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MRE for Exponential Distribution under General Progressive Type-II Censored Samples

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Abstract

By assuming a general progressive Type-II censored sample, we propose the minimum risk estimator (MRE) of the location parameter and the scale parameter of the two-parameter exponential distribution. An example is given to illustrate the methods of estimation discussed in this paper.

Key Words and Phrases: Exponential distribution, General progressive Type-II censored sample, Minimum risk estimator

1. Introduction

Progressive censored samples frequently occur in life and fatigue tests, where individual observations are time ordered and where at various times during a test, some of the survivors are removed (i.e. censored) from further observation either by design or by accident. Thus, censoring times are intermixed with failure times. Such samples arise naturally when, at various times during a life test, certain specimems are withdrawn prior to their failure for use at test objects in other experimentation.

Cohen and Norgaard (1977) discussed the inference problems for a wide range of distributions under this progressive censoring scheme. These developments were summarized by Cohen and Whitten (1988), and more by Cohen (1991). Viveros and Balakrishnan (1994) developed exact conditional inference based on progressive Type-II censored samples. Balakrishnan and Sandhu (1996) obtained the best liner unbiased estimators (BLUEs) and the maximum likelihood estimators (MLE) for exponential distribution under general progressive Type-II censored sample. Kang

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and Cho (1997) derived the MRE and the approximate maximum likelihood estimator (AMLE) of the scale parameter of the one-parameter exponential distribution under general progressive Type-II censored sample.

A Type-II progressively censored life test is conducted as follows; Suppose N randomly selected units were placed on a life test; the failure times of the first r units to fail were not observed; at the time of the $(r+1)$-th failure, R_{r+1} number of surviving units are withdrawn from the test randomly, and so on; at the time of the $(r+i)$-th failure, R_{r+i} number of surviving units are randomly withdrawn from the test; finally, at the time of the m-th failure, the remaining $R_m = N - m - R_{r+1} - R_{r+2} - \cdots - R_{m-1}$ are withdrawn from the test.

Suppose $X_{r+1,N} \leq X_{r+2,N} \leq \cdots \leq X_{m,N}$ are the life-times of the completely observed units to fail, and $R_{r+1}, R_{r+2}, \ldots, R_m$ are the number of units withdrawn from the test at these failure times, respectively. If the failure times are from a continuous population with cumulative distribution function $F(x)$ and probability density function $f(x)$, the joint probability density function for $X_{r+1,N}, \ldots, X_{m,N}$ is given by

$$f_{X_{r+1,N}, \ldots, X_{m,N}}(x_{r+1}, \ldots, x_m) = K_1[F(x_{r+1})]^r \prod_{i=r+1}^{m} f(x_i)[1-F(x_i)]^{R_i},$$

where

$$\begin{aligned} K_1 &= \binom{N}{r}(N-r)(N-r-R_{r+1}-1)(N-r-R_{r+1}-R_{r+2}-2) \times \cdots \\ &\quad \times (N-r-R_{r+1}-R_{r+2}-\cdots-R_{m-r}-(m-r)+1) \\ &= \binom{N}{r}(N-r)\prod_{i=r+2}^{m}\left(N-\sum_{i=r+1}^{j-1} R_i - j + 1\right). \end{aligned}$$

In this paper, by assuming that such a general progressive Type-II censored sample, we derive the minimum risk estimators of the scale and the location parameter of the two-parameter exponential distribution. We will compare the estimators of parameters in terms of the mean squared error (MSE) and calculate the estimators and the MSEs through Nelson's data.

2. Estimation for parameter

The random variable X has a two-parameter exponential distribution if it has a probability density function (pdf) of form;

$$f(x;\sigma) = \frac{1}{\sigma}e^{-\frac{x-\mu}{\sigma}}, \ x > \mu, \ \sigma > 0,$$

where σ is the scale parameter and μ is the location parameter.

