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LINEAR JORDAN DERIVATIONS ON NONCOMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. The purpose of this paper is to prove the following result: Let A be a noncommutative Banach algebra. Suppose that there exist continuous linear Jordan derivations $D : A \to A$, $G : A \to A$ such that $[D^2(x)+G(x), x^n]$ lies in the Jacobson radical of A for all $x \in A$. Then $D(A) \subset rad(A)$ and $G(A) \subset rad(A)$.

1. Introduction

Throughout this paper R will represent an associative ring with center $Z(R)$. We also write $[x,y]$ for $xy - yx$. Recall that a ring R is prime if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$. An additive mapping D from R to R is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. A derivation D is inner if there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x, y \in R$. An additive mapping D from R to R is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. Obviously, every derivation is a Jordan derivation. The converse is in general not true. Brešar [1] showed that every Jordan derivation on a 2-torsion free semiprime ring is a derivation. Sinclair [6] proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals invariant, and that the only linear derivation on a commutative semisimple Banach algebras is zero. Combining the above results, we can conclude as follows: every continuous linear Jordan derivation on

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a commutative Banach algebra maps the algebra into its Jacobson radical. Now it seems natural to ask, under additional assumptions the range of a continuous linear Jordan derivation on a noncommutative Banach algebra is contained in the Jacobson radical. It is the purpose of this paper to present partial answers to the above question.

The next lemma is the result of Chung and Luh [3, Lemma1].

LEMMA 1.1. *Let R be an $n!$-torsion free ring. If $y_1, y_2, \ldots, y_n \in R$ satisfy*

$$ty_1 + t^2 y_2 + \cdots + t^n y_n = 0, \; t = 1, 2, \ldots, n.$$

Then $y_i = 0$ for all i.

The following theorem is due to Vukman [2, Theorem1].

THEOREM 1.2. *Let D and G be continuous linear derivations of a Banach algebra A such that*

$$[D^2(x) + G(x), x] \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subset rad(A)$ and $G(A) \subset rad(A)$.

2. Main Results

For the proof of the main theorem we shall need the following purely algebraic result which is motivated by Theorem 1.2.

THEOREM 2.1. *Let n denote a fixed positive integer. Let R be a noncommutative prime ring with $n!$-torsion free. Suppose that there exist $D : R \to R$, $G : R \to R$ derivations such that*

$$[D^2(x) + G(x), x^n] = 0, \; x \in R.$$

Then we have $D = 0$ and $G = 0$ on R.

Proof. Suppose that

$$[D^2(x) + G(x), x^n] = 0 \tag{1}$$

holds for all $x \in R$.

Replacing $x+ty$ for x in (1), we obtain

$$tP_1(x,y)+t^2P_2(x,y)+\cdots+t^nP_n(x,y)=0, \qquad x,y\in R, \tag{2}$$

where $P_k(x,y)$ is a polynomial in x,y such that $P_k(x,ty)=t^kP_k(x,y)$, $t\in\mathbb{N}$. Then by Lemma 1.1, we have

$$\begin{aligned} P_1(x,y)=&[D^2(y)+G(y),x^n]+[D^2(x)+G(x),x^{n-1}y]\\ &+[D^2(x)+G(x),x^{n-2}yx]+[D^2(x)+G(x),x^{n-3}yx^2]\\ &+\cdots+[D^2(x)+G(x),yx^{n-1}]=0, \qquad x,y\in R. \end{aligned} \tag{3}$$

Let us write xy instead of y in (3). Then we have

$$\begin{aligned} 0=&x[D^2(x)+G(x),x^{n-1}y]+[D^2(x)+G(x),x]x^{n-1}y\\ &+x[D^2(x)+G(x),x^{n-2}yx]+[D^2(x)+G(x),x]x^{n-2}yx\\ &+x[D^2(x)+G(x),x^{n-3}yx^2]+[D^2(x)+G(x),x]x^{n-3}yx^2\\ &+\cdots+x[D^2(x)+G(x),yx^{n-1}]+[D^2(x)+G(x),x]yx^{n-1}\\ &+(D^2(x)+G(x))[y,x^n]+2[D(x)D(y),x^n]+x[D^2(y)\\ &+G(y),x^n], \qquad x,y\in R. \end{aligned} \tag{4}$$

Left multiplication of (3) by x gives

$$\begin{aligned} 0=&x[D^2(y)+G(y),x^n]+x[D^2(x)+G(x),x^{n-1}y]\\ &+x[D^2(x)+G(x),x^{n-2}yx]+x[D^2(x)+G(x),x^{n-3}yx^2]\\ &+\cdots+x[D^2(x)+G(x),yx^{n-1}], \qquad x,y\in R. \end{aligned} \tag{5}$$

Subtracting (5) from (4), we obtain

$$\begin{aligned} 0=&[D^2(x)+G(x),x]x^{n-1}y+[D^2(x)+G(x),x]x^{n-2}yx\\ &+[D^2(x)+G(x),x]x^{n-3}yx^2+\cdots+[D^2(x)+G(x),x]yx^{n-1}\\ &+(D^2(x)+G(x))[y,x^n]+2[D(x)D(y),x^n], \qquad x,y\in R. \end{aligned} \tag{6}$$

