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A NOTE ON S^1-EQUIVARIANT COHOMOLOGY THEORY

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ABSTRACT. We briefly review the S^1-equivariant cohomology theory of a finite dimensional compact oriented S^1-manifold and extend our discussion in infinite dimensional case.

1. Introduction

Let $\tilde{M}^n$ be a smooth compact oriented n-dimensional S^1-manifold and let $\Omega^*_{S^1}(M)$ be the subalgebra of $(\Omega^*(M), d)$ of all invariant forms on M. In [1] we introduced so called periodic and minus equivariant cohomologies of M using $\Omega^*_{S^1}(M)$ and established the Poincaré duality homomorphism $\tilde{D}$ between equivariant homology and cohomology when M is equipped with an invariant metric. And we have seen that the periodic equivariant cohomology measures the failure of $\tilde{D}$ to be an isomorphism.

In this paper we first review the equivariant homology and cohomology theory and discuss the above result in a more general case. And then we extend our discussion when $\tilde{M}$ is infinite dimensional. However, we do not intend to claim any original result.

$H^*_{S^1}(M;\mathbb{R}) = H^*_{S^1}(ES^1 \times_{S^1} M;\mathbb{R})$ is a module over a polynomial ring $H^*_{S^1}(pt;\mathbb{R}) = H^*(BS^1;\mathbb{R}) = \mathbb{R}[u]$, where $\deg u = 2$ and one defines the S^1-*periodic equivariant cohomology* as

$$PH^*_{S^1}(M) = \varinjlim(\cdots \to H^*_{S^1}(M;\mathbb{R}) \xrightarrow{\cdot u} H^{*+2}_{S^1}(M;\mathbb{R}) \to \cdots).$$

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Let $\beta : \Omega^*_{S^1}(M) \to \Omega^{*-1}_{S^1}(M)$ be the contraction along the vector field generated by the S^1-action. It is well-known that

$$H^*_{S^1}(M;\mathbb{R}) = H^*({}_\beta C^*(M), d+\beta)$$

where

$${}_\beta C^*(M) = \oplus_{l\geq 0}\Omega^{*-2l}_{S^1}(M),$$
$$(d+\beta)(\cdots,\omega_{*-4},\omega_{*-2},\omega_*) = (\cdots, d\omega_{*-4}+\beta\omega_{*-2}, d\omega_{*-2}+\beta\omega_*, d\omega_*).$$

Using this it is easy to see that

$$PH^*_{S^1}(M) = H^*(PC^*_{S^1}(M), d+\beta)$$

where

$$PC^*_{S^1}(M) = \oplus_l \Omega^{*+2l}_{S^1}(M),$$
$$(d+\beta)(\cdots,\omega_{*-2},\omega_*,\omega_{*+2},\cdots) = (\cdots, d\omega_{*-2}+\beta\omega_*, d\omega_*+\beta\omega_{*+2},\cdots).$$

We now define so called *minus equivariant cohomology* as

$${}^-H^*_{S^1}(M;\mathbb{R}) = H^*({}^-{}_\beta C^*(M), d+\beta)$$

where

$$\begin{aligned} {}^-{}_\beta C^*(M) &= \oplus_{l\geq 0}\Omega^{*+2l}_{S^1}(M), \\ (d+\beta)(\omega_*,\omega_{*+2},\cdots) &= (d\omega_*+\beta\omega_{*+2}, d\omega_{*+2}+\beta\omega_{*+4},\cdots) \end{aligned}$$

As in [2] we have the following commutative diagram of short exact sequence

$$0 \to {}_\beta C^{*-2}(M) \to PC^*_{S^1}(M) \xrightarrow{p} {}^-{}_\beta C^*(M) \to 0$$

where p is the projection.

This induces the following diagram of a long exact sequence in cohomology.

$$(1) \quad \cdots \to PH^k_{S^1}(M) \to {}^-H^k_{S^1}(M) \to H^{k-1}_{S^1}(M) \to PH^{k+1}_{S^1}(M) \to \cdots$$

2. Poincaré duality

THEOREM 2.1. ${}^-H^l_{S^1}(M;\mathbb{R}) \cong H^{S^1}_{n-l}(M;\mathbb{R})$.

