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THE PAN–GENERALIZED FUZZY INTEGRAL OF A COMMUTATIVE ISOTONIC SEMIGROUP–VALUED FUNCTION

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ABSTRACT. In this paper, we introduce the pan-generalized fuzzy integral of a commutative isotonic semigroup-valued function, which is generalization of the (G) fuzzy integral and investigate the fundamental properties of this kind of fuzzy integral.

1. Introduction

In 1980, D.A. Ralescu and G. Adams generalized the concept of fuzzy integral due to M. Sugeno[3]. For convenience, we will call it (S) fuzzy integral. Following that, D.A. Ralescu and G. Adams [1] and D.A. Ralescu [2] have investigated the basic properties of (S) fuzzy integral. Wang Zhenyuen obtained a series of (S) fuzzy integral convergent theorems in [4]. Meanwhile, Zhao Ruhuai introduced a new definition of fuzzy integral, viz. (N) fuzzy integral in [7]. Wu Congxin, Wang Shuli, and Ma Ming [5] introduced the (G) fuzzy integral using a generalized triangular norm which is a generalization of both (S) fuzzy integral and (N) fuzzy integral. In this paper, we introduce the pan-generalized fuzzy integral of a commutative isotonic semigroup-valued function, which is generalization of the (G) fuzzy integral and investigate the fundamental properties of this kind of fuzzy integral.

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2. Preliminaries

DEFINITION 2.1. Let X be a nonempty set, $\mathcal{A}$ be a σ-algebra of a class of the subsets of X, the mapping $\mu : \mathcal{A} \to [0, \infty]$ is called a fuzzy measure provided

(1) $\mu(\varnothing) = 0$;
(2) if $A \subset B$, then $\mu(A) \leq \mu(B)$;
(3) if $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots, A_n \in \mathcal{A}$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$;
(4) if $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots, A_n \in \mathcal{A}$, and there exists a natural number n_0 such that $\mu(A_{n_0}) < \infty$, then $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

If μ is a fuzzy measure, $(X, \mathcal{A}, \mu)$ is called a fuzzy measure space.

DEFINITION 2.2. Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, $f : X \to [0, \infty]$ is said to be $\mathcal{A}$-measurable function if $N_\alpha(f) \in \mathcal{A}$ for all $\alpha \in (-\infty, \infty)$, where $N_\alpha(f) = \{x : f(x) > \alpha\}$.

DEFINITION 2.3. A fuzzy measure is said to be null-additive if $\mu(A \cup B) = \mu(A)$ whenever $A, B \in \mathcal{A}$ with $\mu(B) = 0$.

In this paper, let $R_+ = [0, \infty)$, $\bar{R}_+ = [0, \infty]$ and a, b, c, d, a_i, $b_i \in \bar{R}_+$.

DEFINITION 2.4([8]). Let $\oplus$ be a binary operation on $\bar{R}_+$. The pair $(\bar{R}_+, \oplus)$ is called a commutative isotonic semigroup and $\oplus$ is called a pan-addition on $\bar{R}_+$ iff $\oplus$ satisfies the following requirements:

(PA1) $a \oplus b = b \oplus a$;
(PA2) $(a \oplus b) \oplus c) = a \oplus (b \oplus c)$;
(PA3) $a \leq b$, then $a \oplus c \leq b \oplus c$ for any c;
(PA4) $a \oplus 0 = a$;
(PA5) if $\lim_n a_n$ and $\lim_n b_n$ exist, then $\lim_n(a_n \oplus b_n)$exists, and $\lim_n(a_n \oplus b_n) = \lim_n a_n \oplus \lim_n b_n$

DEFINITION 2.5([8]). Let $\odot$ be a binary operation on $\bar{R}_+$. The triple $(\bar{R}_+, \oplus, \odot)$, where $\oplus$ is a pan-addition on $\bar{R}_+$, is called a commutative isotonic semiring with respect to $\oplus$ and $\odot$, iff:

(PM1) $a \odot b = b \odot a$;
(PM2) $(a \odot b) \odot c = a \odot (b \odot c)$;
(PM3) $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$;
(PM4) if $a \leq b$, then $(a \odot c) \leq (b \odot c)$ for any c;
(PM5) $a \neq 0$ and $b \neq 0 \Leftrightarrow a \odot b \neq 0$;
(PM6) there exists $e \in \bar{R}_+$ such that $e \odot a = a$ for any $a \in \bar{R}_+$;
(PM7) if $\lim_n a_n$ and $\lim_n b_n$ exist and are finite, then $\lim_n (a_n \odot b_n) = \lim_n a_n \odot \lim_n b_n$.