Substituting yx for y in (6), we have

$$
\begin{aligned}
0 =&[D^2(x)+G(x),x]x^{n-1}yx+[D^2(x)+G(x),x]x^{n-2}yx^2\\
&+[D^2(x)+G(x),x]x^{n-3}yx^3+\cdots+[D^2(x)+G(x),x]yx^n\\
&+(D^2(x)+G(x))[y,x^n]x+2[D(x)D(y),x^n]x+2[D(x)\\
&yD(x),x^n], \qquad x,y\in R.
\end{aligned}
\tag{7}
$$

Right multiplication of (6) by x gives

$$
\begin{aligned}
0 =&[D^2(x)+G(x),x]x^{n-1}yx+[D^2(x)+G(x),x]x^{n-2}yx^2\\
&+[D^2(x)+G(x),x]x^{n-3}yx^3+\cdots+[D^2(x)+G(x),x]yx^n\\
&+(D^2(x)+G(x))[y,x^n]x+2[D(x)D(y),x^n]x, \qquad x,y\in R.
\end{aligned}
\tag{8}
$$

Subtracting (8) from (7), we get

$$D(x)yD(x)x^n - x^nD(x)yD(x) = 0, \qquad x,y\in R, \tag{9}$$

since R is 2-torsion free. Replacing y by $yD(x)z$ in (9), we obtain

$$D(x)yD(x)zD(x)x^n - x^nD(x)yD(x)zD(x) = 0, \qquad x,y,z\in R.$$

By (9) we can write in the above relation $x^nD(x)zD(x)$ instead of $D(x)zD(x)x^n$ and $D(x)yD(x)x^n$ for $x^nD(x)yD(x)$, which gives

$$D(x)y[D(x),x^n]zD(x) = 0, \qquad x,y,z\in R. \tag{10}$$

Putting x^ny for y in (10), we have

$$D(x)x^ny[D(x),x^n]zD(x) = 0, \qquad x,y,z\in R. \tag{11}$$

Left multiplication of (10) by x^n leads to

$$x^nD(x)y[D(x),x^n]zD(x) = 0, \qquad x,y,z\in R. \tag{12}$$

Subtracting (12) from (11), we obtain

$$[D(x), x^n]y[D(x), x^n]zD(x) = 0, \qquad x, y, z \in R. \tag{13}$$

Replacing z by zx^n in (13), we get

$$[D(x), x^n]y[D(x), x^n]zx^nD(x) = 0, \qquad x, y, z \in R. \tag{14}$$

Right multiplication of (13) by x^n gives

$$[D(x), x^n]y[D(x), x^n]zD(x)x^n = 0, \qquad x, y, z \in R. \tag{15}$$

Subtracting (15) from (14), we get

$$[D(x), x^n]y[D(x), x^n]z[D(x), x^n] = 0, \qquad x, y, z \in R. \tag{16}$$

Since R is a prime ring, we see that $[D(x), x^n] = 0$ holds for all $x \in R$, which implies that $D = 0$ holds by [4, Corollary]. By hypothesis we then have $[G(x), x^n] = 0$ for all $x \in R$. And so [4, Corollary] implies that $G = 0$. The proof of the theorem is complete. □

Now, let us prove our main results in using the above algebraic result.

THEOREM 2.2. *Let A be a noncommutative Banach algebra. Suppose that there exist continuous linear Jordan derivations $D : A \to A$ and $G : A \to A$ such that*

$$[D^2(x) + G(x), x^n] \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subset rad(A)$ and $G(A) \subset rad(A)$.

Proof. By [6,Lemma 3.2] every continuous linear Jordan derivation on a Banach algebra leaves all primitive ideals invariant. Since the

Jacobson radical is the intersection of all primitive ideals, we have $D(rad(A)) \subset rad(A)$, and $G(rad(A)) \subset rad(A)$, which means that there is no loss of generality in assuming that A is semisimple. Since D and G leave all primitive ideals invariant, one can introduce for any primitive ideal $P \subset A$ a linear Jordan derivation

$$D_p : A/P \to A/P,\ G_p : A/P \to A/P,$$

where A/P is a factor Banach algebra, by

$$D_P(x+P) = D(x) + P,\ G_p(x+P) = G(x) + P,\ x \in A.$$

The factor algebra A/P is prime, since P is a primitive ideal. Hence by [5,Theorem] D_p is a derivation. The assumption of the theorem gives

$$[D_P^2(x+P) + G_p(x+P),\ (x+P)^n] = 0,\ x+P \in A/P.$$

Thus, in case A/P is noncommutative, we have $D_P = 0$, $G_p = 0$, since all the assumptions of Theorem 2.1 are fulfilled. In case A/P is a commutative Banach algebra, we can conclude that $D_P = 0$, $G_p = 0$ as well since A/P is semisimple and since we know that therefore no nonzero linear Jordan derivations on commutative semisimple Banach algebras. In any case we have $D_p = 0$, $G_p = 0$. Hence we see that $D(A) \subset rad(A)$, $G(A) \subset rad(A)$, since P was any primitive ideal of A. The proof of the theorem is complete. □

By Theorem 2.2, we can obtain the following Corollary.

COROLLARY 2.3. *Let A be a noncommutative semisimple Banach algebra. Suppose that there exist linear Jordan derivations $D : A \to A$, $G : A \to A$ such that*

$$[D^2(x) + G(x), x^n] = 0$$

for all $x \in A$. Then we have $D = 0$ and $G = 0$.

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