Proof. Let $\overline{D} : ({}_\beta{}^-C^l(M), d+\beta) \to ({}_\beta C^{n-l}(M)', d'+\beta')$ be a chain map defined by

$$ {}_\beta C^{n-l}(M)' = \mathrm{Hom}({}_\beta C^{n-l}(M), \mathbb{R}), $$

$$ \overline{D}(\omega_l, \omega_{l+2}, \dots)((\cdots, \eta_{n-l-2}, \eta_{n-l})) = \sum_{k\geq 0} \int_M \omega_{l+2k}(\eta_{n-l-2k}) $$

and d', β' are duals of d and β respectively. Consider the cochain homomorphism $\Phi : {}_\beta{}^-C^l(M) \to {}_\beta{}^-C^{l+2}(M)$ defined by

$$ \Phi(\omega_l, \omega_{l+2}, \dots) = (\omega_{l+2}, \dots). $$

Then one has the following commutative diagram of short exact sequences

$$ \begin{array}{ccccccccc} 0 \longrightarrow & (\Omega^l_{S^1}(M), d) & \longrightarrow & ({}_\beta{}^-C^l(M), d+\beta) & \xrightarrow{\Phi} & ({}^-{}_\beta C^{l+2}(M), d+\beta) & \longrightarrow 0 \\ & \downarrow & & \downarrow{\scriptstyle \overline{D}} & & \downarrow{\scriptstyle \overline{D}} & \\ 0 \longrightarrow & (\Omega^{n-l}_{S^1}(M)', d') & \longrightarrow & ({}_\beta C^{n-l}(M)', d'+\beta') & \longrightarrow & ({}_\beta C^{n-l-2}(M)', d'+\beta') & \longrightarrow 0 \end{array} $$

It induces a commutative diagram of long exact sequence in homologies

$$ \begin{array}{ccccccccc} \cdots \longrightarrow & H^l(M;\mathbb{R}) & \longrightarrow & {}^-H^l_{S^1}(M;\mathbb{R}) & \longrightarrow & {}^-H^{l+2}_{S^1}(M;\mathbb{R}) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \overline{D}^*} & & \downarrow{\scriptstyle \overline{D}^*} & \\ \cdots \longrightarrow & H_{n-l}(M;\mathbb{R}) & \longrightarrow & H^{S^1}_{n-l}(M;\mathbb{R}) & \longrightarrow & H^{S^1}_{n-l-2}(M;\mathbb{R}) & \longrightarrow \cdots \end{array} $$

since $H^{n-l}({}_\beta C^*(M)', d'+\beta') \cong H^{S^1}_{n-l}(M;\mathbb{R})$. Hence by induction each $\overline{D}^*$'s are isomorphisms. □

REMARK 2.1. By replacing ${}^-H^l_{S^1}(M;\mathbb{R})$ with $H^{S^1}_{n-l}(M;\mathbb{R})$ in the exact sequence (1) in the introduction we have

$$ \cdots \to PH^l_{S^1}(M) \to H^{S^1}_{n-l}(M) \xrightarrow{\tilde{D}} H^{l-1}_{S^1}(M) \to PH^{l+1}_{S^1}(M) \to \cdots $$

$\tilde{D}$ is called the S^1-equivariant Poincaré duality homomorphism and $PH^*_{S^1}(M)$ measures the failure of $\tilde{D}$ to be an isomorphism.

3. Infinite dimensional case

Let $\tilde{X}$ be a (possibly infinite dimensional) S^1-space. Assume that X^G is an ANR(Absolute Neighborhood Retract) for all subgroups G in S^1. (Note that X itself should be an ANR). We usually call such $\tilde{X}$ "good" S^1-spaces. We define a category $x(\tilde{X})$ for such $\tilde{X}$ as follows.

Objects of $x(\tilde{X})$ are consisted of all pairs $(\tilde{A}_\alpha, f_\alpha)$ where $\tilde{A}_\alpha$ is a finite dimensional ANR with S^1-action and f_α is an equivariant map from $\tilde{A}_\alpha$ to $\tilde{X}$. We use the notation $\tilde{A}_\alpha \xrightarrow{f_\alpha} \tilde{X}$ for $(\tilde{A}_\alpha, f_\alpha)$ alternatively. A morphism f^β_α from $(\tilde{A}_\alpha, f_\alpha)$ to $(\tilde{A}_\beta, f_\beta)$ is an injective map $\theta^\beta_\alpha : \tilde{A}_\alpha \to \tilde{A}_\beta$ with $\theta^\beta_\alpha(A_\alpha)$ being closed in A_β such that $f_\beta \circ \theta^\beta_\alpha = f_\alpha$. We also use θ^β_α for f^β_α time to time.

Let Ab denote the category of abelian groups. We define a functor $h : x(\tilde{X}) \to Ab$ as

$$h(\tilde{A}_\alpha \xrightarrow{f_\alpha} \tilde{X}) = PH(\tilde{A}_\alpha)$$

and

$$h(f^\beta_\alpha) = PH(\theta^\beta_\alpha) : PH(\tilde{A}_\alpha) \to PH(\tilde{A}_\beta)$$

where f^β_α and θ^β_α are as above.