The operation $\odot$ is called a pan-multiplication on $\bar{R}_+$, and the number e is called the unit element of $(\bar{R}_+, \oplus, \odot)$.

NOTE 2.1. $\bar{R}_+$ with the common addition and the common multiplication of real numbers is a commutative isotonic semiring.

NOTE 2.2. $\bar{R}_+$ with the logical addition and logical multiplication of real numbers is commutative isotonic semiring.

If $(X, \mathcal{A}, \mu)$ is a fuzzy measure space and $(\bar{R}_+, \oplus, \odot)$ is a commutative semiring, $(X, \mathcal{A}, \mu, \bar{R}_+, \oplus, \odot)$ is called a pan-space and if $E \subset X$,

$$\chi_E(x) = \begin{cases} e & \text{if } x \in E \\ 0 & \text{otherwise.} \end{cases}$$

is called the pan-characteristic function of E, where e is the unit element of $(\bar{R}_+, \oplus, \odot)$.

DEFINITION 2.6. Let $(X, a, \mu, \bar{R}_+, \oplus, \odot)$ be a pan-space. A function on X given by $s(x) = \oplus_{i=1}^{n} [a_i \odot \chi_{E_i}(x)$ is called a pan-simple measurable function, where $a_i \in \bar{R}_+$, $i = 1, 2, \cdots, n$ and $\{E_i | i = 1, 2, \cdots, n\}$ is a measurable partition of X.

3. Definition and fundamental properties of (PG) fuzzy integral

DEFINITION 3.1. Denote $D = [0,\infty] \times [0,\infty] \setminus \{(0,\infty),(\infty,0)\}$. The mapping $S : D \to [0,\infty]$ is called a c-generalized triangular norm provided that

(1) $S[0,x] = 0$ for all $x \in [0,\infty)$ and there exists an $e \in (0,\infty]$ such that $S[x,e] = x$ for each $x \in [0,\infty]$, e is called the unit element of S;
(2) $S[x,y] = S[y,x]$ for all $(x,y) \in D$;
(3) $S[a,b] \leq S[c,d]$ whenever $a \leq c$, $b \leq d$;
(4) if $\{(x_n,y_n)\} \subset D$, $(x,y) \in D$, $x_n \to x$, $y_n \to y$, then

$$S[x_n,y_n] \to S[x,y].$$

NOTE 3.1. A generalized triangular norm ([8]) implies a c-generalized triangular norm.

NOTE 3.2. Take $S_1[x,y] = \min(x,y)$, $S_2[x,y] = k(xy)^p$, $(k,p > 0)$ and

$$S_3[x,y] = \begin{cases} 0 & \text{if } \min(x,y) = 0, \\ xy + k(xy)^p & \text{if } \min(x,y) \neq 0 \ (k,p > 0) \end{cases}$$

then S_1, S_2 and S_3 are c-generalized triangular norm.

DEFINITION 3.2. Let S be a c-generalized triangular norm and f be a nonnegative measurable function, $A \in \mathcal{A}$. Then the (PG) fuzzy integral of f on A is defined by

$$(PG)\int_A f\,d\mu = \inf_{0<s\leq f} Q_A(s)$$

where $s = \oplus_{i=1}^n \alpha_i \odot \chi_{A_i}, \alpha_i \neq \alpha_j\,(i \neq j), \alpha_i > 0$, $A_i \in \mathcal{A}$ $(i = 1,2,\cdots,n)$ and $Q_A(s) = \sup_{1\leq i\leq n} S[\alpha_i, \mu(A \cap A_i)]$. Define $\sup\{i : i \in \varnothing\} = 0$.

