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BOUNDARY BEHAVIOR OF GREEN'S POTENTIALS WITHIN TANGENTIAL APPROACH REGION

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ABSTRACT. In this paper, we will study properties of the Green's potential for the Green's function of B which is defined in [8]. In particular, we will investigate boundary behavior of some functions related with Green's function within tangential approach regions that were used in [4].

1. Introduction

Let B denote the open unit ball in $\mathbb{C}^n, n \geq 1$. For $z, w \in \mathbb{C}^n$, let $< z, w > = \sum_{i=1}^n z_i \bar{w}_i$, $|z|^2 =< z, z >$. For $0 < r \leq 1$, let $B_r = \{z \in B : |z| < r\}$ and σ the rotation-invariant positive Borel measure on S with $\sigma(S) = 1$ where $S = \partial B$ is the unit sphere in $\mathbb{C}^n$. For $z = |z|\eta \in B$, the Möbius transformation φ_z on B is defined by

$$\varphi_z(w) = \frac{z- < w, \eta > \eta - \sqrt{1 - |z|^2}(w- < w, \eta > \eta)}{1- < w, z >}, \quad w \in B.$$

Let $\mathcal{M}$ be the group of all biholomorphic maps of B onto B. Then $\varphi_z \in \mathcal{M}$ for $z \in B$. Further, any $\psi \in \mathcal{M}$ has a unique representation $\psi = U \circ \varphi_z$ for some $z \in B$ and $U \in \mathcal{U}$ where $\mathcal{U}$ is the unitary group (See [6] Theorem 2.2.5).

If ν denotes the normalized Lebesgue measure on B, the measure λ is defined on B by $d\lambda(w) = (1 - |w|^2)^{-(n+1)} d\nu(w)$.

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The Bergman metric on the ball B is given by

$$ds_B^2(z) = \sum_{i,j=1}^{n} b_{ij} dz_i d\bar{z}_j$$

where

$$b_{ij} = \frac{\partial^2 z}{\partial z_i \partial \bar{z}_j} \left(\log \frac{1}{(1-|z|^2)^{n+1}} \right) = \frac{n+1}{(1-|z|^2)^2} ((1-|z|^2)\delta_{ij} + \bar{z}_i z_j).$$

The Laplace-Beltrami operator of the metric is given by

$$\tilde{\triangle} = 4 \sum_{i,j=1}^{n} b^{ij} \frac{\partial^2}{\partial \bar{z}_i \partial z_j}$$

where $(b^{ij}) = (b_{ij})^{-1}$.

The Green's function G for the Laplace-Beltrami operator $\tilde{\triangle}$ is given by

$$G(z, w) = (g \circ \varphi_z)(w)$$

where

$$g(z) = \frac{n+1}{2n} \int_{|z|}^{1} t^{-2n+1}(1-t^2)^{n-1} dt. \tag{1}$$

The formula (1) had been derived in 1967 by K.T. Hahn and J. Mitchell[2]. The Green's function on B was used extensively by Ullrich in [8]. For a nonnegative regular Borel measure μ on B, the Green's potential $\mathcal{G}\mu$ of μ is defined by

$$\mathcal{G}\mu(z) = \int_B G(z, w) d\mu(w), z \in B.$$

A function $f : B \to [-\infty, \infty)$ is said to be $\mathcal{M}$-subharmonic on B if it is upper semicontinuous and $f(\varphi(0)) \le \int_S f(\varphi(r\zeta)) d\sigma(\zeta)$ for all $\varphi \in \mathcal{M}$ and $0 \le r < 1$. A function f is $\mathcal{M}$-superharmonic if $-f$

is $\mathcal{M}$-subharmonic. It is well known that the function $g(z)$ in (1) is $\mathcal{M}$-superharmonic on B (See [8], Lemma 2.7).

In section 2, we will study the $\mathcal{M}$-superharmonicity of Green's potential.

