_i > 0, i = 1, 2, \cdots, p$ and for all $a_i > 0$ except $a_j < 0$
the membership function of $Y = (A_1 \otimes X_1) \oplus \cdots \oplus (A_p \otimes X_p)$ is given by

$$\mu_Y(y) = L\left((y - \sum_{i=1}^{p} a_i x_i) / max(x_1 \alpha_1, \cdots, x_p \alpha_p, a_1 \gamma_1, \cdots, a_{j-1} \gamma_{j-1}, -a_j \gamma_j, \cdots, a_p \gamma_p)\right)$$

3. Fuzzy linear regression

In our model all variables and parameters are fuzzy numbers and two types of outputs are considered, i.e. non-fuzzy data and fuzzy data. The following definitions are from Tanaka and Watada(1988). Throughout this section, we consider only symmetric fuzzy numbers $L(\cdot)$.

Definition 3.1 The inclusion of fuzzy numbers with degree $0 \le h < 1$, denoted as $A \supset_h B$, is defined by $[A]_h \supset [B]_h$, where $[A]_h = \{x : \mu_A(x) \ge h\}$.

Note that $[A]_h \supset [B]_h$ is equivalent to

$$a_2 - \left|L^{-1}(h)\right| (\alpha_1 - \alpha_2) \le a_1 \le a_2 + \left|L^{-1}(h)\right| (\alpha_1 - \alpha_2)$$

where $\mu_A(x) = L(\frac{x-a_1}{\alpha_1})$ and $\mu_B(x) = L(\frac{x-a_2}{\alpha_2})$.
It follows from definition 3.1 that $[A]_{h'} \supset [B]_{h'}$, for $h' < h$ if $[A]_h \supset [B]_h$. Let the measure of containment of B in A be denoted as $C_B(A)$.

Definition 3.2 The measure of containment is defined as

$$C_B(A) = sup_{\{h:[A]_h \supset [B]_h\}} h.$$

3.1 Non-fuzzy data

In case of non-fuzzy data, the following conditions are assumed to formulate a possibilistic linear regression model :

(i) The data can be represented by a possibilistic linear model :

$$Y_i^* = (A_1^* \otimes X_{i1}) \oplus \cdots \oplus (A_p^* \otimes X_{ip}) \triangleq A^* \otimes X_i, \quad i = 1, 2, \cdots, n \tag{1}$$

where $X_{ij} = (x_{ij}, \gamma_j)_L$, $x_{ij} > 0$, $j = 1, 2, \cdots, p$, $i = 1, 2, \cdots, n$ and $A_j = (a_j, \alpha_j)_L$, $a_j > 0$, $j = 1, 2, \cdots, p$.

(ii) Given input-output relations (X_i, y_i), $i = 1, 2, \cdots, n$ and a threshold h, it must hold that

$$C_{y_i}(Y_i^*) \geq h, \quad i = 1, 2, \cdots, n \tag{2}$$

Where y_i is a real number and $X_i = (X_{i1}, \cdots, X_{ip})$.

(iii) The index of fuzziness of the possibilistic linear model is

$$J(a, \alpha) = \sum_{i=1}^{n} max_{1 \leq j \leq p}(a_j \gamma_j, x_{ij} \alpha_j). \tag{3}$$

Under the above assumptions, our problem is to obtain fuzzy parameters $A_j = (a_j, \alpha_j)_L$, $a_j > 0$, $\alpha_j > 0$, $j = 1, 2, \cdots, p$ that minimize $J(a, \alpha)$ in (3) subject to the constraint (2). This problem can be modeled as the follows :

$$Min\ J(a, \alpha) = \sum_{i=1}^{n} max_{1 \leq j \leq p}(a_j \gamma_j, x_{ij} \alpha_j) \tag{4}$$

$$subject\ to\ a > 0, \gamma > 0, \alpha > 0, x_1 > 0, \cdots, x_n > 0\ and$$

$$y_i \leq \left|L^{-1}(h)\right| max_{1 \leq j \leq p}(a_j \gamma_j, x_{ij} \alpha_j) + \sum_{j=1}^{p} a_j x_{ij},$$

$$y_i \geq -\left|L^{-1}(h)\right| max_{1 \leq j \leq p}(a_j \gamma_j, x_{ij} \alpha_j) + \sum_{j=1}^{p} a_j x_{ij}$$

Proposition 3.1 Given the data (X_i, y_i), $i = 1, 2, \ldots, n$, there exists an optimal solution $A_j = (a_j, \alpha_j)_L$, $j = 1, 2, \ldots, p$, for $0 \leq h < 1$ in (4).

Proof. For sufficiently large $\alpha_1, \alpha_2, \ldots, \alpha_p$, $A_j = (a_j, \alpha_j)_L$, $j = 1, 2, \ldots, p$ is a feasible solution. Since y_i is finite, an optimal solution exists.