In what follows, $(PG) \int_X f\,d\mu$ will be denoted by $(PG) \int f\,d\mu$. For the notions and results on fuzzy measure and fuzzy measurable functions refer to [8].

THEOREM 3.1. *For (PG) fuzzy integral we have the following equivalent forms:*

$$\begin{aligned}(PG)\int_A f\,d\mu &= \sup_{\alpha>0} S[\alpha, \mu(A\cap N_\alpha(f))] \\ &= \sup_{\alpha>0} S[\alpha, \mu(A\cap N_\alpha^*(f))] \\ &= \sup_{E\in\mathcal{A},\, \inf_{x\in E} f(x)>0} S[\inf_{x\in E} f(x), \mu(A\cap E)]\end{aligned}$$

where $N_\alpha^*(f) = \{x : f(x) \geq \alpha\}$

Proof. The above four expressions are denoted by (1),(2,),(3), and (4) in proper order. Then we infer (1)$\leq$ (4): For any $E \in \mathcal{A}$, $\inf_{x\in E} f(x) > 0$, it is clear that $\inf_{x\in E} f(x) \odot \chi_E \leq f$. By Definition 3.2, we know

$$(PG)\int_A f\,d\mu = \sup_{0<s\leq f} Q_A(s) \geq S[\inf_{x\in E} f(x), \mu(A\cap E)]$$

Since $E \in \mathcal{A}$ is arbitrary, hence

$$(PG)\int_A f\,d\mu \geq \sup_{E\in\mathcal{A},\, \inf_{x\in E} f(x)>0} S[\inf_{x\in E} f(x), \mu(A\cap E)]$$

(4)$\geq$(3): By $N_\alpha^*(f) \in \mathcal{A}$ for any $\alpha > 0$ and $\inf_{x\in N_\alpha^* f(x)} f \geq \alpha$, therefore we have

$$\sup_{E\in\mathcal{A},\, \inf_{x\in E} f(x)>0} S[\inf_{x\in E} f(x), \mu(A\cap E)] \geq S[\alpha, \mu(A\cap N_\alpha^*(f)]$$

Since α is arbitrary, hence we have

$$\sup_{E\in\mathcal{A},\, \inf_{x\in E} f(x)>0} S[\inf_{x\in E} f(x), \mu(A\cap E)] \geq \sup_{\alpha>0} S[\alpha, \mu(A\cap N_\alpha^*(f)]$$

(3)$\leq$(2):$N_\alpha(f) \subset N^*_\alpha(f)$ for any $\alpha > 0$ and the monotonicity of fuzzy measure μ and c-generalized triangular norm, we infer that $S[\alpha, \mu(A \cap N^*_\alpha(f)] \geq S[\alpha, \mu(A \cap N_\alpha(f)]$, thus we have

$$\sup_{\alpha>0} S[\alpha, \mu(A \cap N^*_\alpha(f)] \geq \sup_{\alpha>0} S[\alpha, \mu(A \cap N_\alpha(f)]$$

(2)$\geq$(1): If $(PG) \int_A f \, d\mu < \infty$, then for arbitrary $\epsilon > 0$, there exist $0 < s \leq f$, $s = \oplus_{i=1}^n \alpha_i \odot \chi_{A_i}$ and i_0 such that

$$\begin{aligned}(PG) \int_A f \, d\mu < Q_A(s) + \frac{\varepsilon}{2} &= \sup_{1 \leq i \leq n} S[\alpha_i, \mu(A \cap A_i)] + \frac{\varepsilon}{2} \\ &= S[\alpha_{i_0}, \mu(A \cap A_{i_0})] + \frac{\varepsilon}{2}\end{aligned}$$

On the other hand, by the property of c-generalized triangular norm we have

$$\lim_{n \to \infty} S[\alpha_{i_0} - 1/n, \mu(A \cap A_{i_0})] = S[\alpha_{i_0}, \mu(A \cap A_{i_0})]$$

Therefore, there exists an n_0 with $\alpha_{i_0} - 1/n_0 > 0$ such that

$$S[\alpha_{i_0} - 1/n, \mu(A \cap A_{i_0})] > S[\alpha_{i_0}, \mu(A \cap A_{i_0})] - \frac{\varepsilon}{2}$$