Definition 3.1.

$$\begin{aligned} \mathbb{P}H(\tilde{X}) &\equiv \varinjlim{}_{x(\tilde{X})} h \\ &= \oplus h(\tilde{A}_\alpha \xrightarrow{f_\alpha} \tilde{X}) / \sim \end{aligned}$$

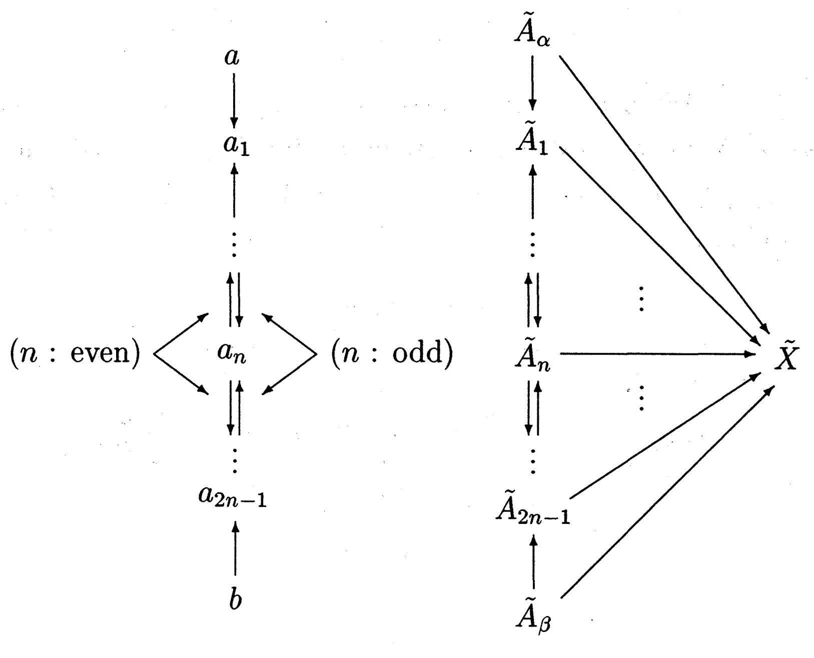
where $\sim$ is an equivalence relation generated by a relation R;

$$\text{for each} \quad a \in h(\tilde{A}_\alpha \xrightarrow{f_\alpha} \tilde{X}) \text{ and } b \in h(\tilde{A}_\beta \xrightarrow{f_\beta} \tilde{X})$$

$$aRb \Longleftrightarrow \exists\, \theta^\beta_\alpha : \tilde{A}_\alpha \to \tilde{A}_\beta \text{ such that } b = \theta^\beta_\alpha(a).$$

LEMMA 3.1. *Let $a \in h(\tilde{A}_\alpha \xrightarrow{f_\alpha} \tilde{X})$ and let $b \in h(\tilde{A}_\beta \to \tilde{X})$. Let $\overline{a}$ and $\overline{b}$ be equivalence classes of a and b respectively in $\mathbb{P}H(\tilde{X})$. Then*

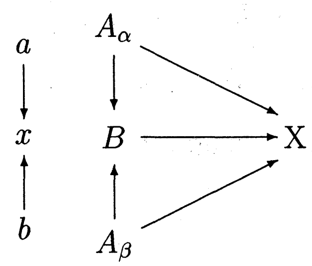
$\overline{a} = \overline{b} \iff \exists$ *a finite number of objects* $(\tilde{A}_1, f_1), \cdots, (\tilde{A}_{2n-1}, f_{2n-1})$ *and elements* $a_1, \ldots, a_{2n-1}$, *in* $h(\tilde{A}_i \xrightarrow{f_i} \tilde{X})$ *satisfying the following diagram.*



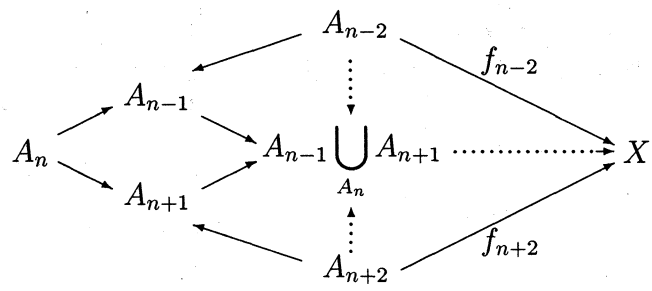
Proof. ($\Longleftarrow$) clear.

