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JACOBI FIELDS AND CONJUGATE POINTS IN A COMPLETE RIEMANNIAN MANIFOLD

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ABSTRACT. In this paper, we investigate some properties of Jacobi fields and conjugate points in a complete Riemannian manifold M. Also we get a necessary and sufficient condition about a geodesic without conjugate points in the manifold with non-negative curvature.

1. Jacobi Fields on Space of Constant Curvature

In this paper, we shall make a survey of some relations between the two basic concepts, namely, geodesics and curvature. The curvature $K(p,\sigma)$, determines how fast geodesics, that start from p and arc tangent to σ, spread apart. In order to formalize precisely this velocity of variation of the geodesics, it is necessary to introduce the so called Jacobi fields.

M will denote a n-dimensional complete Riemannian manifold. We shall begin by making precise the idea of neighboring curves of a given curve. We are particularly interested in studying the behavior of the geodesics neighboring $\gamma : [0,a] \to M$, which start from $\gamma(0)$. Thus, we shall consider variation $h : [0,a] \times (-\epsilon,\epsilon) \to M$ that satisfy the condition $h(0,t) = \gamma(0)$ for all $t \in (-\epsilon,\epsilon)$. Therefore, the corresponding Jacobi field satisfies the condition $J(0) = 0$. Now we are going to relate the rate of spreading of the geodesics that start from $p \in M$ with the curvature R at p.

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Now, let $e_1(0), \cdots, e_n(0)$ be unit orthogonal vectors at $p = \gamma(0) \in M$ and $e_1(s), \cdots, e_n(s)$ be the parallel transport of $e_1(0), \cdots, e_n(0)$, respectively, along the geodesic $\gamma(s)$ on M. Then we can write

$$J(s) = \sum f_i(s) e_i(s)$$
$$a_{ij} = < R(\gamma'(s), e_i(s))\gamma'(s), e_j(s) >, \quad (i, j = 1, 2, \cdots, n).$$

Thus

$$\frac{D^2 J(s)}{ds^2} = \sum f_i''(s) e_i(s),$$
$$R(\gamma', J)\gamma' = \sum_j < R(\gamma', J)\gamma', e_j > e_j = \sum_{i,j} f_i a_{ij} e_j.$$

Therefore, the Jacobi equation is equivalent to the system

$$f_j''(s) + \sum_i a_{ij}(s) f_i(s) = 0, \quad (j = 1, \cdots, n).$$

In this section, we obtain some informations on the behavior of geodesics neighboring a given geodesic $\gamma : [0, a] \to M$, and derive some results on M with constant sectional curvature K_0.

Now let $J(s)$ be a Jacobi field along $\gamma(s), < J, \gamma' >= 0$ and $|\gamma'| = 1$, then for all vector field W along γ we have

$$\begin{aligned} < R(\gamma', J)\gamma', J > &= K_0 < \gamma', \gamma' >< J, W > - K_0 < \gamma', J >< \gamma', W > \\ &= K_0 < J, W > . \end{aligned}$$

Thus we have the following theorems.

THEOREM 1.1. ([2], Theorem 3.1) *We have*

$$J(s) = \begin{cases} \frac{1}{\sqrt{K_0}} \sin(s\sqrt{K_0}) w(s) & \text{for } K_0 > 0, \\ s w(s) & \text{for } K_0 = 0, \\ \frac{1}{\sqrt{-K_0}} \sinh(s\sqrt{-K_0}) w(s) & \text{for } K_0 < 0, \end{cases}$$

where $w(s)$ be a parallel vector field along a geodesic $\gamma(s)$ with $< \gamma'(s), w(s) >= 0, |\gamma'(s)| = 1$, and $|w(s)| = 1$.

COROLLARY 1.2. ([2], Corollary 3-2). *For $K > 0, K = 0$, and $K < 0$ we have, respectively,*

$$J(s) = \begin{cases} [a\cos(\sqrt{K_0}|\gamma'(s)|s) + b\sin(\sqrt{K_0}|\gamma'(s)|s)]w(s), \\ (a+bs)w(s), \\ [a\cosh(\sqrt{-K_0}|\gamma'(s)|s) + b\sinh(\sqrt{-K_0}|\gamma'(s)|s)]w(s). \end{cases}$$

where $w(s)$ be a parallel field along a geodesic $\gamma(s)$ ($\neq$ constant) with $<\gamma'(s), w(s)>= 0$ and $|w(s)| = 1$.