In section 3, we will investigate some properties of the tangential approach regions that were used in [4]. Also, we apply this result to show that some functions related with Green's function has zero boundary limits a.e. on S within tangential approach regions.

2. Some properties of Green's potential

$f(z) \approx h(z)$ means that there exist positive constants C_1, C_2 such that $C_1 h(z) \leq f(z) \leq C_2 h(z)$ for all indicated z.

LEMMA 1. *Let $0 < \delta < \frac{1}{2}$ be fixed. Then $g(z) \leq C_\delta (1 - |z|^2)^n$ for all $z \in B, |z| \geq \delta$ where C_δ is positive constant depending only on δ. Furthermore, for all z, $|z| \leq \delta$,*

$$g(z) \approx \begin{cases} |z|^{-2n+2} & n > 1, \\ \log \dfrac{1}{|z|} & n = 1. \end{cases}$$

Proof. The proof is a routine estimation of the integral (1), and thus is omitted. □

LEMMA 2. *Let $a \in B$. Then*

(i) $\varphi_a(0) = a$, $\varphi_a(a) = 0$ *and* $\varphi_a(\varphi_a(z)) = z$.

(ii) *For all $z, w \in \overline{B}$, we have*

$$1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1 - |a|^2)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)}.$$

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a \rangle|^2}. \tag{2}$$

Proof. See [6. Theorem 2.2.2]. □

For $z \in B$, let $E(z) = \{w \in B : |\varphi_z(w)| < \frac{1}{2}\} = \varphi_z(B_{\frac{1}{2}})$. Fix an r, $0 < r < 1$, and let $A_r = \{z : r < |z| < 1\}$.

LEMMA 3. *For $0 < \varrho < 1$, there exists r_0 such that $E(z) \subset A_\varrho$ for all z, $|z| \geq r_0$.*

Proof. We first note that $w \in E(z)$ if and only if $w = \varphi_z(\xi)$ with $|\xi| < \frac{1}{2}$. Since $w \in E(z)$, $(1 - |w|^2) \leq 3(1 - |z|^2)$ by Lemma 2.

If $3(1 - |z|^2) \leq 1 - \varrho^2$, then $|w| \geq \varrho$. If we let $r_0 = \frac{2}{3} + \frac{1}{3}\varrho^2$, then $E(z) \subset A_\varrho$ for all $|z| \leq r_0$. □

THEOREM 4. *If μ is a nonnegative regular Borel measure on B such that*

$$\int_B (1 - |w|^2)^n d\mu(w) < \infty,$$

then Green's potential $\mathcal{G}\mu$ of μ is $\mathcal{M}$-superharmonic.

Proof. Fix $\varrho, 0 < \varrho < 1$, and let $\mu_1 = \mu|_{\varrho B}$ and $\mu_2 = \mu - \mu_1$. put $U_1(z) = \mathcal{G}\mu_1, U_2(z) = \mathcal{G}\mu_2$. Consider the function $U_1(z)$. By Fatou's Lemma,

$$\begin{aligned}\varliminf_{z \to z_0} \mathcal{G}\mu_1(z) &= \varliminf_{z \to z_0} \int_{\varrho B} G(z, w) d\mu(w) \\ &\geq \int_{\varrho B} \varliminf_{z \to z_0} G(z, w) d\mu(w) = \int_{\varrho B} G(z_0, w) d\mu(w).\end{aligned}$$

Therefore, for all positive measure μ, $\mathcal{G}\mu_1(z)$ is lower semicontinuous. For fixed ϱ, there exists r_0 such that $E(z) \subset A_\varrho$ for all z, $|z| \geq r_0$, by Lemma 3.