($\Longrightarrow$) We can find k elements $a_1, \ldots, a_k$ which "connects" a and b, i.e., aRa_{i_1} (We may assume this rather than $a_{i_1}Ra$ since aRa), $\cdots a_{i_j}Ra_{i_{j+1}}$(or $a_{i_{j+1}}Ra_{i_j}$), $\cdots$, bRa_{i_k}. We discard a_i's if we have xRa_i and a_iRy for some x and y, and we replace them by a single relation xRy. Continuing this we can reach to a diagram as in the lemma. And it is easy to see that k should be odd. □

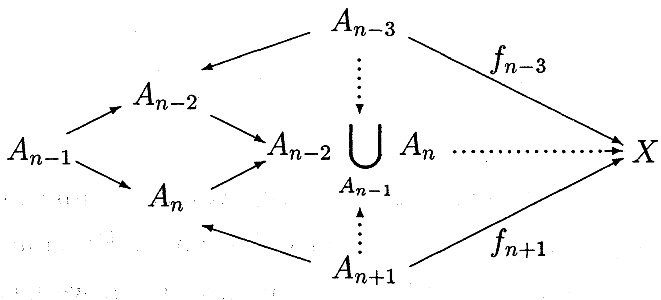
LEMMA 3.2. *Under the same hypothesis as in the Lemma 3.1, $\overline{a} = \overline{b}$ iff there exists a finite dimensional $\tilde{B}$ and $x \in B$ such that the following diagram holds.*



Proof. We use the induction on n. If $n = 1$, there is nothing to prove. We can construct a new finite dimensional ANR and maps out of $A_{n-1} \leftarrow A_n \rightarrow A_{n+1}$ if n is even (or $A_{n-2} \leftarrow A_{n-1} \rightarrow A_n$ if n is odd) as follows



(n : even)



(n : odd)

Hence we decreased the number of A_i's by 2. Applying the induction hypothesis we complete the proof. □

Now we claim that $\mathbb{P}H$ is an *extension* of PH.

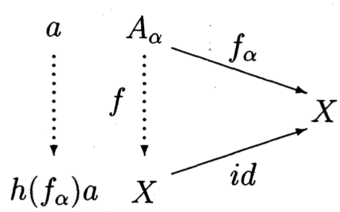
THEOREM 3.3. *If $\tilde{X}$ is a finite dimensional ANR, then* $\mathbb{P}H(\tilde{X}) = PH(\tilde{X})$.

Proof. We have a canonical map

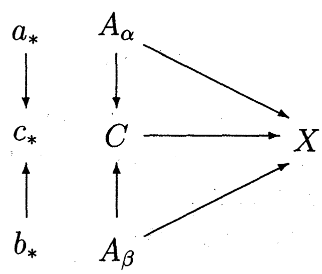
$$\begin{aligned} PH(\tilde{X}) \xrightarrow{\text{can}} \mathbb{P}H(\tilde{X}) &= \oplus h(\tilde{A}_\alpha \xrightarrow{f_\alpha} \tilde{X})/\sim \\ &= \oplus PH(A_\alpha)/\sim \end{aligned}$$

We claim that this is an isomorphism.

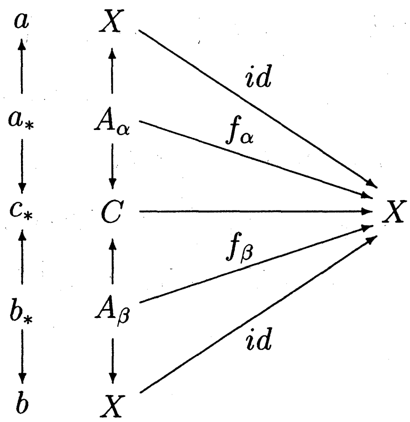
Surjectivity: Let $a \in h(\tilde{A}_\alpha \xrightarrow{f_\alpha} \tilde{X})$. Then we get an element $h(f_\alpha)a$ in $h(X \xrightarrow{id} X) = PH(\tilde{X})$. Clearly $\overline{h(f_\alpha)a} = \overline{a}$ and $\text{can}(\overline{a}) = \overline{a}$.



Injectivity: Let $a, b \in h(\tilde{X} \xrightarrow{id} \tilde{X}) = PH(\tilde{X})$ and suppose $\overline{a}_* = \overline{b}_*$ where $a_* = \text{can}(a)$, $b_* = \text{can}(b)$. Then by lemma 3.2 there exists $\tilde{C}$ and c_* such that $c_* \in h(\tilde{C} \to \tilde{X})$ described as in the diagram.



Hence we have the following diagram and correspondence between elements. And applying PH in each finite dimensional ANR we see that $a = b$.



□

REMARK 3.1. i) It can be seen that $\mathbb{P}H$ is a homotopy functor from the category of "good" S^1-spaces to Ab and satisfies the Mayer-Vietoris sequence.

ii) The discussions in the previous sections can be generalized in infinite dimensional case by passing to the inductive limit.

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