Therefore we have

$$\begin{aligned}(PG) \int_A f \, d\mu &< S[\alpha_{i_0}, \mu(A \cap A_{i_0})] + \frac{\varepsilon}{2} \\ &\leq S[\alpha_{i_0} - 1/n_0, \mu(A \cap A_{i_0})] + \varepsilon \\ &\leq S[\alpha_{i_0} - 1/n_0, \mu(A \cap N_{\alpha_{i_0} - 1/n_0}(f))] + \varepsilon \\ &\leq \sup_{\alpha>0} S[\alpha, \mu(A \cap N_\alpha(f))] + \varepsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this implies that $\sup_{\alpha>0} S[\alpha, \mu(A \cap N_\alpha(f))] \geq (PG) \int_A f \, d\mu$

If $(PG)\int_A f\,d\mu = \infty$, then for each $M > 0$, there exists $0 < s^{(M)} \leq f$, $s^{(M)} = \oplus_{i=1}^{n} \alpha_i \odot \chi_{A_i}$ such that $Q_A(s^{(M)}) > M$. Similar to the case of $(PG)\int_A f\,d\mu < \infty$, we infer that there exists an n_0 such that $S[\alpha_{i_0} - 1/n_0, \mu(A \cap A_{i_0})] > M$. It follows that $\sup_{\alpha>0} S[\alpha, \mu(A \cap N_\alpha(f))] \geq S[\alpha, \mu(A \cap A_{i_0})] > M$. Since $M > 0$ is arbitrary, we have $\sup_{\alpha>0} S[\alpha, \mu(A \cap N_\alpha(f))] = \infty$. From the preceding proof, we conclude the proof of this theorem. □

NOTE 3.3. By the preceding theorem, it is easy to show that (S) fuzzy integral, (N) fuzzy integral , and (G) fuzzy integral are special kinds of (PG) fuzzy integral.

THEOREM 3.2. *For (PG) fuzzy integral, we have*

(1) *if* $f_1 \leq f_2$, *then* $(PG)\int_A f_1\,d\mu \leq (PG)\int_A f_2\,d\mu$;
(2) *if* $A_1 \subset A_2$, *then* $(PG)\int_{A_1} f\,d\mu \leq (PG)\int_{A_2} f\,d\mu$;
(3) *if* $\mu(A) = 0$, *then* $(PG)\int_A f\,d\mu = 0$;
(4) $(PG)\int_A f\,d\mu = (PG)\int f \odot \chi_A\,d\mu$;
(5) $(PG)\int_A c\,d\mu = S[c, \mu(A)]$ *for any* $A \in \mathcal{A}$ *and constant* $c \in (0, \infty)$;
(6) $(PG)\int_A f_1 \vee f_2\,d\mu \geq (PG)\int_A f_1\,d\mu \vee (PG)\int_A f_2\,d\mu$;
(7) $(PG)\int_A f_1 \wedge f_2\,d\mu \leq (PG)\int_A f_1\,d\mu \wedge (PG)\int_A f_2\,d\mu$;
(8) $(PG)\int_A (c \vee f\,d\mu = (PG)\int_A c\,d\mu \wedge (PG)\int_A f\,d\mu$ *for any constant* $c \in (0, \infty)$.

Proof. The proof of (1),(2),(3),(6),and (7) is deduced directly from Theorem 3.1. (4) By Theorem 3.1, we have

$$\begin{aligned}(PG)\int_A f\,d\mu &= \sup_{\alpha>0} S[\alpha, \mu(A \cap N_\alpha(f)] \\ &= \sup_{\alpha>0} S[\alpha, \mu(N_\alpha(f \odot \chi_A))] \\ &= (PG)\int f \odot \chi_A\,d\mu\end{aligned}$$

(5) Let $f = c$. For any $\alpha > 0$, we have

$$N_\alpha(f) = \begin{cases} X, & \alpha \leq c \\ \varnothing, & \alpha > c \end{cases}$$

Hence Theorem 3.1 gives that

$$(PG)\int_A f\,d\mu = \sup_{\alpha>0} S[\alpha, \mu(A \cap N_\alpha(f))] = S[c, \mu(A)]$$