If $|z| \geq r_0$, then $E(z) = \varphi_z(\delta B) \subset A_\varrho$. If $w \in \varrho B$, then $w \notin \varphi_z(\delta B)$. Since $|\varphi_z(w)| \geq \delta$, $0 < \delta < \frac{1}{2}$,

$$g(\varphi_z(w)) \approx (1 - |\varphi_z(w)|^2)^n \tag{3}$$

by Lemma 1. Also

$$(4) \qquad |1-<z,w>| \geq 1-|z||w| \geq 1-\varrho|z| \geq 1-\varrho.$$

By (3),

$$\begin{aligned} G(z,w) &= g(\varphi_z(w)) \\ &\leq C(1-|\varphi_z(w)|^2)^n \\ &\leq C\frac{(1-|z|^2)^n(1-|w|^2)^n}{|1-<z,w>|^{2n}}, \quad w \in \varrho B, \quad |z| \geq r_0 \end{aligned}$$

By (4), $U_1(z) \leq C_\varrho(1-|z|^2)^n \int_{\varrho B}(1-|w|^2)^n d\mu(w)$ for $|z| \geq r_0$. Therefore $U_1 \not\equiv \infty$ on B.

Since $g \circ \varphi_z$ is $\mathcal{M}$-superharmonic and $|\varphi_z(w)| = |\varphi_w(z)|$,

$$\begin{aligned} U_1(z) &= \int_{\varrho B}(g \circ \varphi_w)(z)d\mu(w) \\ &\geq \int_{\varrho B}\int_S (g \circ \varphi_w \circ \varphi_z)(r\zeta)d\sigma(\zeta)d\mu(\omega) \\ &= \int_S\int_{\varrho B}(g \circ \varphi_w \circ \varphi_z)(r\zeta)d\mu(\omega)d\sigma(\zeta) \\ &= \int_S U_1(\varphi_z(r\zeta))d\sigma(\zeta). \end{aligned}$$

So U_1 is $\mathcal{M}$-superharmonic on B.

$$U_2(z) = \int_{A_\varrho} G(z,w)d\mu(w).$$

$$U_2(0) = \int_{A_\varrho} g(w)d\mu(w) \leq C\int_{A_\varrho}(1-|w|^2)^n d\mu(w) < \infty.$$

Hence $U_2 \not\equiv \infty$ and is $\mathcal{M}$-superharmonic on B. □

3. Tangential approach region

For $\zeta \in S, c > 1$ and $\tau > 1$, the tangential approach regions that were used in [4] are

$$\Omega_{c,\tau}(\zeta) = \{z = |z|\eta \in B : |1 - <\eta, \zeta>|^\tau < c(1 - |z|)\}.$$

These regions were also considered by Nagel et al.[5] for $n = 1$ and by Shaw[7] for any n.

THEOREM 5. *If $z \in \Omega_{c,\tau}(\zeta)$, then $E(z) \subset \Omega_{c',\tau}(\zeta)$ for some constant c'.*

Proof. Suppose $z \in B, \zeta \in S$. By Lemma 2,

$$\begin{aligned} 1 - <\varphi_z(w), \zeta> &= 1 - <\varphi_z(w), \varphi_z(\varphi_z(\zeta))> \\ &= \frac{(1 - |z|^2)(1 - <w, \varphi_z(\zeta)>)}{(1 - <w, z>)(1 - <z, \varphi_z(\zeta)>)}. \end{aligned}$$

$$1 - <z, \varphi_z(\zeta)> = 1 - <\varphi_z(0), \varphi_z(\zeta)> = \frac{(1 - |z|^2)}{1 - <z, \zeta>}.$$

Therefore,

$$1 - <\varphi_z(w), \zeta> = \frac{(1 - <z, \zeta>)(1 - <w, \varphi_z(\zeta)>)}{1 - <w, z>}. \tag{5}$$

Suppose $w \in B_\delta,\ 0 < \delta < \frac{1}{2},\ z \in \Omega_{c,\tau}(\zeta)$. Put $\eta = \frac{z}{|z|}$.