Remark. If $h = 1$, them $L^{-1}(h) = 0$. Hence the following equations must hold :

$$y_i = \sum_{j=1}^{p} a_j x_{ij}, \quad i = 1, 2, \ldots, n$$

Thus, in general, there is no solution because given data do not usually satisfy the equations.

3.2 Min problem in fuzzy data

Let us consider a possibilistic linear regression model with fuzzy inputs $X_i = (x_{ij}, \gamma_j)_L$ and fuzzy outputs $Y_i = (y_i, e_i)_L$, $i = 1, 2, \ldots, n$. With the same idea as discribed in 3.1, the problem is to determine fuzzy parameters $\bar{A}^*{}_1, \ldots, \bar{A}^*{}_p$ such that

$$\bar{Y}_i^* = (\bar{A}_1^* \otimes X_{il}) \oplus \cdots \oplus (\bar{A}_p^* \otimes X_{ip}) \supset_h Y_i, \quad i = 1, 2, \cdots, n$$

where assumption (ii) in 3.1 is changed into (ii)$'$ in 3.2 as follows :
(ii)$'$ Given input-output relations (X_i, Y_i), $i = 1, 2, \cdots, n$ and a threshold h, it must hold that

$$C_{Y_i}(\bar{Y}_i^*) \geq h, \quad i = 1, 2, \cdots, n.$$

Under the assumptions (i),(ii)$'$, and (iii), Our problem can be modeled as follows :

$$Min\ J(a, \alpha) = \sum_{i=1}^{n} \max_{1 \leq j \leq p} (a_j \gamma_j, x_{ij} \alpha_j) \tag{5}$$

subject to $a > 0, \gamma > 0, \alpha > 0, x_1 > 0, \cdots, x_n > 0$ and for $i = 1, 2, \cdots, n$

$$y_i \leq \sum_{j=1}^{p} a_j x_{ij} - \left|L^{-1}(h)\right| e_i + \left|L^{-1}(h)\right| \max_{1 \leq j \leq p} (a_j \gamma_j, x_{ij} \alpha_j),$$
$$y_i \geq \sum_{j=1}^{p} a_j x_{ij} + \left|L^{-1}(h)\right| e_i - \left|L^{-1}(h)\right| \max_{1 \leq j \leq p} (a_j \gamma_j, x_{ij} \alpha_j).$$

This problem will be called Min problem. (5) indicates that the model constructed by fuzzy data is fuzzier than the model constructed by crisp data.

Proposition 3.2 Given the data (X_i, Y_i), $i = 1, 2, \ldots, n$, there exists an optimal solution $\bar{A}_j^* = (a_j^*, \alpha_j^*)_L$, $j = 1, 2, \ldots, p$, of the problem (5) for $0 \leq h < 1$.
Proof. Similar to proposition 3.1 it can be proved.

3.3 Max problem in fuzzy data

Let us consider the same model in 3.2. However the idea to the solution is converse to the idea in 3.2, i.e. the problem is to determine fuzzy parameters $\underline{A_1^*}, \ldots, \underline{A_p^*}$ such that

$$\underline{Y_i^*} = (\underline{A_i^*} \otimes X_{i1}) \oplus \cdots \oplus (\underline{A_p^*} \otimes X_{ip}) \subset_h Y_i.$$

Note that the inclusion relation is opposite to Min problem. Thus, We want to determine Y_i^*, $i = 1, 2, \cdots, n$ which are near to Y_i subject to $\underline{Y_i^*} \subset_h Y_i$. Hence Assumption (ii) in 3.1 is changed into (ii)$''$ in 3.3 as follows :
(ii)$''$. Given input-output relations (X_i, Y_i), $i = 1, 2, \cdots, n$, and a threshold h, it must hold that

$$C_{Y_i^*}(Y_i) \geq h, \quad i = 1, 2, \ldots, n$$

Under the assumptions (i),(ii)$''$, and (iii), our problem can be modeled as follows :

$$Max\ J(a, \alpha) = \sum_{i=1}^{n} \max_{1 \leq j \leq p} (a_j \gamma_j, x_{ij} \alpha_j) \tag{6}$$

subject to $a > 0,\ \gamma > 0,\ \alpha > 0,\ x_1 > 0, \cdots, x_n > 0$ and for $i = 1, 2, \ldots, n$

$$y_i \leq \sum_{j=1}^{p} a_j x_{ij} + \left|L^{-1}(h)\right| e_i - \left|L^{-1}(h)\right| \max_{1 \leq j \leq p} (a_j \gamma_j, x_{ij} \alpha_j),$$

$$y_i \geq \sum_{j=1}^{p} a_j x_{ij} - \left|L^{-1}(h)\right| e_i + \left|L^{-1}(h)\right| \max_{1 \leq j \leq p} (a_j \gamma_j, x_{ij} \alpha_j).$$

This problem will be called Max problem and it will not assure the existence of a solution.

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