COROLLARY 1.3. ([2], Corollary 3-3). *We have*

$$J(s) = \begin{cases} [\frac{1}{\sqrt{K_0}}\sin(\sqrt{K_0}|\gamma'(s)|s)]w(s) & \text{for } K_0 > 0, \\ sw(s) & \text{for } K_0 = 0, \\ [\frac{1}{\sqrt{-K_0}}\sinh(\sqrt{-K_0}|\gamma'(s)|s)]w(s) & \text{for } K_0 < 0. \end{cases}$$

THEOREM 1.4. *Let M be a Riemannian manifold with constant negative sectional curvature $K_0 < 0$. Let $\gamma : [0, a] \to M$ be a normalized geodesic, and let $v \in T_{\gamma(a)}M$ such that $< v, \gamma'(a) >= 0$ and $|v| = 1$. Then the Jacobi field J along γ determined by $J(0) = 0, J(a) = v$ is given by*

$$J(s) = \frac{\sinh(s\sqrt{-K_0})}{\sinh(a\sqrt{-K_0})}w(s),$$

where $w(s)$ is the parallel transport along γ of the vector $w(0) = \frac{u_0}{|u_0|}, u_0 = (d\exp_p)^{-1}_{a\gamma'(0)}(v)$, and where u_0 is considered as a vector $T_{\gamma(0)}M$ by the identification $T_{\gamma(0)}M \approx T_{a\gamma'(0)}(T_{\gamma(0)}M)$.

Proof. From Theorem 1.1, the Jacobi field J_1 along γ satisfying $J_1(0) = 0$ and $J_1'(0) = w(0) = \frac{u_0}{|u_0|}$ is given by

$$J_1(s) = \frac{\sinh s\sqrt{-K_0}}{\sqrt{-K_0}}w(s),$$

and a Jacobi field J_1 along γ with $J_1(0) = 0$ is given by

$$J_1(a) = (d\exp_p)_{a\gamma'(0)}(aw(0)) = \frac{a}{|u_0|}(d\exp_p)_{a\gamma'(0)}(u_0),$$

$$w(0) = J'(0).$$

Thus

$$J(a) = v = (d\exp_p)_{a\gamma'(0)}(u_0) = \frac{|u_0|}{a}J_1(a),$$

or

$$u_0 = (d\exp_p)^{-1}_{a\gamma'(0)}(v).$$

Therefore

$$J(s) = \frac{|u_0|}{a}J_1(s) = \frac{|u_0|}{a}\frac{\sinh s\sqrt{-K_0}}{\sqrt{-K_0}}w(s).$$

On the other hand, since

$$1 = |v| = |J(a)| = \frac{|u_0|}{a}\frac{\sinh a\sqrt{-K_0}}{\sqrt{-K_0}}$$

we obtain

$$\frac{|u_0|}{a} = \frac{\sqrt{-K_0}}{\sinh a\sqrt{-K_0}}.$$

Therefore we have the desired result. □

2. Conjugate Points of a Complete Riemannian Manifold.

Let $\gamma : [0, a] \to M$ be a geodesic starting from $\gamma(0)$. The point $\gamma(s_0), s_0 \in (0, a]$, is called conjugate point of $\gamma(0)$ along γ if there exists a Jacobi field $J(s)$ which is not identically zero along γ with $J(0) = J(s_0) = 0$, and we say that 0 and s_0 are conjugate values along γ. If the dimension of M is n, there exist exactly n−linealy independent Jacobi fields along $\gamma : [0, a] \to M$, which is not zero at $\gamma(0)$. This follows from the fact that the Jacobi fields $J_1, \cdots, J_k$ with

$J_i(0) = 0$ are linearly independent if and only if $J_0'(0), \cdots, J_k'(0)$ are linearly independent. In addition, the Jacobi field $J(s) = s\gamma'(s)$ never vanishes for $s \neq 0$. So we deduce that the multiplicity of a conjugate point never exceed $n - 1$.

Let $\gamma : [0, a] \to M$ be a geodesic and Ω_γ denotes the set of all piecewise vector fields W along γ with $W(0) = W(a) = 0$. Then we have a function $F : \Omega_\gamma \times \Omega_\gamma \to \mathbb{R}$ defined by

$$F(W_1, W_2) = \int_a^0 < W_2, \frac{D^2 W_1}{ds^2} + R(\gamma', W_1)\gamma' > ds - < W_2, \frac{DW_1}{ds} > .$$

THEOREM 2.1. ([5], Theorem 8.6). *Let $\gamma : [0, a] \to M$ be a geodesic with conjugate points. Then there is some $W \in \Omega_\gamma$ with $F(W, W) < 0$.*

THEOREM 2.2. ([5], Proposition 8.9). *Let $\gamma : [0, a] \to M$ be a geodesic without conjugate points. Then $F(W, W) > 0$ for every non-zero $W \in \Omega_\gamma$.*

THEOREM 2.3. ([5], Proposition 8.11). *If all sectional curvature of M are ≤ 0, then no two points of M are conjugate along any geodesic.*

Proof. Let $p \in M$ and let $\gamma : [0, a] \to M$ be a geodesic of M with $\gamma(0) = p$. Assume that there exists a non-vanishing Jacobi field along $\gamma(s)$ with $J(0) = J(a) = 0$. Then we have

$$< J(s), \gamma'(s) >= 0 \quad \text{for all} \quad s \in [0, a].$$

On the other hand, since $K(p, \sigma) \leq 0$, we have

$$\begin{aligned} \frac{d}{ds} < \frac{DJ}{ds}, J > &=< \frac{D^2 J}{ds^2}, J > + < \frac{DJ}{ds}, \frac{DJ}{ds} > \\ &= -K(p, \sigma) < J, J > + |\frac{DJ}{ds}|^2 \geq 0, \end{aligned}$$

which means that $< \frac{DJ}{ds}, J >$ is increasing.