$$\begin{aligned} | <\zeta, \eta - z> | &= | <\zeta, \frac{z}{|z|} - z> | \\ &= | <\zeta, \left(\frac{1}{|z|} - 1\right) z> | \\ &= \left(\frac{1}{|z|} - 1\right) | <\zeta, z> | \\ &\le \left(\frac{1}{|z|} - 1\right) |\zeta||z| \le (1 - |z|). \end{aligned}$$

Since $z \in \Omega_{c,\tau}(\zeta)$,

$$
\begin{aligned}
|1-<z,\zeta>|^{\tau} &= |1-<\eta,\zeta>-<z-\eta,\zeta>|^{\tau} \\
&\le (|1-<\eta,\zeta>|+|<z-\eta,\zeta>|)^{\tau} \\
&\le (c^{\frac{1}{\tau}}(1-|z|)^{\frac{1}{\tau}}+(1-|z|))^{\tau} \\
&\le (c^{\frac{1}{\tau}}+1)^{\tau}(1-|z|).
\end{aligned} \tag{6}
$$

By (5) and (6),

$$
\begin{aligned}
&|1-<\varphi_z(w),\zeta>| \\
&\le c_1(1-|z|^2)^{\frac{1}{\tau}}\frac{|1-<w,\varphi_z(\zeta)>|}{|1-<w,z>|} \\
&\le c_1\frac{(1-|\varphi_z(w)|^2)^{\frac{1}{\tau}}|1-<z,w>|^{\frac{2}{\tau}-1}}{(1-|w|^2)^{\frac{1}{\tau}}}|1-<w,\varphi_z(\zeta)>| \\
&\le c_1(1-|\varphi_z(w)|^2)^{\frac{1}{\tau}}3^{\frac{1}{\tau}}.
\end{aligned}
$$

where the second inequality follows by Lemma 2. Therefore,

$$
|1-<\varphi_z(w),\zeta>|^{\tau} \le c(1-|\varphi_z(w)|^2). \tag{7}
$$

$$
\begin{aligned}
|1-<\eta,\zeta>| &\le \left(|1-<z,\zeta>|^{\frac{1}{2}}+|1-<\eta,z>|^{\frac{1}{2}}\right)^2 \\
&\le \left(|1-|z||^{\frac{1}{2}}+|1-<z,\zeta>|^{\frac{1}{2}}\right)^2 \\
&\le \left(2|1-<z,\zeta>|^{\frac{1}{2}}\right)^2.
\end{aligned} \tag{8}
$$

By (7) and (8) ,

$$
\begin{aligned}
|1-<\frac{\varphi_z(w)}{|\varphi_z(w)|},\zeta>|^{\tau} &\le 4^{\tau}|1-<\varphi_z(w),\zeta>|^{\tau} \\
&\le 4^{\tau}c(1-|\varphi_z(w)|^2)^2 \le c'(1-|\varphi_z(w)|).
\end{aligned}
$$

Hence $\varphi_z(w) \in \Omega_{c',\tau}(\zeta)$. □

LEMMA 6. *For $n \geq 1$, there exists a constant p_n such that for all δ, $0 \leq \delta \leq 2$,*

$$\left(\frac{\delta}{2}\right)^n \leq \sigma(Q(\zeta,\delta)) \leq p_n\delta^n.$$

where $Q(\zeta,\delta) = \{\eta \in S : |1- <\zeta,\eta> | < \delta\}$ is the Koranyi ball centered at ζ with radius $\sqrt{\delta}$.

Proof. See [6. Proposition 5.1.4]. □

THEOREM 7. *If μ is nonnegative regular Borel measure on B such that*

$$\int_B (1-|z|^2)^{\frac{n}{\tau}} d\mu(z) < \infty$$

for some $\tau \geq 1$, then $\mu(\Omega_{\tau,c}(\zeta)) < \infty$ for a.e. $\zeta \in S$, for all $c > 0$. Furthermore, for a.e. $\zeta \in S$