Let $X_{0,N} = 0$,

$$D_{i,N} = (N - i + 1)(X_{i,N} - X_{i-1,N}),\ i = 1, 2, \ldots, r + 1,$$
$$S_{j,N} = \left(N - \sum_{i=r+1}^{j-1} R - i - j + 1\right)(X_{j,N} - X_{j-1,N}),\ j = r + 2, \ldots, m.$$

It is then known that $D_{i,N}/\sigma,\ i = 1, 2, \ldots, r + 1$ are independent standard exponential random variables. By using similar arguments, Viveros and Balakrishnan (1994) established that $S_{j,N}/\sigma,\ j = r + 2, \ldots, m$ are also independent standard exponential random variables.

Now, by writing

$$X_{r+1,N} = \sum_{i=1}^{r+1} \frac{D_{i,N}}{N - i + 1}$$

and

$$X_{j,N} = X_{r+1,N} + \sum_{i=r+2}^{j} \frac{S_{i,N}}{N - \sum_{k=r+1}^{i-1} R_k - i + 1},\ j = r + 2, \ldots, m.$$

The expectation, the variance, and the covariance of the order statistic (Balakrishnan and Sandhu (1996)) are given by

$$E(X_{r+1,N}) = \mu + \sigma \sum_{i=1}^{r+1} \frac{1}{N - i + 1} = \mu + \sigma\alpha_{r+1},$$

$$E(X_{j,N}) = \mu + \sigma\left[\alpha_{r+1} + \sum_{i=r+2}^{j} \frac{1}{N - \sum_{k=r+1}^{i-1} R_k - i + 1}\right],\ j = r + 2, \ldots, m,$$

$$\mathrm{Var}(X_{r+1,N}) = \sigma^2 \sum_{i=1}^{r+1} \frac{1}{(N - i + 1)^2} = \sigma^2\beta_{r+1},$$

$$\mathrm{Var}(X_{j,N}) = \sigma^2\left[\beta_{r+1} + \sum_{i=r+2}^{j} \frac{1}{(N - \sum_{k=r+1}^{i-1} R_k - i + 1)^2}\right],\ j = r + 2, \ldots, m,$$

and

$$\mathrm{Cov}(X_{i,N}, X_{j,N}) = \mathrm{Var}(X_{j,N}),\ r + 1 \le i < j \le m.$$

For the case $r > 0$, Balakrishnan and Sandhu (1996) derived the MLEs of μ and σ from a two-parameter exponential distribution. The results are

$$\widehat{\mu}_{MLE} = X_{r+1,N} + \widehat{\sigma}_{MLE} \ln\Big(1 - \frac{r}{N}\Big) \tag{1}$$

and

$$\widehat{\sigma}_{MLE} = \frac{1}{m-r} \sum_{i=r+2}^{m} (R_i + 1)(X_{i,N} - X_{r+1,N}). \tag{2}$$

Further, the variances of these estimators are given by

$$\mathrm{Var}(\widehat{\mu}_{MLE}) = \Big(\beta_{r+1} + \Big[\ln(1 - \frac{r}{N})/(m-r)\Big]^2 (m-r-1)\Big)\sigma^2 \tag{3}$$

and

$$\mathrm{Var}(\widehat{\sigma}_{MLE}) = \frac{m-r-1}{(m-r)^2}\sigma^2. \tag{4}$$

Then, by making use of the formulae for the BLUE's of μ and σ given by David (1981, pp. 128-130) and Arnold, Balakrishnan and Nagaraja (1992, pp. 171-173), Balakrishnan and Sandhu (1996) derived the BLUE's of μ and σ to be

$$\widehat{\mu}_{BLUE} = X_{r+1,N} - \frac{\alpha_{r+1}}{m-r-1} \sum_{j=r+2}^{m} (R_j + 1)(X_{j,N} - X_{r+1,N}) \tag{5}$$

and

$$\widehat{\sigma}_{BLUE} = \frac{1}{m-r-1} \sum_{j=r+2}^{m} (R_j + 1)(X_{j,N} - X_{r+1,N}). \tag{6}$$

The variances of these estimators are given by

$$\mathrm{Var}(\widehat{\mu}_{BLUE}) = \sigma^2\Big(\frac{\alpha_{r+1}^2}{m-r-1} + \beta_{r+1}\Big) \tag{7}$$

and

$$\mathrm{Var}(\widehat{\sigma}_{BLUE}) = \frac{\sigma^2}{m-r-1}. \tag{8}$$

We propose the minimum risk estimator of the location parameter μ, the MRE can be derived by minimizing the MSE among the class of estimators of the form