(8) For any $\alpha > 0$, we have

$$N_\alpha(c \vee f) = \begin{cases} X, & \alpha \leq c \\ N_\alpha(f), & \alpha > c \end{cases}$$

Hence Theorem 3.1 gives that

$$\begin{aligned}(PG)\int_A (c \vee f)\,d\mu &= \sup_{0<\alpha\leq c} S[\alpha, \mu(A \cap N_\alpha(c \vee f)))] \\ &\quad \vee \sup_{\alpha>c} S[\alpha, \mu(A \cap N_\alpha(c \vee f))]] \\ &= S[c, \mu(A)] \vee \sup_{\alpha>c} S[\alpha, \mu(A \cap N_\alpha(f))]\end{aligned}$$

In addition, $\sup_{\alpha\leq c} S[\alpha, \mu(A \cap N_\alpha(f))] = \sup_{\alpha\leq c} S[\alpha, \mu(A)]$. Therefore we have

$$(PG)\int_A (c \vee f)\,d\mu = (PG)\int_A c\,d\mu \vee (PG)\int_A f\,d\mu$$

□

THEOREM 3.3. *Let f, g be nonnegative measurable functions. Then $(PG)\int f\,d\mu = (PG)\int g\,d\mu$ whenever $f = g$ a.e., if and only if μ is null-additive.*

Proof. Sufficiency: Suppose that μ is null-additive and $f = g$ *a.e.* Put $B = \{x; f(x) \neq g(x)\}$. Then $\mu(B) = 0$ and $\mu(N_\alpha(g)) = \mu(N_\alpha(g) \cup B)$. So we have $\mu(N_\alpha(f)) \leq \mu(N_\alpha(g) \cup B) = \mu(N_\alpha(g))$ for any $\alpha > 0$.

The converse inequality holds as well, we have $\mu(N_\alpha(f)) = \mu(N_\alpha(g))$. By Theorem 3.1, we have that $(PG)\int f\,d\mu = (PG)\int g\,d\mu$.
Necessity: For any $A \in \mathcal{A}$, $B \in \mathcal{A}$ with $\mu(B) = 0$, if $\mu(A) = \infty$, then by the monotonicity of μ, we have $\mu(A \cup B) = \infty = \mu(A)$. Now, we assume that $\mu(A) < \infty$, define

$$f(x) = \begin{cases} e, & x \in A \cup B \\ 0, & x \notin A \cup B \end{cases} \text{ and } g(x) = \begin{cases} e, & x \in A \\ 0, & x \notin A \end{cases}$$

where e is the unit of S, then $f = g$ *a.e.* So, by hypothesis, $(PG)\int f\,d\mu = (PG)\int g\,d\mu$. Therefore we have $S[e, \mu(A \cup B)] = S[e, \mu(A)]$. It follows that $\mu(A \cap B) = \mu(A)$. Hence μ is null-additive. □

COROLLARY 3.1. *If μ is null-additive, then $(PG)\int_A f\,d\mu = (PG)\int_A g\,d\mu$ whenever $f = g$ a.e. on A.*

Proof. If $f = 0$ *a.e.* on A, then $f \odot \chi_A = g \odot \chi_B$ *a.e.* ¿From Theorem 3.3 and Theorem 3.2(4), we get the conclusion. □

COROLLARY 3.2. *If μ is null-additive, then $(PG)\int_{A\cup B} f\,d\mu = (PG)\int_A f\,d\mu$ whenever $A \in \mathcal{A}$, $B \in \mathcal{A}$ with $\mu(B) = 0$.*

Proof. Since $f \odot \chi_{A\cup B} = f \odot \chi_A$ *a.e.*, by Theorem 3.3 and Theorem 3.2(4), we get the conclusion. □

THEOREM 3.4. *Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space and f be nonnegative measurable function. Then for $A \in \mathcal{A}$,*

$$(PG)\int_A f\,d\mu = (PG)\int_0^\infty g_A(\alpha)\,dm = (PG)\int_0^\infty g_A^*(\alpha)\,dm$$

where m is the Lebesgue measure, $g_A(\alpha) = \mu(A \cap N_\alpha(f))$, $g_A^*(\alpha) = \mu(A \cap N_\alpha^*(f))$,