$$\lim_{r\to 1} \mu(\Omega_{\tau,c}(\zeta) \cap A_r) = 0.$$

Proof. Let $\tilde{\Omega}_{c,\tau}(z) = \{\zeta \in S : z \in \Omega_{c,\tau}(\zeta)\}$ and $\eta = \frac{z}{|z|}$. If $\zeta \in \tilde{\Omega}_{c,\tau}(z)$, then $|1- <\zeta,\eta> | \leq 2|1- <z,\eta> | \leq 2c^{\frac{1}{\tau}}(1-|z|)^{\frac{1}{\tau}}$. Hence $\zeta \in Q(\eta, 2c^{\frac{1}{\tau}}(1-|z|)^{\frac{1}{\tau}})$. Since $\tilde{\Omega}_{c,\tau}(z) \subset Q(\eta, 2c^{\frac{1}{\tau}}(1-|z|)^{\frac{1}{\tau}})$,

$$\sigma(\tilde{\Omega}_{c,\tau}(z)) \leq \sigma(Q(\eta, 2c^{\frac{1}{\tau}}(1-|z|)^{\frac{1}{\tau}}) \leq C_n(1-|z|)^{\frac{n}{\tau}}$$

by Lemma 6. Put $\mathcal{S}\mu(\zeta) = \mu(\Omega_{c,\tau}(\zeta))$.

$$\begin{aligned}\int_S \mathcal{S}\mu(\zeta)d\sigma(\zeta) &= \int_S\int_B \mathcal{X}_{\Omega_{c,\tau}(\zeta)}(z)d\mu(z)d\sigma(\zeta)\\ &= \int_B\int_S \mathcal{X}_{\tilde{\Omega}_{c,\tau}(z)}(\zeta)d\sigma(\zeta)d\mu(z)\\ &= \int_B \sigma(\tilde{\Omega}_{c,\tau}(z))d\mu(z)\\ &\leq \int_B c_n(1-|z|)^{\frac{n}{\tau}}d\mu(z) < \infty.\end{aligned}$$

Hence $\mathcal{S}\mu \in L^1(S)$. Therefore $\mu(\Omega_{c,\tau}(z)) < \infty$ for a.e. z and

$$\lim_{r\to 1} \mu(\Omega_{c,\tau}(z) \cap A_r) = 0$$

for a.e. $\zeta \in S$. □

Define the functions V on B as follows;

$$V(z) = \int_{E(z)} G(z,w)d\lambda(w)$$

THEOREM 8. *Let μ be a nonnegative regular Borel measure on B satisfying*

$$\int_B (1-|w|^2)^n d\mu(w) < \infty.$$

Then

$$\lim_{z\to\zeta, z\in\Omega_{c,\tau}(\zeta)} V(z) = 0$$

for a.e. $\zeta \in S$.

proof.

$$\int_{\delta B} g(z)^q d\lambda(z) \leq C_1 \int_{\delta B} |z|^{-q(2n-2)} d\lambda(z)$$
$$\leq C_2 \int_0^{\frac{1}{2}} r^{2n-q(2n-2)-1} dr < \infty$$

provided $2n - q(2n-2) > 0$. Thus, for all $q < \frac{n}{n-1}$,

$$\sup_{z\in B} \int_{E(z)} G^q(z,w)d\lambda(z) < \infty.$$

By Holder's inequality,

$$V(z) \leq C\left[\int_{E(z)} d\lambda(w)\right]^{\frac{1}{p}}$$

Suppose $z \in \Omega_{\tau,c}(\zeta)$. Since $E(z) \subset \Omega_{\tau,c'}(\zeta) \cap A_r$ for any $c' \geq c3^{\tau+1}$ and $r^2 = 3|z|^2 - 2$. Thus $V(z) \leq C\left[\int_{\Omega_{\tau,c'}(\zeta)\cap A_r} d\lambda(w)\right]^{\frac{1}{p}}$. By Theorem7,

$$\lim_{z\to\zeta, z\in\Omega_{c,\tau}(\zeta)} V(z) = 0$$

for a.e. $\zeta \in S$.

□

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