$c_1 X_{r+1,N} + c_2 \sum_{j=r+2}^{m} (R_j + 1)(X_{j,N} - X_{r+1,N})$ where c_1 and c_2 are constants. We can obtain the MRE as follows;

$$\widehat{\mu}_{MRE} = X_{r+1,N} - \frac{\alpha_{r+1}}{m-r} \sum_{j=r+2}^{m} (R_j + 1)(X_{j,N} - X_{r+1,N}). \tag{9}$$

And its the MSE of the MRE $\widehat{\mu}_{MRE}$ to be

$$\mathrm{MSE}(\widehat{\mu}_{MRE}) = \sigma^2 \Big(\frac{\alpha_{r+1}^2}{m-r} + \beta_{r+1} \Big). \tag{10}$$

Also we propose the minimum risk estimator of the scale parameter σ, the MRE can be derived by minimizing the MSE among the class of estimators of the form $c \sum_{j=r+2}^{m} (R_j + 1)(X_{j,N} - X_{r+1,N})$ where c is constant.

We can obtain the MRE as follows;

$$\widehat{\sigma}_{MRE} = \frac{1}{m-r} \sum_{j=r+2}^{m} (R_j + 1)(X_{j,N} - X_{r+1,N}). \tag{11}$$

And its the MSE of the MRE $\widehat{\sigma}_{MRE}$ to be

$$\mathrm{MSE}(\widehat{\sigma}_{MRE}) = \frac{\sigma^2}{m-r}. \tag{12}$$

It turns out that the MLE $\widehat{\sigma}_{MLE}$ of the scale parameter σ is the MRE $\widehat{\sigma}_{MRE}$. It is important to note here that the precision of the MRE's of μ and σ, as given by (10) and (12), depends only on r, m and N, and not on the progressive censoring scheme $(R_{r+1}, \ldots, R_m)$. From (7) and (10), the proposed estimator $\widehat{\mu}_{MRE}$ is more efficient than $\widehat{\mu}_{BLUE}$ in terms of the MSE. From (8) and (12), the proposed estimator $\widehat{\sigma}_{MRE}$ is more efficient than $\widehat{\sigma}_{BLUE}$ in terms of the MSE.

3. Numerical illustration

For the purpose of illustration, let us consider Nelson's data (1982, p. 228, Table 6.1) which gives data on times to breakdown of an insulating fluid in an accelerated test conducted at various test voltages.

In analyzing the complete data, Nelson assumed a scaled Weibull distribution for the times to breakdown(from the 90% confidence interval [0.459, 1.381] that he determined for the shape parameter, it is quite clear that an exponential model is also appropriate for this data). For the purpose of illustrating the methods of inference presented sample from the $N = 19$ observations recorded at 34 kV in Nelson's Table 6.1 (with one smallest observation censored and three stages of progressive censoring).

i	1	2	3	4	5	6	7	8
$X_{i,N}$	-	0.78	0.96	1.31	2.78	4.85	6.50	7.35
R_i	-	0	3	0	3	0	0	5

Balakrishnan and Sandhu (1996) obtained from (2) and (6) that $\widehat{\sigma}_{MLE} = 8.3514$ and $\widehat{\sigma}_{BLUE} = 9.7433$, and their relative MSE (RMSE) to be RMSE($\widehat{\sigma}_{BLUE}$) = MSE($\widehat{\sigma}_{BLUE}$)/σ^2 = 0.1667, and RMSE($\widehat{\sigma}_{MLE}$) = 0.1429. Also Balakrishnan and Sandhu (1996) obtained from (1) and (5) that $\widehat{\mu}_{MLE} = 0.3285$ and $\widehat{\mu}_{BLUE} = -0.2741$, and their relative MSE to be RMSE($\widehat{\mu}_{MLE}$) = 0.0169 and RMSE ($\widehat{\mu}_{BLUE}$) = 0.0078. We get from (9) and (10) that $\widehat{\mu}_{MRE} = -0.1235$, RMSE($\widehat{\mu}_{MRE}$) = 0.0075.

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