Proof. Since $g_A^*(\alpha) \geq g_A(\alpha)$ for every $\alpha \in (0, \infty)$, it follows that

$$(PG)\int_0^\infty g_A^*(\alpha)\,dm \geq (PG)\int_0^\infty g_A^*(\alpha)\,dm$$

In addition, $g_A(x) \geq g_A(\alpha)$ for every $x \in [0, \alpha]$. Therefore we have $[0, \alpha] \subset \{x : g_A(x) \geq g_A(\alpha)\}$. It follows that

$$(PG) \int_A f\, d\mu = \sup_{\alpha>0} S[\alpha, \mu(A \cap N_\alpha(f))] = \sup_{\alpha>0} S[\alpha, g_A(\alpha)]$$
$$\leq \sup_{\alpha>0} S[m(N_{g_A(\alpha)}(g_A)), g_A(\alpha)] \leq (PG) \int_0^\infty g_A(\alpha)\, dm$$

Hence we have $(PG) \int_A f\, d\mu \geq (PG) \int_0^\infty g_A^*(f)\, dm$.
In what follows, we will show that

$$(PG) \int_A f\, dm \geq (PG) \int_0^\infty g_A^*(f)\, dm.$$

If $(PG) \int_A f\, d\mu = \infty$ or $g_A^* = 0$ *a.e.*, then the inequality is trivial. Suppose that there exists an $A_0 \in \mathcal{A}$, $A_0 \subset A$ with $g_A^*(x) > 0$ for all $x \in A_0$. For any pan-simple $s = \oplus_{i=1}^n \alpha_i \odot \chi_{A_i}$ with $0 < s \leq g_A^*$, there exists an i_0 such that $S[\alpha_{i_0}, m(A_{i_0})] = \sup_{1 \leq i \leq n} S[\alpha_i, m(A_i)]$. In what follows, we conclude the proof in two cases.

(a) If $m(A_{i_0}) < \infty$, denote $\beta = \sup\{x : x \in A_{i_0}\}$, then $m(A_{i_0}) \leq \beta$. For $\beta < \infty$, there exists $x_k \in A_{i_0}$ such that $\lim x_k = \beta$. For any $\varepsilon > 0$, there exists a k_0 such that

$$S[\alpha_{i_0}, \beta] \leq S[\alpha_{i_0}, x_k] + \epsilon \leq S[g_A^*(x_k), x_k] + \epsilon \leq (PG) \int_A f\, d\mu$$

for all $k \geq k_0$. Hence we have that

$$(PG) \int_0^\infty g_A(f)\, dm \leq (PG) \int_A f\, d\mu.$$

When $\beta < \infty$, there exists a $x_0 \in A_{i_0}$ such that $m(A_{i_0}) \leq x_0$. Since

$$(PG) \int_A f\, d\mu = \sup_{\alpha>0} S[\alpha, \mu(A \cap N_\alpha^*(f))] = \sup_{\alpha>0} S[\alpha, m(g_A^*(\alpha))]$$

Hence we have $(PG) \int_0^\infty g_A^*(\alpha)\, dm \leq (PG) \int_A f\, d\mu$.

(b) If $m(A_{i_0}) = \infty$, then there exist $x_n \in A_{i_0}$ $(n = 1, 2, \cdots)$ such that $\lim x_n = \infty$. If $g_A^*(x_n) = \infty$ for all $n \in N$. Otherwise, there exists an n_0 such that $g_A^*(x_{n_0}) < \infty$. In addition, $g_A^*(x_n) \geq \alpha_{i_0}$ implies $g_A^*(\infty) \geq \alpha_{i_0}$. Hence

$$S[\alpha_{i_0}, m(A_{i_0})] \leq S[g_A^*(\infty), \infty] = \lim S[g_A^*(x_n), x_n] \leq (PG) \int_A f \, d\mu$$

Hence we have $(PG) \int_0^\infty g_A^* \, dm \leq (PG) \int_A f \, d\mu$. □

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