Now if J vanishes at two points, $s = 0$ and $s = a$, then $< \frac{DJ}{ds}, J >=$ 0 at $s = 0$ and $s = a$, so $< \frac{DJ}{ds}, J >$ must be 0 on $[0, a]$.

Finally, since $\frac{d}{ds} < J, J >=< \frac{DJ}{ds}, J >= 0$, we have $< J, J >=$ constant. Remembering the initial condition $J(0) = 0$, we have

$$|J(s)| = 0 \quad \text{for all} \quad s \in [0, a].$$

This contradicts to the hypothesis. Therefore M with $K \leq 0$ does not have conjugate points. □

REMARK. Corollary 1.3 shows that a geodesic γ on a space of constant curvature $K_0 \leq 0$ has no conjugate point. If $K_0 > 0$, then there are conjugate points at precisely $s = \frac{n\pi}{\sqrt{K_0}}|\gamma'|, n = 1, 2, \cdots$. Hence there are no conjugate points if $L(\gamma : [0, a] \to M) = |\gamma'(s)| < \frac{\pi}{\sqrt{K_0}}$, whereas for $L(\gamma : [0, a] \to M) = |\gamma'(s)| \geq \frac{\pi}{\sqrt{K_0}}$, there are conjugate points.

THEOREM 2.4. *Let M be a complete surface with non-negative curvature. A necessary and sufficient condition that a geodesic γ : $[0, a] \to M$ has no conjugate points is*

$$< J', J' > \geq < R(\gamma', J)\gamma', J > .$$

Proof. Suppose there exists a Jacobi field J along γ, not identically zero, with $J(0) = 0 = J(s_0), s \in (0, a]$, and suppose $< J', J' > - < R(\gamma', J)\gamma', J > \geq 0$. Let $f : [0, a] \to < J(s), J(s) >$, then

$$\begin{aligned} f' &= 2 < J', J > (s), \\ f'' &= 2(< J', J' > + < J'', J >)(s) \\ &= 2(< J', J' > - < R(\gamma', J)\gamma', J >)(s) \geq 0. \end{aligned}$$

Thus we have $f(s) \leq 0$ for $s \in [0, s_0]$, which is contradict the hypothesis. Therefore $\gamma(s)$ has no conjugate points.

To prove the converse, suppose now that $\gamma(0) = p$ and there exists a Jacobi field satisfying

$$< J_0', J_0' > (s_0) \quad < \quad < R(\gamma', J_0)\gamma', J_0 > (s_0), \quad s_0 \in (0, a].$$

Then, there are two number s_1 and s_2 satisfying

$$< J_0', J_0' > (s) - < R(\gamma'', J_0)\gamma', J_0 > (s) < 0 \text{ for all } s \in [s_1, s_2].$$

$$I(J_0, J_0)|_{[s_1,s_2]} = \int_{s_1}^{s_2} (< J_0', J_0' > - < R(\gamma', J_0)\gamma', J_0 >)(s)dt < 0.$$

Now put, for any $\epsilon > 0$,

$$W_\epsilon = \begin{cases} J_{-\epsilon} & \text{for } s \in [s_1 - \epsilon, s_1], \\ J_0 & \text{for } s \in [s_1, s_2], \\ J_{+\epsilon} & \text{for } s \in [s_2, s_2 + \epsilon]. \end{cases}$$

Then we have

$$J_{-\epsilon}(s_1 - \epsilon) = 0,$$
$$J_{-\epsilon}(s_1) = J_0(s_1),$$
$$J_{+\epsilon}(s_2) = J_0(s_2),$$
$$J_{+\epsilon}(s_1 + \epsilon) = 0.$$

Since $J_{-\epsilon}$ and $J_{+\epsilon}$ are unique, $s_1 - \epsilon$ and $s_2 + \epsilon$ are not conjugate values along γ. Therefore, since γ has no conjugate points, we obtain

$$\begin{aligned} 0 &\leq I(W_\epsilon, W_\epsilon)|_{[s_1-\epsilon, s_2+\epsilon]} \\ &= I(J_{-\epsilon}, J_{-\epsilon})|_{[s_1-\epsilon, s_1]} + I(J_0, J_0)|_{[s_1,s_2]} + I(J_{+\epsilon}, J_{+\epsilon})|_{[s_2,s_2+\epsilon]} \\ &= < J_\epsilon', J_{-\epsilon} > (s_1) + I < J_0, J_0 > |_{[s_1,s_2]} - < J_{+\epsilon}', J_{+\epsilon} > (s_2). \end{aligned}$$

As $a \to \infty$ we have

$$< J_{-\epsilon}', J_{-\epsilon} > (s_1) \to 0, \quad < J_{+\epsilon}', J_{+\epsilon} > (s_2) \to 0.$$

Thus

$$0 \leq I(J_0, J_0)|_{[s_1,s_2]} = \int_{s_1}^{s_2} (< J_0', J_0' > - < R(\gamma', J_0)\gamma', J_0 >)(s)ds,$$

which contradicts the condition. □

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